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Mathematical structures for Quantum Mechanics and connections to operational principles

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Operational Probabilistic Theories as Foils to Quantum Theory

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July 4th 2007

Postulates

- **Postulate 1 (Independent systems)** *There exist **independent** systems.*

- **Postulate 2 (Symmetric faithful state)** *For every composite system made of two identical physical systems there exists a **symmetric joint state** that is both **dynamically** and **preparationally faithful**.*

- **Postulate 3 (Local observability principle)** *For composite systems **local informationally complete observables** provide **global informationally complete observables**.*

- **Postulate 4 (Info-complete discriminating observable)** *For every system there exists a minimal info-complete observable that can be achieved using a joint **discriminating observable** on system+ ancilla.*

P1÷P4 ⟶ Hilbert space

P1,P2 ⟶ C*-algebra

Postulates (in progress)

- **Postulate 1 (Independent systems)** *There exist* ***independent*** *systems.*
- **Postulate 2 (Symmetric faithful state)** *For every composite system made of two identical physical systems there exists a* ***symmetric joint state*** *that is both* ***dynamically*** *and* ***preparationally faithful****.*
- **Postulate 3 (Pure symmetric faithful state)** *If there exists a* <u>*pure*</u> *symmetric faithful state then we have Quantum Mechanics*

Actions and outcomes

Experiment (or "action"): every experiment is described by a set $\mathbb{A} \equiv \{\mathscr{A}_j\}$ of possible transformations $\mathscr{A}_j$ having overall unit probability, with the apparatus signaling the outcome j labeling which transformation actually occurred.

States

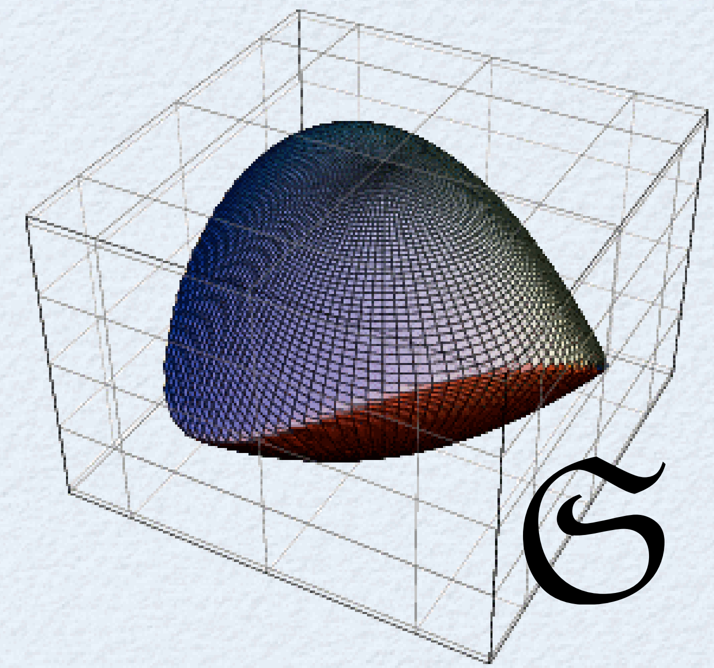
State: A state ω for a physical system is a rule which provides the probability for any possible transformation within an experiment, namely:

$$\omega : \boldsymbol{state}, \quad \omega(\mathscr{A}) : probability\ that\ the\ transformation\ \mathscr{A}\ occurs$$

Normalization: $$\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$$

Identity transformation: $\omega(\mathscr{I}) = 1$

States and transformations



- ***States*** make a convex set $\mathfrak{S}$
- ***Transformations*** make a monoid $\mathfrak{T}$

Independent systems and local transformations

Independent systems and local experiments: two physical systems are "independent" if on each system it is possible to perform "local experiments" for which on every joint state one has the commutativity of the pertaining transformations

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}$$

$$(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) \doteq \mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} \circ \mathscr{C}^{(3)} \circ \ldots$$

Multipartite system: a collection of independent systems

Local state

For a multipartite system we define the local state $\omega|_n$ of the *n*-th system the state that gives the probability of any local transformation $\mathscr{A}$ on the *n*-th system with all other systems untouched, namely

$$\omega|_n(\mathscr{A}) \doteq \Omega(\mathscr{I}, \ldots, \mathscr{I}, \underbrace{\mathscr{A}}_{nth}, \mathscr{I}, \ldots)$$

Conditional state

When composing two transformations $\mathscr{A}$ and $\mathscr{B}$ the probability that $\mathscr{B}$ occurs conditioned that $\mathscr{A}$ occurred before is given by

$$p(\mathscr{B}|\mathscr{A}) = \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

Conditional state: the conditional state $\omega_{\mathscr{A}}$ gives the probability that a transformation $\mathscr{B}$ occurs on the physical system in the state ω after the transformation $\mathscr{A}$ occurred, namely

$$\boxed{\omega_{\mathscr{A}}(\mathscr{B}) \doteq \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}}$$

No-signaling from the future

[Ozawa] The definition of conditional state needs to assume that

$$\sum_{\mathscr{B}_j \in \mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A}) = \omega(\mathscr{A}), \quad \forall \mathbb{B}, \forall \mathscr{A}.$$

This is no-signaling from the future.

Weights and Operations

Weight: un-normalized state $\boxed{\omega = \dfrac{\tilde{\omega}}{\tilde{\omega}(\mathscr{I})}}$

$$0 \leqslant \tilde{\omega}(\mathscr{A}) \leqslant \tilde{\omega}(\mathscr{I}) < +\infty$$

convex cone of weights: $\mathfrak{W}$

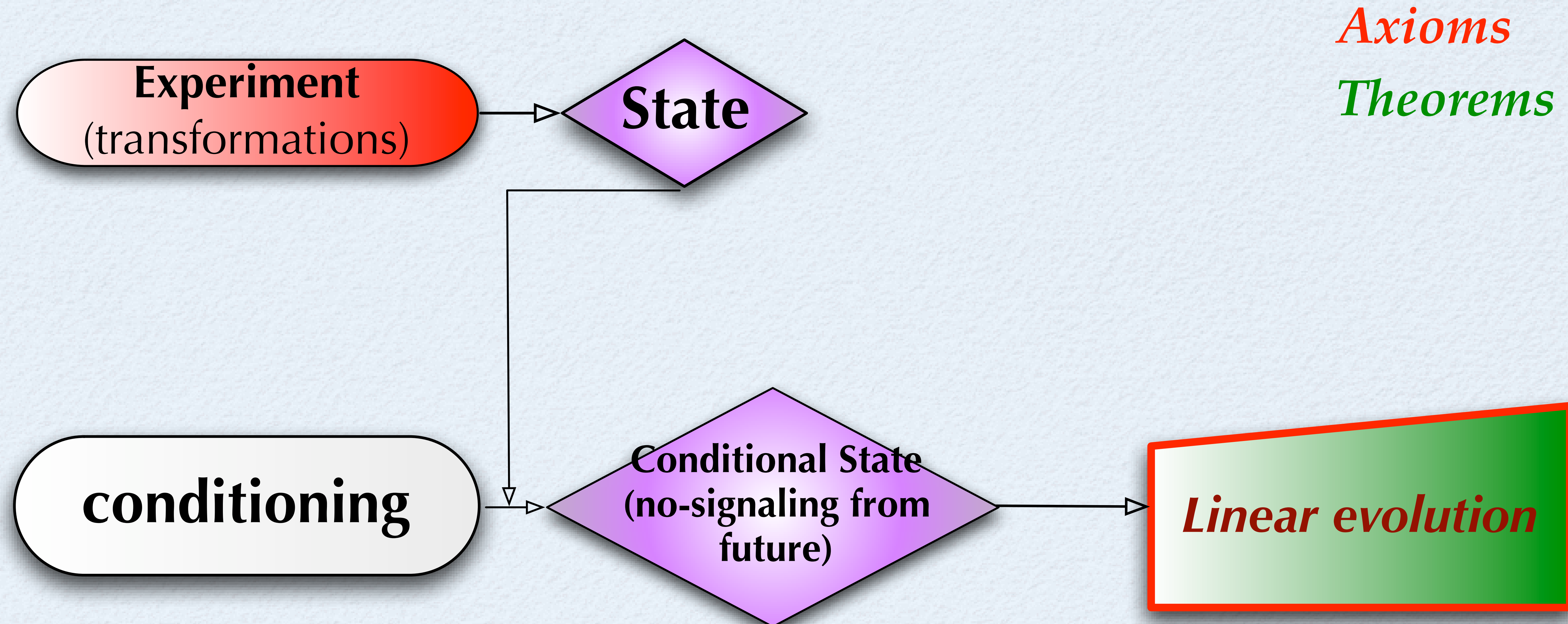
Operation: $\boxed{\mathrm{Op}_{\mathscr{A}}\,\omega \doteq \tilde{\omega}_{\mathscr{A}} = \omega(\cdot \circ \mathscr{A})}$ $\qquad \tilde{\omega}_{\mathscr{A}}(\mathscr{B}) = \omega(\mathscr{B} \circ \mathscr{A})$

Action of a transformation over a state ("Schrödinger picture"):

$$\boxed{\mathscr{A}\,\omega := \mathrm{Op}_{\mathscr{A}}\,\omega}$$

$$(\mathscr{A}\,\omega)(\mathscr{B}) := \omega(\mathscr{B} \circ \mathscr{A})$$

Evolution as conditioning



Dynamical and informational equivalence

From the definition of conditional state we have:

- there are different transformations which always produce the **same state change, but** generally occur with **different probabilities**
- there are different transformations which always occur with the **same probability, but** generally affect a **different state change**

Dynamical and informational equivalence

Dynamical equivalence of transformations: two transformations $\mathscr{A}$ and $\mathscr{B}$ are dynamically equivalent if

$$\boxed{\omega_{\mathscr{A}} = \omega_{\mathscr{B}}} \qquad \forall \omega \in \mathfrak{S}$$

Informational equivalence of transformations: two transformations $\mathscr{A}$ and $\mathscr{B}$ are informationally equivalent if

$$\boxed{\omega(\mathscr{A}) = \omega(\mathscr{B})} \qquad \forall \omega \in \mathfrak{S}$$

A transformation is completely specified by the two classes

Addition of transformations

Two transformations $\mathscr{A}$ and $\mathscr{B}$ are ***informationally compatible*** (or coexistent) if for every state ω one has

$$\omega(\mathscr{A}) + \omega(\mathscr{B}) \leq 1$$

For any two coexistent transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ we define the transformation $\boxed{\mathscr{A}_1 + \mathscr{A}_2}$ as the transformation corresponding to the event $e = \{1, 2\}$ namely the apparatus signals that either $\mathscr{A}_1$ or $\mathscr{A}_2$ occurred, but doesn't specify which one:

$$\forall \omega \in \mathfrak{S} \qquad \omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2)$$ ***(info-class)***

$$\forall \omega \in \mathfrak{S} \qquad \omega_{\mathscr{A}_1 + \mathscr{A}_2} = \frac{\omega(\mathscr{A}_1)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_1} + \frac{\omega(\mathscr{A}_2)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_2}$$ ***(dyn-class)***

$$(\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{A}_1 \omega + \mathscr{A}_2 \omega$$

$$\boxed{\circ, + \text{ distributive}}$$

Rescaling of transformations

Multiplication by a scalar: for each transformation $\mathscr{A}$ the transformation $\lambda\mathscr{A}$ for $0 \leq \lambda \leq 1$ is defined as the transformation which is dynamically equivalent to $\mathscr{A}$ but occurs with probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$

Convex structure for transformations $\mathfrak{T}$
and for actions

Effect

We call ***effect*** an informational equivalence class $\underline{\mathscr{A}}$ of transformations $\mathscr{A}$

***"Heisenberg picture"*:**
(from the notion of conditional state)

$$\boxed{\mathrm{Op}_{\mathscr{A}}\,\underline{\mathscr{B}} = \underline{\mathscr{B}\circ\mathscr{A}} = \underline{\mathscr{B}\circ\underline{\mathscr{A}}}}$$

duality

effects as positive linear functionals l over states:

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

Convex structure for effects $\mathfrak{P}$

No-signaling

The occurrence of the transformation $\mathscr{B}$ on system 1 generally affects the local state on system 2, i. e.

$$\Omega_{\mathscr{B},\mathscr{I}}|_2 \neq \Omega_2$$

However a local action $\mathbb{A} \equiv \{\mathscr{A}_j\}$ on system 2 does not affect the local state on system 1, more precisely:

acausality of local actions: any local action on a system is equivalent to the identity transformation on another independent system.

$$\boxed{\mathbb{A} \equiv \mathscr{S}(\mathbb{A}) := \sum_{\mathscr{A}_j \in \mathbb{A}} \mathscr{A}_j}$$

$$\forall \Omega \in \mathfrak{S}^{\times 2}, \forall \mathbb{A}, \qquad \Omega_{\mathbb{A},\mathscr{I}}|_2 = \Omega|_2$$

No-signaling

Theorem 1 (No-signaling) *Any local action on a system does not affect another independent system. More precisely, any local action on a system is equivalent to the identity transformation when viewed from another independent system. In equations one has*

$$\forall \Omega \in \mathfrak{S}^{\times 2}, \forall \mathbb{A}, \qquad \Omega_{\mathbb{A},\mathscr{I}}|_2 = \Omega|_2. \tag{1}$$

Proof. Since the two systems are dynamically independent, for every two local transformations one has $\mathscr{A}^{(1)} \circ \mathscr{A}^{(2)} = \mathscr{A}^{(2)} \circ \mathscr{A}^{(1)}$, which implies that $\Omega(\mathscr{A}^{(1)} \circ \mathscr{A}^{(2)}) = \Omega(\mathscr{A}^{(2)} \circ \mathscr{A}^{(1)}) \equiv \Omega(\underline{\mathscr{A}}^{(1)}, \underline{\mathscr{A}}^{(2)})$. By definition, for $\mathscr{B} \in \mathfrak{T}$ one has $\Omega|_2(\mathscr{B}) = \Omega(\mathscr{I}, \mathscr{B})$, and using the addition rule for transformations and reminding the identification $\mathbb{A} \equiv \sum_j \mathscr{A}_j$, one has

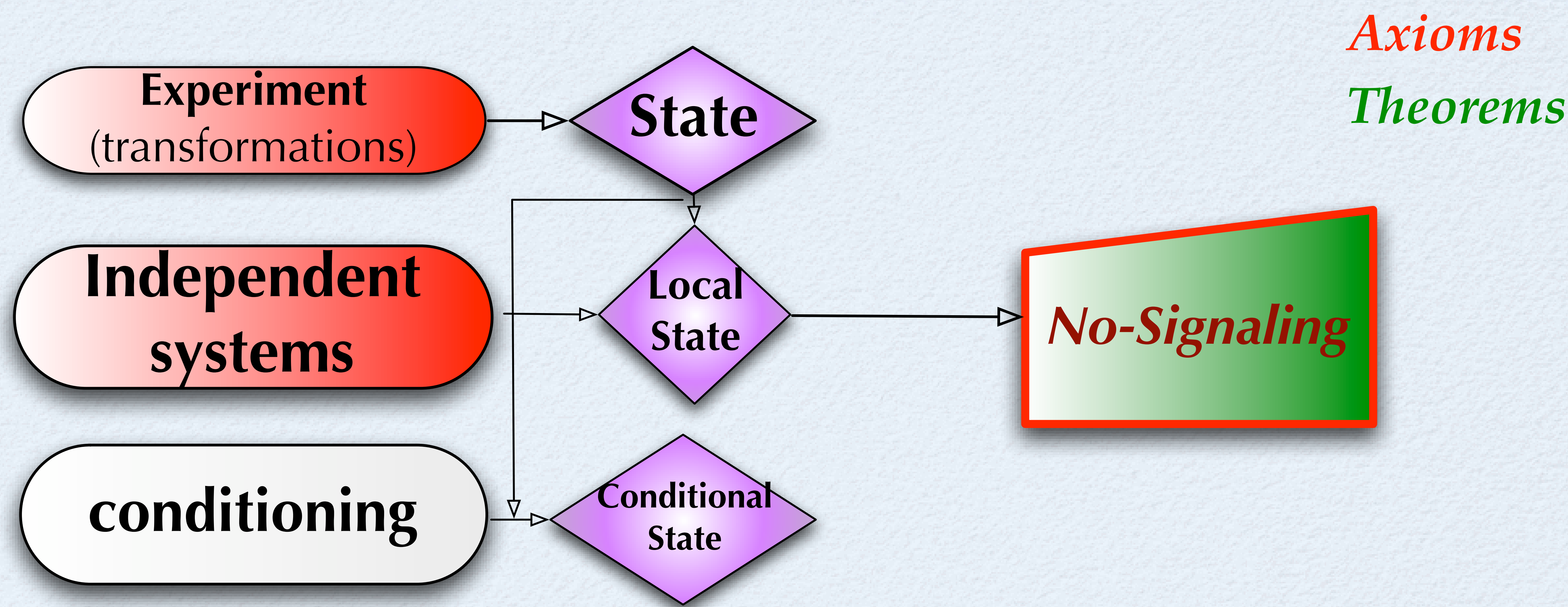
$$\Omega(\mathbb{A}, \mathscr{B}) = \Omega(\underline{\mathbb{A}}, \underline{\mathscr{B}}) = \Omega(\underline{\mathscr{I}}, \underline{\mathscr{B}}) =: \Omega|_2(\mathscr{B}). \tag{2}$$

On the other hand, we have

$$\Omega_{\mathbb{A},\mathscr{I}}|_2(\mathscr{B}) = \Omega((\mathscr{I}, \mathscr{B}) \circ (\mathbb{A}, \mathscr{I})) = \Omega(\mathbb{A}, \mathscr{B}), \tag{3}$$

namely the statement. ■

No-signaling from dynamical independence



Generalized weights, transformations, and effects

Generalize by taking differences:

convex sets/cones → (affine) linear spaces

weights $\mathfrak{W}$ → gen. weights $\mathfrak{W}_{\mathbb{R}}$

transformations $\mathfrak{T}$ → gen. transformations $\mathfrak{T}_{\mathbb{R}}$
(real algebra)

effects $\mathfrak{P}$ → gen. effects $\mathfrak{P}_{\mathbb{R}}$

Real Banach spaces

norms:

gen. effects $\mathfrak{P}_{\mathbb{R}}$: $\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$

gen. weights $\mathfrak{W}_{\mathbb{R}}$: $\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$

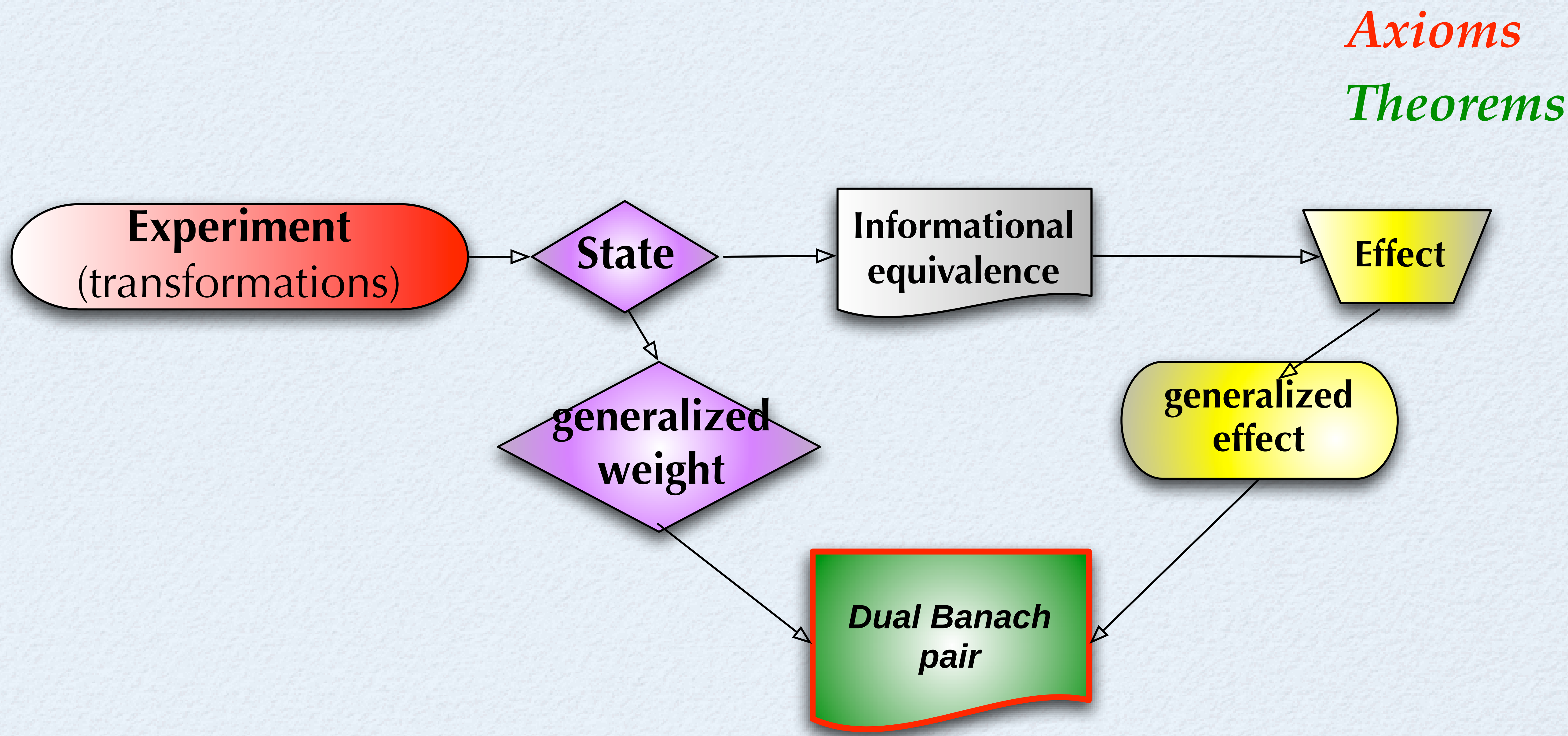
gen. transformations $\mathfrak{T}_{\mathbb{R}}$: $\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\|$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual Banach pair*** under the pairing

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

$\mathfrak{T}_{\mathbb{R}}$ ***Banach algebra***

Banach-space structures



Observable

Observable: a complete set of effects $\mathbb{L} = \{l_i\}$

$$\sum_j l_j = \underline{\mathscr{I}}$$

Informationally complete observable

Informationally complete observable: an observable $\mathbb{L} = \{l_i\}$ is informationally complete if any effect l can be written as linear combination of elements of $\mathbb{L}$, namely there exist coefficients $c_i(l)$ such that

$$l = \sum_{i=1}^{|\mathbb{L}|} c_i(l) l_i$$

affine dimension: $\dim(\mathfrak{S}) = |\mathbb{L}| - 1$, for $\mathbb{L}$ minimal informationally complete on $\mathfrak{S}$

Bloch representation

$$l_{\underline{\mathscr{A}}} = \sum_j m_j(\underline{\mathscr{A}})n_j \qquad l_{\underline{\mathscr{A}}}(\omega) = \boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})$$

Conditioning: fractional affine transformation

$$\boldsymbol{n}(\omega) \longrightarrow \boldsymbol{n}(\omega_{\mathscr{A}})$$

$$\boldsymbol{n}(\omega_{\mathscr{A}}) = \frac{\boldsymbol{M}(\mathscr{A})\boldsymbol{n}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})}$$

$$M_{ij}(\mathscr{A}) = \begin{pmatrix} \boxed{q(\underline{\mathscr{A}})} & \boxed{\boldsymbol{m}(\underline{\mathscr{A}})} \\ \boxed{\boldsymbol{k}(\mathscr{A})} & \boxed{\boldsymbol{M}(\mathscr{A})} \end{pmatrix}$$

Informationally complete observable

Theorem: *there always exists a minimal informationally complete observable.*

Proof. By definition $\mathfrak{P}_\mathbb{R} = \mathsf{Span}_\mathbb{R}(\mathfrak{P})$, whence there must exists a spanning set for $\mathfrak{P}_\mathbb{R}$ that is contained in $\mathfrak{P}$. The maximal number of elements of this set that are linearly independent will constitute a *basis*, which we suppose has finite cardinality $\dim(\mathfrak{P}_\mathbb{R})$. It remains to be shown that it is possible to have a basis with sum of elements equal to $\underline{\mathscr{I}}$, and that such basis is obtained operationally starting from the available observables from which we constructed $\mathfrak{P}$.

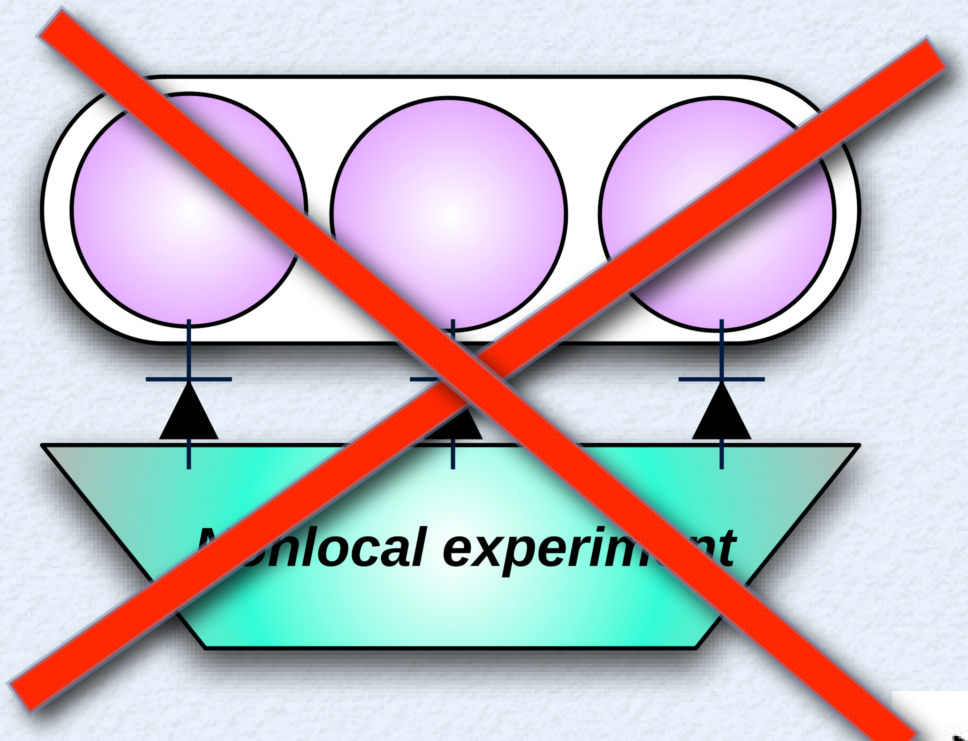
If all observables are *uninformative* (i. e. with all effects proportional to $\underline{\mathscr{I}}$) , then $\mathfrak{P}_\mathbb{R} = \mathsf{Span}(\underline{\mathscr{I}})$, $\underline{\mathscr{I}}$ is a minimal infocomplete observable, and the statement of the theorem is proved. Otherwise, there exists at least an observable $\mathbb{E} = \{l_i\}$ with $n \geqslant 2$ linearly independent effects. If this is the only observable, again the theorem is proved. Otherwise, take a new binary observable $\mathbb{E}_2 = \{x, y\}$ from the set of available ones (you can take different binary observables out of a given observable with more than two outcomes by summing up effects to yes-no observables). If $x \in \mathsf{Span}(\mathbb{E})$ discard it. If $x \notin \mathsf{Span}(\mathbb{E})$, then necessarily also $y \notin \mathsf{Span}(\mathbb{E})$ [since if there exists coefficients λ_i such that $y = \sum_i \lambda_i l_i$, then $x = \sum_i (1-\lambda_i) l_i$]. Now, consider the observable

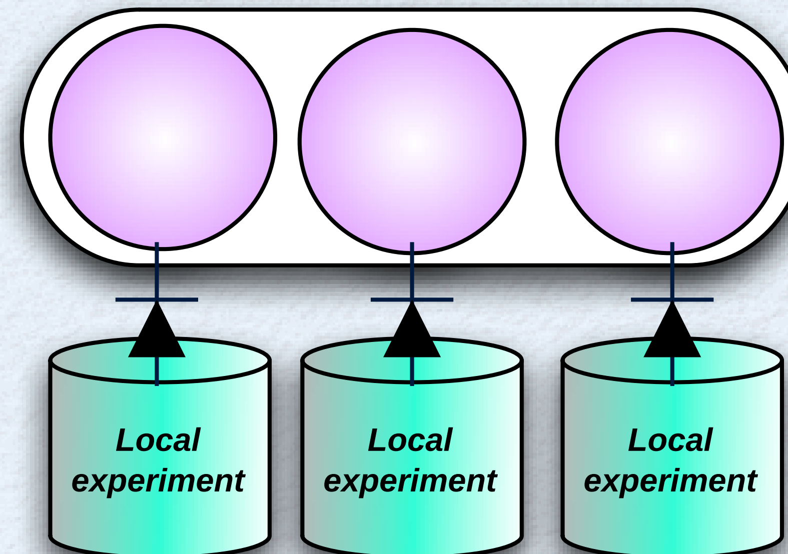
$$\mathbb{E}' = \left\{\tfrac{1}{2}y, \tfrac{1}{2}(l_1 + x), \tfrac{1}{2}l_2, \ldots, l_n\right\} \tag{1}$$

(which operationally corresponds to the random choice between the observables $\mathbb{E}$ and $\mathbb{E}_2$ with probability $\frac{1}{2}$, and with the events corresponding to x and l_1 made indistinguishable). This new observable has now $|\mathbb{E}'| = n+1$ linearly independent effects (since y is linearly independent on the l_i and one has $y = \sum_{i=1}^n l_i - x = \sum_{i=2}^n l_i + l_1 - x$). By iterating the above procedure we reach $|\mathbb{E}'| = \dim(\mathfrak{P}_\mathbb{R})$, and we have so realized an apparatus that measures a minimal informationally complete observable. ■

Local observability principle

For composite systems local info-complete observables provide global info-complete observables.





Holism

Reductionism

identity for the affine dimension of composite systems

$$\boxed{\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)}$$

Local observability principle

identity for the affine dimension of composite systems

$$\boxed{\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)}$$

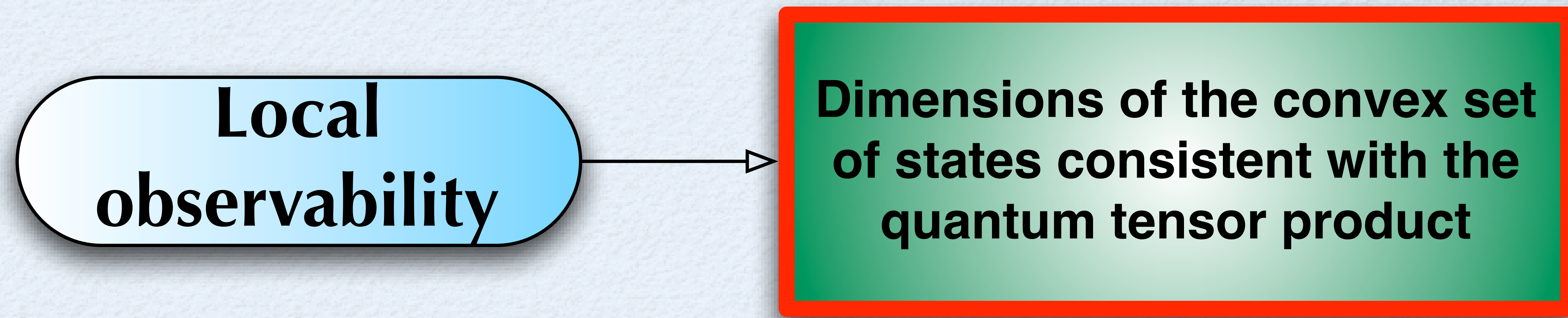
Proof. We first prove that the left side is a lower bound for the right side. Indeed, the number of outcomes of a minimal informationally complete observable is $\dim(\mathfrak{S}) + 1$, since it equals the dimension of the affine space embedding the convex set of states $\mathfrak{S}$ plus an additional dimension for normalization. Now, consider a global informationally complete measurement made of two local minimal informationally complete observables measured jointly. It has number of outcomes $[\dim(\mathfrak{S}_1) + 1][\dim(\mathfrak{S}_2) + 1]$. However, we are not guaranteed that the joint observable is itself minimal, whence the bound.

The opposite inequality can be easily proved by considering that a global informationally incomplete measurement made of minimal local informationally complete measurements should belong to the linear span of a minimal global informationally complete measurement. ■

In Quantum Mechanics we have: $\dim(\mathfrak{S}) = \dim(\mathsf{H})^2 - 1$

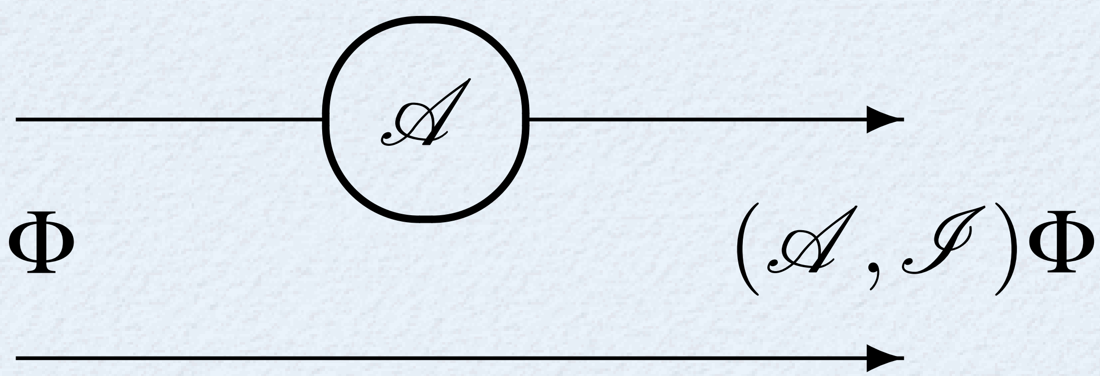
Local observability principle

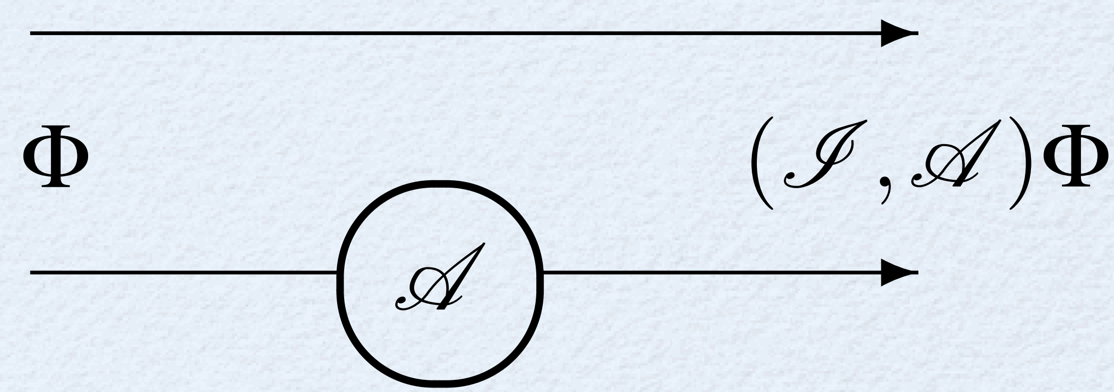
Postulates
Axioms
Theorems



Faithful states

Dynamically faithful state: we say that a state Φ of a bipartite system is dynamically faithful if when acting on it with a local transformation $\mathscr{A}$ on one system the output conditioned weight $(\mathscr{A},\mathscr{I})\Phi$ is in 1-to-1 correspondence with the transformation $\mathscr{A}$

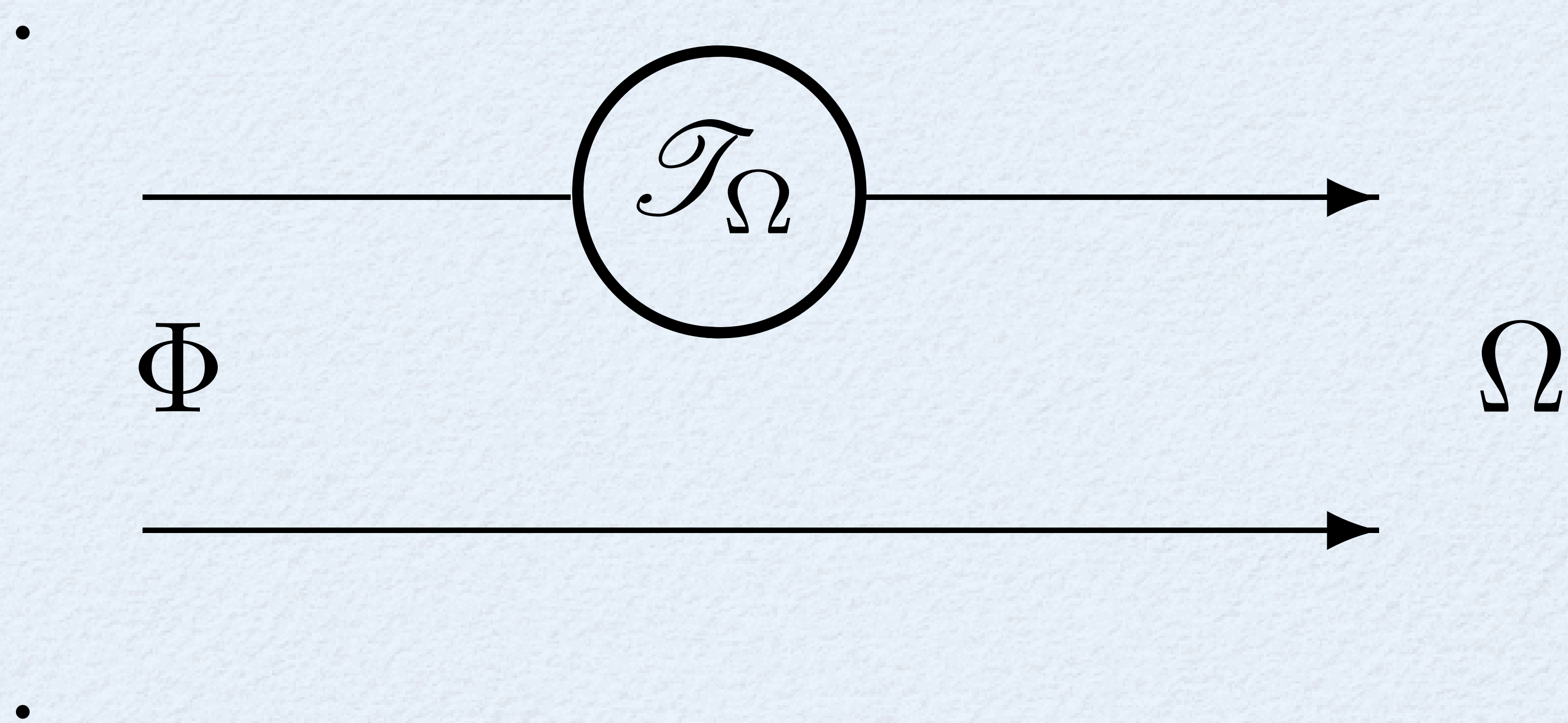




$$(\mathscr{A},\mathscr{I})\Phi = 0 \Longrightarrow \mathscr{A} = 0, \qquad \forall \mathscr{A} \in \mathfrak{T}_{\mathbb{R}}$$

Faithful states

Preparationally faithful state: we say that a state Φ of a bipartite system is preparationally faithful if every joint state Ω can be achieved by a suitable local transformation $\mathscr{T}_\Omega$ on one system occurring with nonzero probability



Faithful states

Symmetric bipartite state: we call a joint state Φ of a bipartite system symmetric if

$$\Phi(\mathscr{A},\mathscr{B}) = \Phi(\mathscr{B},\mathscr{A})$$

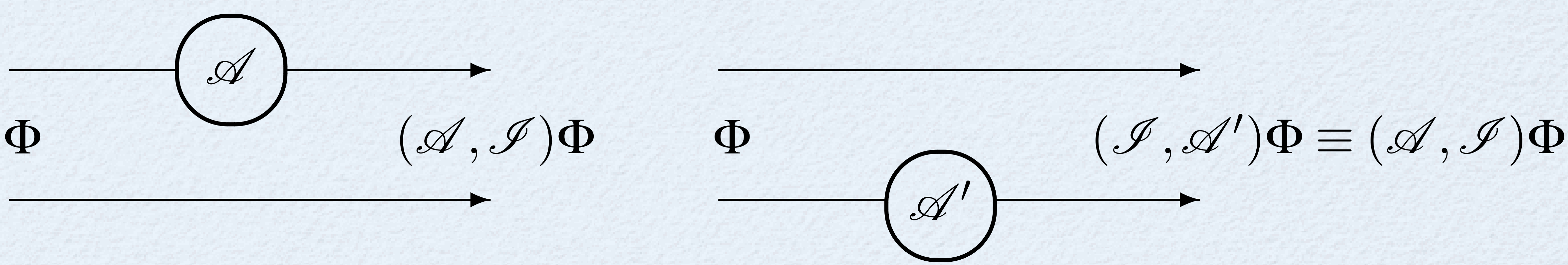
Construction of a C*-algebra of transformations

Operational definition of *transposed*

Existence of symmetric faithful states

"transposition" over the real algebra $\mathcal{A}$ of (generalized) transformations

$$\boxed{\mathscr{A} \Longleftrightarrow \mathscr{A}'}$$



$$\boxed{\Phi(\mathscr{B}\circ\mathscr{A},\mathscr{C}) = \Phi(\mathscr{B},\mathscr{C}\circ\mathscr{A}')}$$

Operational definition of *transposed*

For *symmetric* faithful state it is easy to check that the involution $\mathscr{A} \Longleftrightarrow \mathscr{A}'$ satisfies the properties of the transposed:

1. $(\mathscr{A} + \mathscr{B})' = \mathscr{A}' + \mathscr{B}'$
2. $(\mathscr{A}')' = \mathscr{A}$,
3. $(\mathscr{A} \circ \mathscr{B})' = \mathscr{B}' \circ \mathscr{A}'$

Positive bilinear form

Positive form over generalized effects: *Jordan decomposition* of the real symmetric form Φ over generalized effects $\mathfrak{P}_{\mathbb{R}}$ (finite dimension)

$$|\Phi| := \Phi_+ - \Phi_-.$$

$$|\Phi|(\underline{\mathscr{A}}, \underline{\mathscr{B}}) = \Phi(\varsigma(\underline{\mathscr{A}}), \underline{\mathscr{B}}), \quad \varsigma(\underline{\mathscr{A}}) = (\mathscr{P}_+ - \mathscr{P}_-)(\underline{\mathscr{A}})$$

$$\varsigma^2 = \mathscr{I}$$

$|\Phi|(\underline{\mathscr{A}}, \underline{\mathscr{B}})$ *strictly positive* scalar product over $\mathfrak{P}_{\mathbb{R}}$

The complex conjugation

The involution ς corresponds to a generalized transformation

$$\varsigma(\underline{\mathscr{A}}) = \underline{\mathscr{A}} \circ \mathscr{Z}$$

Extend ς to transformations as follows

$$\underline{\mathscr{A}} \circ \varsigma(\mathscr{B}) := \varsigma(\varsigma(\underline{\mathscr{A}}) \circ \mathscr{B}) = \underline{\mathscr{A}} \circ \mathscr{Z} \circ \mathscr{B} \circ \mathscr{Z}.$$

Correspondingly the involution over transformations reads

$$\varsigma(\mathscr{A}) = \mathscr{Z} \circ \mathscr{A} \circ \mathscr{Z}$$

which is composition preserving, namely

$$\varsigma(\mathscr{B} \circ \mathscr{A}) = \varsigma(\mathscr{B}) \circ \varsigma(\mathscr{A}).$$

The involution ς will play the role of a *complex conjugation*.

The complex conjugation

In term of a canonical basis $[c_i]$ for $\mathfrak{P}_{\mathbb{R}}$ or which

$$\Phi(c_i, c_j) = s_j \delta_{ij}$$

the involution ς writes

$$\varsigma(\underline{\mathscr{A}}) = \underline{\mathscr{A}} \circ \mathscr{Z} = \sum_k \Phi(c_k, \underline{\mathscr{A}}) c_k$$

One has: $\mathscr{Z} = \mathscr{Z}' \Rightarrow \varsigma(\mathscr{A})' = \varsigma(\mathscr{A}')$

$$\Rightarrow \quad (\mathscr{A}^\dagger)^\dagger = \mathscr{A},$$

where $\boxed{\mathscr{A}^\dagger := \varsigma(\mathscr{A}')}$

The adjoint

Scalar product over $\mathfrak{P}_{\mathbb{R}}$:

$$ {}_{\Phi}\langle \underline{\mathscr{B}} | \underline{\mathscr{A}} \rangle_{\Phi} := \Phi(\varsigma(\underline{\mathscr{B}'}), \underline{\mathscr{A}'}) = \Phi|_1(\mathscr{B}^{\dagger} \circ \mathscr{A}) $$

$\mathscr{A}^{\dagger} := \varsigma(\mathscr{A}')$ works as an adjoint with respect to the scalar product

$$ {}_{\Phi}\langle \mathscr{C}^{\dagger} \circ \underline{\mathscr{A}} | \underline{\mathscr{B}} \rangle_{\Phi} = {}_{\Phi}\langle \underline{\mathscr{A}} | \mathscr{C} \circ \underline{\mathscr{B}} \rangle_{\Phi} $$

The C*-algebra of generalized transformations

Take complex linear combinations of generalized transformations and define $\varsigma(c\mathscr{A}) = c^*\varsigma(\mathscr{A})$ for $c \in \mathbb{C}$.

c-generalized transformations: $\mathfrak{T}_{\mathbb{C}}$
c-generalized effects: $\mathfrak{P}_{\mathbb{C}}$

complex Banach spaces

complex C*-algebra

GNS-like construction: the generalized transformations act as complex operators over the (pre)Hilbert space of generalized effects $\mathfrak{P}_{\mathbb{C}}$

$\mathfrak{T}_{\mathbb{C}}$ becomes a C*-algebra with respect to the norm induced by the scalar product on $\mathfrak{P}_{\mathbb{C}}$

GNS construction for representing transformations

Representations π_Φ of transformations $\mathscr{A} \in \mathcal{A}$ over effects $\mathcal{A}/\mathfrak{I}$

$$\pi_\Phi(\mathscr{A})|\underline{\mathscr{B}}\rangle_\Phi \doteq |\underline{\mathscr{A} \circ \mathscr{B}}\rangle_\Phi$$

The Born rule rewrites in the form of pairing:

$$\omega(\underline{\mathscr{A}}) = {}_\Phi\langle\underline{\mathscr{A}^\dagger}|\varrho\rangle_\Phi$$

with representation of states given by

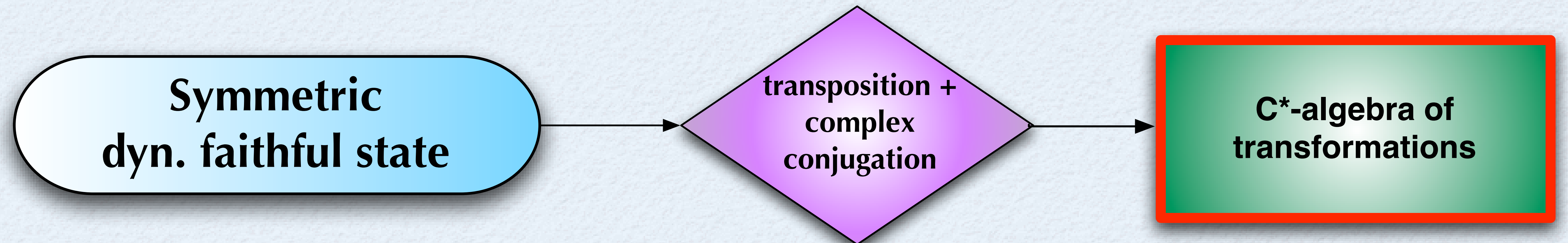
$$\varrho = \underline{\mathscr{T}}'_\omega/\Phi(\underline{\mathscr{T}}_\omega, \mathscr{I})$$

The representation of transformations is given by

$$\omega(\underline{\mathscr{B}} \circ \mathscr{A}) = {}_\Phi\langle\underline{\mathscr{B}^\dagger}|\mathscr{A}|\rho\rangle_\Phi :=$$
$${}_\Phi\langle\underline{\mathscr{B}^\dagger}|\mathscr{A} \circ \rho\rangle_\Phi \equiv {}_\Phi\langle\mathscr{A}^\dagger \circ \underline{\mathscr{B}^\dagger}|\rho\rangle_\Phi$$

C*-algebra of transformations

Postulates
Axioms
Theorems



An explicit representation

$$\Phi = d^{-1}|I\rangle\!\rangle\langle\!\langle I|$$

$$\Phi(\mathscr{A},\mathscr{B}) = \tfrac{1}{d}\,\mathrm{Tr}[P_{\mathscr{A}}P^*_{\mathscr{B}}] \qquad \langle\underline{\mathscr{A}}|\underline{\mathscr{B}}\rangle := \Phi(\mathscr{A}^\dagger,\mathscr{B}') = \tfrac{1}{d}\,\mathrm{Tr}[\tilde{P}^*_{\mathscr{A}}\tilde{P}^*_{\mathscr{B}}]$$

$$\boxed{\langle\underline{\mathscr{A}}|\underline{\mathscr{B}}\rangle := \tfrac{1}{d}\langle\!\langle I|\check{A}^\dagger\check{B}|I\rangle\!\rangle} \qquad \check{A} := \textstyle\sum_n A_n \otimes A_n^*$$

$$\boxed{\check{A}|I\rangle\!\rangle = |\mathscr{A}^\dagger(I)\rangle\!\rangle = |\tilde{P}^*_{\mathscr{A}}\rangle\!\rangle =: |\underline{\mathscr{A}}\rangle}$$

	$\mathscr{A}$	$\underline{\mathscr{A}}$
$\mathscr{A}$	$\sum_n A_n \cdot A_n^\dagger$	$\sum_n A_n^\dagger A_n =: P$
$\mathscr{A}'$	$\sum_n A_n^\intercal \cdot A_n^*$	$\sum_n A_n^* A_n^\intercal := \tilde{P}$
$\varsigma(\mathscr{A}) := \mathscr{A}^*$	$\sum_n A_n^* \cdot A_n^\intercal$	$\sum_n A_n^\intercal A_n^* := P^*$
$\varsigma(\mathscr{A}') \equiv \mathscr{A}^\dagger$	$\sum_n A_n^\dagger \cdot A_n$	$\sum_n A_n A_n^\dagger := \tilde{P}^*$

An explicit representation

$$\Phi(c_i, c_j) = \frac{1}{d}\,\mathrm{Tr}[W_i W_j^*] = \delta_{ij} s_j$$

$$\Phi(c_i, c_j) \doteq \frac{1}{2}\,\mathrm{Tr}[\sigma_i \sigma_j^*] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} := \delta_{ij} s_j \qquad c_j \Longleftrightarrow \frac{1}{\sqrt{2}}\sigma_j$$

$$R_\varsigma = \mathscr{Z} \otimes \mathscr{I}\left(|I\rangle\!\rangle\langle\!\langle I|\right) = \check{R}\varsigma = E$$

$$E|W_j\rangle\!\rangle = |W_j^*\rangle\!\rangle = s_j |W_j\rangle\!\rangle$$

$$\mathscr{Z} = \sum_i s_j W_j \cdot W_j$$

Quantum vs Classical C*-algebras (in progress)

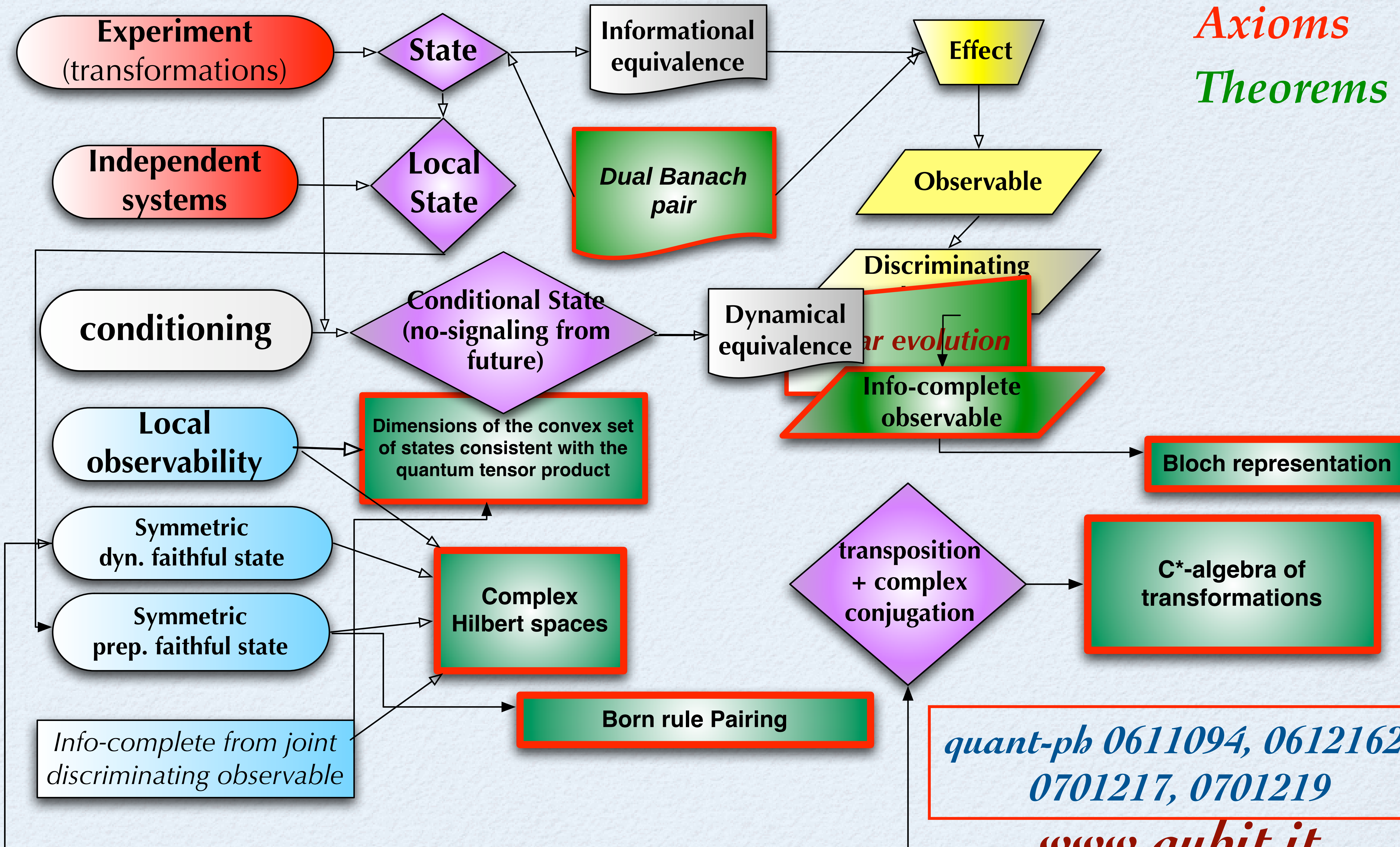
- The C*-algebra of transformations is isometrically *-homomorphic to the usual operator C*-algebra.
- Then the GNS representation is irreducible if the faithful state (cyclic vector) is pure, corresponding to QM
- The representation is abelian if the faithful state is separable corresponding to CM

C*-algebra of transformations

state-effect duality	$\dim(\mathfrak{P}) = \dim(\mathfrak{S}) + 1$	(D1)
P2 (prep. faith.)	$\boxed{\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1}$	($\mathfrak{T}$)
($\mathfrak{T}$)+GNS	$\boxed{\dim(\mathfrak{S}^{\times 2}) + 1 = (\dim(\mathfrak{S}) + 1)^2}$	($\mathfrak{T}_4$)≡(D2)
P3 (loc. observability)	$\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$	(D2)

Summary

Postulates
Axioms
Theorems



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Open problems

$\boxed{\dim(\mathfrak{P}) = \infty}$ Existence of ς (i.e. existence of the decomposition of the Banach space $\mathfrak{P}_{\mathbb{R}}$ into positive and negative parts for the symmetric real form Φ

$\boxed{\dim(\mathfrak{P}) \leqslant \infty}$ Extrapolation:

$$\dim(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S}^{\times 2})^2 - 1 \quad \Longrightarrow \quad \dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S})^2 - 1$$

Find a simple postulate discriminating the quantum from the classical C*-algebras

Exploit purity of Φ

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