Group theoretical quantum tomography

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The paper is devoted to the mathematical foundation of quantum tomography using the theory of square-integrable representations of unimodular Lie groups. © *2000 American Institute of Physics.* [S0022-2488(00)02512-3]

I. INTRODUCTION

In quantum mechanics to any physical system there is associated a Hilbert space $\mathcal{H}$: the states are described by positive trace-class trace-one operators T on $\mathcal{H}$, the physical quantities by self-adjoint operators A on $\mathcal{H}$, and the physical content of the theory is given by the expectation values $\operatorname{Tr}(AT)$. The state T is completely determined by $\operatorname{Tr}(Q_n T)$ for Q_n running on a suitable set $\{Q_n\}$ of observables and, for arbitrary operator A, $\operatorname{Tr}(AT)$ can be computed in terms of $\operatorname{Tr}(Q_n T)$. In order to implement this scheme one has to estimate $\operatorname{Tr}(Q_n T)$ experimentally, facing the problems arising from statistical errors and instrumental noise. Moreover, the number of experimental observations is clearly finite, while A and T are operators on an infinite dimensional Hilbert space and the set $\{Q_n\}$ is infinite.

The problem of determining the state of a quantum system entered the realm of experiments, in the last decade, in the domain of quantum optics. Many authors, see e.g., Refs. 1–4, proposed and used various techniques to reconstruct the density operator of a single mode of the electromagnetic field from the probability distributions of its quadratures. These methods were originally based on the use of the Radon transform, as in medical tomographic imaging. Due to this analogy the name *quantum tomography* is currently used to refer to these techniques. Their common feature, for a review see Ref. 5, is the use of a set of observables $\{Q_n : n \in X\}$, called *quorum*, parametrized by a space X endowed with a probability measure μ. The fundamental property of the quorum is that any observable A can be expressed as an *integral transform* on the space X,

$$A = \int_X \mathcal{E}[A](n)\, d\mu(n),$$

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in such a way that, for all $n\in X$, the operator $\mathcal{E}[A](n)$ is a function of Q_n in the sense of the functional calculus. Then, if T is the state, one has

$$\mathrm{Tr}\,(AT)=\int_{X\times\mathbb{R}}\sigma(A)(n,\lambda)\,\omega(n,\lambda)\,d\mu(n)\,d\lambda, \tag{1}$$

where $\lambda\mapsto\omega(n,\lambda)$ is the probability density of Q_n in the state T, i.e.,

$$\mathrm{Tr}\,(TQ_n)=\int_{\mathbb{R}}\lambda\,\omega(n,\lambda)\,d\lambda,$$

and $\lambda\mapsto\sigma(A)(n,\lambda)$ is the function defined by $\mathcal{E}[A](n)$ using the functional calculus, i.e.,

$$\mathrm{Tr}\,(T\mathcal{E}[A](n))=\int_{\mathbb{R}}\sigma(A)(n,\lambda)\,\omega(n,\lambda)\,d\lambda$$

(in the above-mentioned formulas we assumed for simplicity that each Q_n has an absolutely continuous spectrum). Selecting randomly Q_n in the quorum according to the probability measure μ and measuring it, the probability of obtaining a value in the interval $(\lambda-\frac{1}{2}d\lambda,\lambda+\frac{1}{2}d\lambda)$ is given by $\omega(n,\lambda)d\mu(n)d\lambda$. Then, by means of Eq. (1), the expectation value $\mathrm{Tr}\,(AT)$ can be reconstructed, by averaging the function $\sigma(A)$ over $X\times\mathbb{R}$ endowed with the probability measure $\omega\ d\mu\ d\lambda$. We notice that the function $\sigma(A)$, called the *estimator* of A, does not depend on T, and that the same set of data can be used to estimate all the expectation values $\mathrm{Tr}\,(AT)$.

In Refs. 6 and 7 a general method has been proposed to realize a quorum and define estimators in terms of suitable unitary representations of Lie groups (for a self-contained concise exposition see Refs. 8 and 9). The present paper is concerned with laying the mathematical foundations of this method based on the theory of square-integrable representations of unimodular Lie groups. In Sec. II we present the mathematical theory and in Sec. III we apply it to two examples: the homodyne tomography related to the Weyl–Heisenberg group and the angular momentum tomography associated with the rotation group.

II. GROUP-DYNAMICAL QUORUM

In this section we define a quorum associated with a square-integrable representation of a Lie group.

Let G be a unimodular connected Lie group G and K a central closed subgroup. The quotient space $H=G/K$ is a unimodular connected Lie group. We denote by $\mathfrak{H}$ its Lie algebra, by $m+1$ the (real) dimension of $\mathfrak{H}$ as a vector space, by dv a Lebesgue measure on $\mathfrak{H}$; and by dh a Haar measure on H, uniquely defined up to a positive constant, which will be fixed in the following.

Denote by exp the exponential map from $\mathfrak{H}$ to H; we assume that there is an open subset V of $\mathfrak{H}$ such that exp(V) is open in H, its complement has zero measure with respect to dh, and exp is a diffeomorphism from V onto exp(V). This hypothesis implies that, given $f\in L^1(H,dh)$,

$$\int_H f(h)\,dh=D\int_{\mathfrak{H}}f(\exp(v))\left|\det\,(d(\exp)_v)\right|\chi_V(v)\,dv, \tag{2}$$

where $d(\exp)_v$ is the differential of the exponential map at $v\in\mathfrak{H}$, i.e.,

$$d(\exp)_v(w)=\left(\frac{d}{dt}\exp(-v)\exp(v+tw)\right)_{t=0}\quad w\in\mathfrak{H},$$

$\det\,(\cdot)$ is the determinant and D is a positive constant, see, e.g., Theorem 1.14, Chapter I of Ref. 10. We normalize the Haar measure dh of H in such a way that $D=1$.

Remark 1: The density $\det(d(\exp)_v)$ can be easily computed observing that, if $\lambda_1,\ldots,\lambda_{m+1}$ are the (possibly repeated) eigenvalues of $d(\exp)_v$, viewed as linear operator on $\mathfrak{H}$, then

$$\det(d(\exp)_v)=\frac{1-e^{-\lambda_1}}{\lambda_1}\cdots\frac{1-e^{-\lambda_{m+1}}}{\lambda_{m+1}},$$

with

$$\frac{1-e^{-0}}{0}=1,$$

see, e.g., Theorem 1.7, Chap. I of Ref. 11.

Let U be an irreducible continuous unitary representation of G. We denote by $\mathcal{H}$ the (complex separable) Hilbert space where the representation acts and by $\langle\cdot,\cdot\rangle$ the scalar product on $\mathcal{H}$, linear in the second argument.

We assume that the representation U is square-integrable modulo K, i.e., there is a nonzero vector $v\in\mathcal{H}$ such that

$$\int_H |\langle U_{c(h)}v,v\rangle|^2\,dh<\infty, \tag{3}$$

where c is a section from H to G, i.e., a measurable map $c:H\to G$ such that

$$c(e_H)=e_G,$$

$$\pi(c(h))=h,\quad h\in H,$$

with π being the canonical projection from G to H. Notice that the value of the integral in Eq. (3) is independent of the choice of the section and that Eq. (3) implies that the function $h\mapsto\langle U_{c(h)}u,w\rangle$ is square integrable for all $u,w\in\mathcal{H}$.[12]

We will discuss briefly the meaning and generality of the above-mentioned assumptions in remark 3 in the following.

Remark 2: In many examples K is trivial, i.e., $K=e_G$, so that $H=G$ and Eq. (3) reduces to the usual notion of square integrability. Nevertheless, there are cases, such as the Weyl–Heisenberg group, that require the full theory. Moreover, in this framework one can easily consider projective representations. Indeed, let $\hat{U}$ be a projective representation of a Lie group $\hat{H}$ with multiplier m. Define G as the central extension of the torus K by $\hat{H}$ associated with m. Then K is a central closed subgroup of G, H is canonically isomorphic with $\hat{H}$, and there is a unitary representation U of G such that

$$\hat{U}_{\pi(g)}=U_g,\quad g\in G.$$

Clearly, the fact that U is square-integrable modulo K is equivalent to the fact that $\hat{U}$ is a square-integrable projective representation of $\hat{H}$.

If U is square-integrable modulo K, one can prove[12] that there is a constant $d_U>0$, called *the formal degree* of U, such that, for all $u_1,u_2,v_1,v_2\in\mathcal{H}$,

$$\int_H \overline{\langle U_{c(h)}v_1,u_1\rangle}\,\langle U_{c(h)}v_2,u_2\rangle\,dh=\frac{1}{d_U}\langle u_1,u_2\rangle\langle v_2,v_1\rangle. \tag{4}$$

Using the above-mentioned relation we can represent the Hilbert–Schmidt operators on $\mathcal{H}$ as square integrable functions on H. Indeed, let $\mathcal{L}^2(\mathcal{H})$ be the Hilbert space of the Hilbert–Schmidt operators on $\mathcal{H}$ with the scalar product

$$(A,B) \mapsto \mathrm{Tr}\,(A^*B),$$

where $\mathrm{Tr}\,(\cdot)$ denotes the trace and A^* is the adjoint operator of A. If $u,v \in \mathcal{H}$, let $u \otimes v^*$ be the operator in $\mathcal{L}^2(\mathcal{H})$,

$$(u \otimes v^*)(w) = \langle v,w \rangle u, \quad w \in \mathcal{H}.$$

Given a section c, we define $\Sigma(u \otimes v^*)$ as the function from H to $\mathbb{C}$ given by

$$\Sigma(u \otimes v^*)(h) = \langle U_{c(h)} v,u \rangle, \quad h \in H.$$

From Eq. (4), it follows that $\Sigma(u \otimes v^*)$ is square integrable with respect to dh and

$$\|\Sigma(u \otimes v^*)\|^2_{L^2(H,dh)} = \frac{1}{d_U}\|u\|^2 \|v\|^2 = \frac{1}{d_U}\|u \otimes v^*\|^2_{\mathcal{L}^2(\mathcal{H})}.$$

Taking into account that the set $\{u \otimes v^* : u,v \in \mathcal{H}\}$ is total in $\mathcal{L}^2(\mathcal{H})$, it follows that Σ is defined uniquely by continuity on $\mathcal{L}^2(\mathcal{H})$ and, if $A,B \in \mathcal{L}^2(\mathcal{H})$,

$$\mathrm{Tr}\,(A^*B) = d_U \langle \Sigma(A), \Sigma(B) \rangle_{L^2(H,dh)}. \tag{5}$$

Moreover, if A is of trace-class, then for almost all $h \in H$,

$$\Sigma(A)(h) = \mathrm{Tr}\,(U_{c(h)^{-1}} A). \tag{6}$$

Indeed, let

$$A = \sum_i \lambda_i e_i \otimes f_i^*$$

be the canonical decomposition of A, where (e_i) and (f_i) are orthonormal sequences in $\mathcal{H}$, (λ_i) is an ℓ_1-sequence, and the series converges in trace-norm and, hence, in the Hilbert–Schmidt norm. Since Σ is continuous, then

$$\Sigma(A) = \sum_i \lambda_i \Sigma(e_i \otimes f_i^*),$$

where the series converges in $L^2(H,dh)$. On the other hand, fixed $h \in H$, since A is of trace class, so is $U_{c(h)^{-1}} A$, hence

$$\begin{aligned}\mathrm{Tr}\,(U_{c(h)^{-1}} A) &= \sum_i \langle f_i, U_{c(h)^{-1}} A f_i \rangle \\ &= \sum_i \lambda_i \langle U_{c(h)} f_i, e_i \rangle \\ &= \sum_i \lambda_i \Sigma(e_i \otimes f_i^*)(h),\end{aligned}$$

where the series converges pointwise. The claim is now clear.

We are now ready to define a quorum associated with the square-integrable (modulo K) representation U of G.

Let T be a state of $\mathcal{H}$, i.e., a positive trace-class operator of trace one, and A a Hilbert–Schmidt operator on $\mathcal{H}$. Taking into account Eqs. (5) and (6),

$$\begin{aligned}\mathrm{Tr}\,(TA) &= d_U\langle \Sigma(T),\Sigma(A)\rangle_{L^2(H,dh)}\\ &= d_U\int_H \overline{\mathrm{Tr}\,(U_{c(h)^{-1}}T)}\,\Sigma(A)(h)\,dh,\end{aligned}$$

so that

$$\mathrm{Tr}\,(AT) = d_U\int_H \Sigma(A)(h)\mathrm{Tr}\,(TU_{c(h)})\,dh.$$

By means of Eq. (2), the above-mentioned equation becomes

$$\mathrm{Tr}\,(AT) = d_U\int_{\mathfrak{H}} \Sigma(A)(\exp v)\mathrm{Tr}\,(TU_{c(\exp v)})\chi_V(v)|\det(d(\exp)_v)|\,dv.$$

Let S^m be the sphere in $\mathfrak{H}$. Then, for all $n\in S^m$, the map

$$t\mapsto U_{c(\exp(tn))}$$

is a projective representation of $\mathbb{R}$. Since all the multipliers of $\mathbb{R}$ are exact, there is a self-adjoint unbounded operator Q_n and a measurable complex function α_n with modulo 1 such that, for all $t\in\mathbb{R}$,

$$U_{c(\exp(tn))} = \alpha_n(t)e^{itQ_n}. \tag{7}$$

Using polar coordinates in Eq. (7), one has that

$$\begin{aligned}\mathrm{Tr}\,(AT) = d_U\,C_m\int_{S^m} d\Omega(n)\int_0^\infty dt\,t^m\Sigma(A)(\exp(tn))\alpha_n(t)\\ \times \mathrm{Tr}\,(Te^{itQ_n})\chi_V(tn)|\det(d(\exp)_{tn})|,\end{aligned} \tag{8}$$

where $d\Omega$ is the normalized surface measure on the sphere S^m, C_m is the volume of S^m, and dt is the Lebesgue measure on the real line. The set of self-adjoint operators $\{Q_n : n\in S^m\}$, labeled by the probability space $(S^m,d\Omega)$, is called the *quorum* defined by the representation U. We notice that Eq. (7) defines Q_n uniquely up to an additive constant, see, also, Remark 4 in the following.

Since Q_n is self-adjoint, we can find by the spectral theorem a projection valued measure $E\mapsto P_n(E)$ defined on $\mathbb{R}$ such that

$$\mathrm{Tr}\,(TQ_n) = \int_{\mathbb{R}} \lambda\, d\,\mathrm{Tr}\,(TP_n(\lambda)),$$

where $d\,\mathrm{Tr}\,(TP_n(\lambda))$ is the positive bounded measure

$$E\mapsto \mathrm{Tr}\,(TP_n(E))$$

on $\mathbb{R}$. Using this equation, one obtains

$$\begin{aligned}\mathrm{Tr}\,(AT) = d_U\,C_m\int_{S^m} d\Omega(n)\int_0^\infty dt\int_{\mathbb{R}} d\,\mathrm{Tr}\,(TP_n(\lambda))\\ \times e^{i\lambda t}\Sigma(A)(\exp(tn))\alpha_n(t)\chi_V(tn)|\det(d(\exp)_{tn})|t^m.\end{aligned} \tag{9}$$

In order to obtain a reconstruction formula for $\mathrm{Tr}\,(AT)$, we would like to interchange the integrals in dt and in $d\,\mathrm{Tr}\,(TP_n(\lambda))$.

We consider first the case when $\Sigma(A)$, which is only square integrable, is in fact integrable with respect to dh, i.e.,

$$\int_H |\Sigma(A)(h)|\, dh < \infty. \tag{10}$$

By means of Fubini theorem, this condition implies that, for almost all $n \in S^m$, the map $t \mapsto \Sigma(A)(\exp(tn))$ is integrable with respect to the measure

$$dt_n = \chi_V(tn) |\det(d(\exp)_{tn})| t^m\, dt. \tag{11}$$

Then the map from $S^m \times \mathbb{R}$ to $\mathbb{C}$,

$$\sigma(A)(n,\lambda) = d_U\, C_m \int_0^\infty e^{i\lambda t} \Sigma(A)(\exp(tn))\, \alpha_n(t)\, \chi_V(tn) |\det(d(\exp)_{tn})| t^m\, dt, \tag{12}$$

is well-defined and it is called the *estimator* of the observable A. We notice that the estimator does not depend on T and, given the representation U, can be computed analytically.

Since the measure $d\,\mathrm{Tr}(TP_n(\lambda))$ is bounded, by means of Fubini theorem, one can interchange the integrals in Eq. (9) obtaining

$$\mathrm{Tr}(AT) = \int_{S^m} d\Omega(n) \int_{\mathbb{R}} d\,\mathrm{Tr}(TP_n(\lambda))\, \sigma(A)(n,\lambda). \tag{13}$$

The above-mentioned integral transform is the core of the *quantum tomography* and is a concrete realization of the scheme proposed in Sec. I; cf. Eq. (1). Indeed, $d\Omega(n) d\,\mathrm{Tr}(TP_n(\lambda))$ is the probability of obtaining a value in $(\lambda - \frac{1}{2}d\lambda, \lambda + \frac{1}{2}d\lambda)$ when one measures the observable Q_n, chosen randomly in the quorum according to $d\Omega$. Moreover, by means of Eq. (13), the expectation value $\mathrm{Tr}(AT)$ can be reconstructed as average of the estimator $\sigma(A)$ over many random measures of the observables Q_n in the quorum.

Remark 3: Equation (13) is the mathematical justification of quantum tomography and is essentially based on formulas (2) and (4). The assumptions on the existence of the set V, the unimodularity of G, and the square integrability of U modulo a central subgroup are sufficient to deduce in a simple way these formulas in a fairly general framework. In particular, they guarantee the existence of the map Σ that allows one to represent the (Hilbert–Schmidt) operators on $\mathcal{H}$ as (square-integrable) functions on the space H. In other words, Σ defines a family of coherent states in the space of operators. We stress that the existence of a family of coherent states in the space $\mathcal{H}$ is not sufficient to define Σ. Indeed, the square integrability of the representation provides *a set* of families of coherent states $\{U_g v : g \in G\}$ parametrized by the analyzing vector v running on a dense subset of $\mathcal{H}$.

Furthermore, the physical interpretation of Eq. (13) relies on the fact that $d\,\mathrm{Tr}(TP_n(\lambda))$ is a *probability measure*. This holds since, in Eq. (4), the formal degree is a number. If G is not unimodular and/or K is not central, then the formal degree is replaced by an operator and Eq. (4) becomes

$$\int_H \overline{\langle U_{c(h)} v_1, u_1 \rangle} \langle U_{c(h)} v_2, u_2 \rangle\, dh = \langle u_1, u_2 \rangle \langle C v_2, C v_1 \rangle,$$

where C is a positive, possibly unbounded, operator on $\mathcal{H}$, see Refs. 13 and 14. The map Σ can be defined in this more general setting, however one deduces that

$$\overline{\Sigma(T)(\exp(tn))} = \int_{\mathbb{R}} e^{it\lambda} d\,\mathrm{Tr}(CTP_n(\lambda)).$$

If T and C do not commute $d\,\mathrm{Tr}\,(CTP_n(\lambda))$ is not a probability measure and Eq. (13) loses its physical meaning.

Remark 4: There is a choice for the section that simplifies the expression of the estimator. Indeed, denote by $\mathfrak{G}$ the Lie algebra of G; since the differential $d\pi$ of π is a surjective linear map from $\mathfrak{G}$ onto $\mathfrak{H}$, there is an injective linear map j from $\mathfrak{H}$ to $\mathfrak{G}$ such that $d\pi(j(v))=v$ for all $v \in \mathfrak{H}$. Since exp is a diffeomorphism from V onto exp(V), there is defined a smooth map $\hat{c}$ from exp(V) to G such that

$$\hat{c}(\exp(v))=\exp(j(v)), \quad v \in \mathfrak{H}.$$

Clearly $\hat{c}$ is a section and the relation $U_{\hat{c}(\exp(tn))}=U_{\exp(tj(n))}$ shows that one can always choose $\alpha_n(t)=1$ in Eq. (7). Hence $U_{\hat{c}(\exp(tn))}=e^{itQ_n}$.

One can easily prove that, if one changes $j\mapsto j+l$ in such a way that $d\pi(j(v)+l(v))=v$, then the quorum transforms according to $Q_n\mapsto Q_n+q_nI$. However, in most of the cases, there is a *natural* choice for the map j, so that the quorum Q_n is, in fact, defined uniquely by the representation U.

Remark 5: Once the quorum $\{Q_n\}$ is fixed, Eq. (12) is independent of the choice of the section c. Indeed if c' is another section, then, for all $h\in H$, $c'(h)=k(h)c(h)$ and $k(h)\in K$. Since K is central in G and U is irreducible, then $U_{k(h)}=\beta(h)I$, where $\beta(h)$ is a complex number of modulus one. Hence, with obvious notations, for almost all $h\in H$ and for all $t\in\mathbb{R}$,

$$\Sigma'(A)(h)=\overline{\beta(h)\Sigma(A)}(h),$$

$$\alpha_n'(t)=\beta(h)\alpha_n(t),$$

so that $\sigma(A)$ is invariant with respect to the change $c\mapsto c'$.

Remark 6: If A is of trace class and satisfies Eq. (10), then, using Eq. (6), one obtains a more explicit formula for the estimator of A,

$$\sigma(A)(n,\lambda)=d_U\,C_m\int_0^\infty e^{i\lambda t}\mathrm{Tr}\,(Ae^{-itQ_n})\chi_V(tn)\left|\det\,(d(\exp)_{tn})\right|t^m\,dt.$$

Moreover, in most examples the set V is sufficiently nice so that the map $n\mapsto\chi_V(tn)$ is continuous for almost all $t\in\mathbb{R}$. In this case, if one chooses the section $\hat{c}$ as in Remark 4, taking into account that the function $g\mapsto\mathrm{Tr}\,(TU_g)$ is continuous [since the ultraweak operator topology is equivalent to the weak operator topology on the unit ball of $\mathcal{L}(\mathcal{H})$], it follows that the estimator $\sigma(A)$ is continuous on $S^m\times\mathbb{R}$. This property is important in order to approximate the integral of Eq. (13) by a finite sum.

Remark 7: We notice that this procedure is *unbiased* since the observables Q_n are chosen randomly and the integral given by Eq. (13) can be approximated by a finite sum as $d\Omega(n)d\,\mathrm{Tr}\,(TP_n(\lambda))$ is a probability measure. This means that this approach is not affected by the systematic errors that were present in the first tomographic scheme[1,2] due to the cutoff needed in the inversion of the Radon transform; see Ref. 3.

Remark 8: If H is compact then dh is finite and any irreducible representation is square integrable. Since the Hilbert space $\mathcal{H}$ where the representation acts is finite dimensional, $\mathcal{L}^2(\mathcal{H})$ coincides with the space of all the operators. Moreover, since $L^2(H,dh)\subset L^1(H,dh)$, Eq. (10) holds for every operator.

Remark 9: If U is an integrable representation (modulo K), there exists a dense set S in $\mathcal{H}$ such that, if $u,v\in S$, then $\Sigma(u\otimes v^*)$ satisfies Eq. (10).

If condition (10) does not hold, it may happen that, for a non-negligible set of $n\in S^m$, the map $t\mapsto\Sigma(A)(\exp(tn))$ is not integrable with respect to the measure dt_n defined by Eq. (11) (it is only square integrable), so that the estimator $\sigma(A)$ given by Eq. (12) is not well defined.

In these cases, in order to define the estimator one has to use a suitable regularization procedure. For example, for a fixed $L>0$ and all $n\in S^m$, $\lambda\in\mathbb{R}$, let

$$\sigma_L(A)(n,\lambda)=d_U\,C_m\int_0^L e^{i\lambda t}\Sigma(A)(\exp(tn))\,\alpha_n(t)\chi_V(tn)\left|\det(d(\exp)_{tn})\right|t^m\,dt. \tag{14}$$

It may be the case that there exists a function $\sigma(A)$ such that

$$\lim_{L\to\infty}\int_{S^m}d\Omega(n)\int_{\mathbb{R}}d\,\mathrm{Tr}\,(TP_n(\lambda))\sigma_L(A)(n,\lambda)=\int_{S^m}d\Omega(n)\int_{\mathbb{R}}d\,\mathrm{Tr}\,(TP_n(\lambda))\sigma(A)(n,\lambda).$$

Then, as an easy consequence of dominated convergence theorem, one has

$$\mathrm{Tr}\,(AT)=\int_{S^m}d\Omega(n)\int_{\mathbb{R}}d\,\mathrm{Tr}\,(TP_n(\lambda))\sigma(A)(n,\lambda).$$

Analogous regularization procedures could be used to extend $\Sigma(A)$ to non-Hilbert–Schmidt operators. Although this problem is physically relevant (many observables of interest are unbounded) it is beyond the scope of the present paper.

III. EXAMPLES

A. The Weyl–Heisenberg group

Let G be the Weyl–Heisenberg group, i.e., $G=\mathbb{R}^3$ with the composition law

$$(\eta_1,a_1,b_1)(\eta_2,a_2,b_2)=\left(\eta_1+\eta_2+\frac{b_1a_2-a_1b_2}{2},a_1+a_2,b_1+b_2\right).$$

It is known that G is a connected simply connected nilpotent (hence unimodular) Lie group.

The set $K=\{(\eta,0,0):\eta\in\mathbb{R}\}$ is clearly a central closed subgroup of G and the quotient group $H=G/K$ can be identified with the vector group $\mathbb{R}^2$. One has the following facts.

(1) The canonical projection π is given by $\pi(\eta,a,b)=(a,b)$.
(2) A smooth section c is given by $c(a,b)=(0,a,b)$.
(3) A Haar measure on H is the Lebesgue measure $da\,db$ of $\mathbb{R}^2$.
(4) The Lie algebra $\mathfrak{H}$ of H can be identified with $\mathbb{R}^2$ so that the exponential map is the identity and, for all $v\in\mathfrak{H}$, $\det(d(\exp)_v)=1$.
(5) The constant D in Eq. (2) is equal to 1.

It follows that the choice $V=\mathfrak{H}$ satisfies the assumptions of Sec. II.

Let U be the representation of G acting in $\mathcal{H}=L^2(\mathbb{R},dx)$ as

$$(U_{(\eta,a,b)}u)(x)=l^{i\left(\eta+\frac{ab}{2}\right)}e^{ixa}u(x+b),$$

where $x\in\mathbb{R},u\in L^2(\mathbb{R},dx)$, and $(\eta,a,b)\in G$. It is known that U is a unitary continuous irreducible representation of G, called the *Schrödinger representation*. It is in fact square-integrable modulo K and its formal degree is $d_U=1/2\pi$, see for instance Ref. 15. According to Sec. II, it defines a quorum.

In order to make it explicit, we observe that, with the notation of the Sec. II,

$$S^m=\{n_\Phi:=(\cos(\Phi),\sin(\Phi)):\Phi\in[0,2\pi]\},$$

$m=1$, $C_1=2\pi$, and $d\Omega=d\Phi/2\pi$. Moreover, since $t\mapsto U_{c(tn_\Phi)}$ is a one parameter subgroup, we have

$$U_{c(tn_\Phi)} = e^{itY_\Phi},$$

where Y_Φ is a self-adjoint operator [in this example $\alpha_{n_\Phi}(t)=1$]. If u is a Schwartz function, we have

$$Y_\Phi u = \cos(\Phi) Q u + \sin(\Phi) P u,$$

where Q is the operator of multiplication by x, i.e., the position operator, and P is $-i$ times the weak derivative operator, i.e., the momentum operator. Hence the quorum defined by U is given by the set of self-adjoint operators

$$\{Y_\Phi : \Phi \in [0,2\pi]\}$$

labeled by the space $[0,2\pi]$ with the uniform measure $d\Phi/2\pi$.

The above-mentioned quorum has the following property. For each $\Phi \in [0,2\pi]$, there is a unitary operator W_Φ such that

$$Y_\Phi = W_\Phi Q W_\Phi^{-1}. \tag{15}$$

To prove it, given $\Phi \in [0,2\pi]$, let f_Φ from G to G,

$$f_\Phi(\eta,a,b) = (\eta, \cos(\Phi) a - \sin(\Phi) b, \sin(\Phi) a + \cos(\Phi) b).$$

One can easily check that f_Φ is a continuous automorphism of the group G, so that $g \mapsto U_g^{f_\Phi} := U_{f_\Phi(g)}$ is a unitary irreducible continuous representation of G and the restriction to K is the character $\eta \mapsto e^{i\eta}$. From the unicity of the Schrödinger representation, it follows that there exists a unitary operator W_Φ such that

$$U^{f_\Phi} = W_\Phi U W_\Phi^{-1}.$$

Then

$$U_{c(tn_\Phi)} = U^{f_\Phi}_{(0,t,0)} = W_\Phi U_{(0,t,0)} W_\Phi^{-1},$$

and Eq. (15) follows by Stone's theorem.

Now let T be a state of $\mathcal{H}$. Recalling that the spectral measure P_Q of Q is the one given by the operators of multiplication by characteristic functions, then, by means of Eq. (15), for each $\Phi \in [0,2\pi]$ there is a $L^1(\mathbb{R}, d\lambda)$ function $\lambda \mapsto \omega(\Phi,\lambda)$ such that

$$\mathrm{Tr}\,(T P_\Phi(E)) = \mathrm{Tr}\,(W_\Phi^{-1} T W_\Phi P_Q(E)) = \int_E \omega(\Phi,\lambda)\, d\lambda,$$

where $E \mapsto P_\Phi(E)$ is the spectral measure associated with Y_Φ. The map ω can always be chosen to be measurable as a function on $[0,2\pi]\times\mathbb{R}$ and then it is a probability density on $[0,2\pi]\times\mathbb{R}$ with respect to the measure $(d\Phi/2\pi)\, d\lambda$.

Finally, fix a Hilbert–Schmidt operator A in $\mathcal{H}$ such that $\Sigma(A)$ is integrable with respect to $da\, db$. According to Eq. (12), the estimator of A is

$$\sigma(A)(\Phi,\lambda) = \int_0^\infty t\Sigma(A)(t\cos(\Phi), t\sin(\Phi)) e^{i\lambda t}\, dt$$

for $\Phi \in [0,2\pi]$ and $\lambda \in \mathbb{R}$, and the reconstruction formula Eq. (13) is explicitly given by

$$\mathrm{Tr}\,(AT) = \int_0^{2\pi} \int_{\mathbb{R}} \sigma(A)(\Phi,\lambda)\, \omega(\Phi,\lambda) \frac{d\Phi}{2\pi}\, d\lambda.$$

The representation U is actually integrable and, if (u_n) is the basis of eigenvectors of the number operator, then $\Sigma(u_n\otimes u_{n+l}^*)\in L^1(H,dh)$ and one has the explicit formula

$$\sigma(u_n\otimes u_{n+l}^*)(\Phi,\lambda)=\frac{(-i)^l}{2^{l/2}}\sqrt{\frac{n!}{(n+l)!}}e^{il\Phi}\int_0^\infty t^{l+1}L_n^l\left(\frac{t^2}{2}\right)\exp\left(-\frac{t^2}{4}+i\lambda t\right)dt, \tag{16}$$

where L_m^k are the associated Laguerre polynomials. The statistical reliability of Eq. (16) has been verified in Ref. 3.

This example is physically realized by homodyne tomography.[5] The quantum system is the harmonic oscillator representing a single mode of the e.m. field with annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. In terms of such operators, one has the following *dictionary:*

$$Q=\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}},$$

$$P=\frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i},$$

$$U_{(\eta,a,b)}=e^{i\eta}e^{(\alpha\hat{a}^\dagger-\overline{\alpha}\hat{a})},$$

$$Y_\Phi=\sqrt{2}\,\frac{\hat{a}^\dagger e^{i\Phi}+\hat{a}e^{-i\Phi}}{2}=:\sqrt{2}X_\Phi,$$

where

$$\left\{e^{(\alpha\hat{a}^\dagger-\overline{\alpha}\hat{a})};\ \alpha=\frac{-b+ia}{\sqrt{2}}\in\mathbb{C}\right\}$$

is the so-called *displacement group* and X_Φ is the quadrature with phase $\Phi\in[0,2\pi]$.

The measuring apparatus is a homodyne detector with tunable phase with respect to the local oscillator. The function $\sqrt{2}\omega(\Phi,\sqrt{2}\lambda)$ is the probability density (with respect to $d\lambda$) to obtain the value λ measuring the quadrature X_Φ, chosen randomly according to the measure $d\Phi/2\pi$. Moreover, the explicit form of the estimator of A, A being of trace class, is

$$\sigma(A)(\Phi,\sqrt{2}\lambda)=\frac{1}{2}\int_0^\infty \mathrm{Tr}\,(Ae^{-it(X_\Phi-\lambda)})t\,dt.$$

One could consult[16] for an example of an experimental realization of the above-mentioned tomographic method.

Remark 10: In this example one is able to obtain an estimator also for monomials in $\hat{a}$ and $\hat{a}^\dagger$.[17,6] For example, one has that

$$\sigma(\hat{a}^\dagger\hat{a})(\Phi,\sqrt{2}\lambda)=2\lambda^2-\tfrac{1}{2}.$$

B. The group SU(2)

Let SU(2) be the group of the unitary 2×2 complex matrices with determinant 1. It is a unimodular connected simply connected compact Lie group. The corresponding Lie algebra is

$$su(2)=\left\{\frac{i}{2}(x\sigma_1+y\sigma_2+z\sigma_3):x,y,z\in\mathbb{R}\right\}$$

where σ_i are the Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In the following we identify $su(2)$ with $\mathbb{R}^3$ using the basis $(i\sigma_k/2)_{k=1}^3$. Let $V=\{(x,y,z)\in\mathbb{R}^3: \sqrt{(x^2+y^2+z^2)}<2\pi\}$; it is known that V is an open neighborhood of 0 such that the exponential map restricted to V is a diffeomorphism from V onto the open set $\exp(V)$ and the complement of $\exp(V)$ is negligible with respect to the Haar measure of SU(2). Moreover one can check that

$$\det d(\exp)_{(x,y,z)}=4\frac{\sin^2\left(\frac{\sqrt{x^2+y^2+z^2}}{2}\right)}{x^2+y^2+z^2}.$$

If we choose the Haar measure on SU(2) in such a way that the constant D in Eq. (2) is 1 one has that

$$\int_H 1\,dh=\int_V |d(\exp)_{(x,y,z)}|\,dx\,dy\,dz=16\pi^2 \tag{17}$$

(usually the Haar measure on compact groups is normalized to 1).

Given j such that $2j\in\mathbb{N}$, let D^j be the irreducible representation of SU(2) acting on $\mathcal{H}=\mathbb{C}^{2j+1}$. Since the group is compact, D^j is square integrable and the space of the Hilbert–Schmidt operators coincides with the space of all operators $\mathcal{L}(\mathbb{C}^{2j+1})$.

Since the measure of SU(2) is normalized according to Eq. (17), it is well known that the formal degree is $d_{D^j}=(2j+1)/16\pi^2$, see, e.g., Ref. 12.

For all $n\in S^2$, define J_n as the hermitian matrix such that

$$D^j(\exp(tn))=e^{itJ_n}\quad t\in\mathbb{R}.$$

Then, the quorum defined by D^j is the set of spin operators $\{J_n : n\in S^2\}$ labeled by the space S^2 with the measure $dn/4\pi$, dn being the area element of the sphere. It is known that the (simple) eigenvalues of each J_n are $\lambda=-j,\ldots,j$ and there exists a unitary operator W_n, unique up to a phase, such that

$$J_n=W_n^{-1}J_zW_n,$$

where $J_z=J_{(0,0,1)}$.

Now let $A\in\mathcal{L}(\mathbb{C}^{2j+1})$; then, according to Eq. (12) and taking into account that $C_2=4\pi$, the corresponding estimator is

$$\sigma(A)(n,\lambda)=\frac{2j+1}{\pi}\int_0^{2\pi} e^{i\lambda t}\,\mathrm{Tr}\,(Ae^{-itJ_n})\sin^2\left(\frac{t}{2}\right)dt,$$

where $n\in S^2$ and $\lambda=-j,\ldots,j$. Equation (13) becomes

$$\mathrm{Tr}\,(TA)=\sum_{\lambda=-j}^{j}\int_{S^2}\sigma(A)(n,\lambda)|\langle W_ne_\lambda, TW_ne_\lambda\rangle|^2\frac{dn}{4\pi},$$

where $(e_\lambda)_{\lambda=-j}^{j}$ is a basis of eigenvectors of J_z.

This example is realized experimentally by a Stern–Gerlach machine. The quantum system is the spin degree of freedom of an elementary particle with spin j and the number $|\langle W_n e_\lambda, T W_n e_\lambda\rangle|^2$ is the probability to obtain the value λ measuring the spin along the axis n, chosen randomly according to the measure $dn/4\pi$.

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