

Optimal superbroadcasting of mixed qubit states

Francesco Buscemi *
buscemi@fisicavolta.unipv.it

Giacomo Mauro D'Ariano *
dariano@unipv.it

Chiara Macchiavello *
chiara@unipv.it

Paolo Perinotti * †
perinotti@fisicavolta.unipv.it

*QUIT Group, Dipartimento di Fisica "A. Volta"

†INFM-CNR, Unità di Pavia

via Bassi 6, I-27100, Pavia, Italy

TEL: +39-0382-987675 FAX: +39-0382-987563

Abstract— "Broadcasting", namely distributing information over many users, suffers in principle limitations when the information is quantum. This poses a critical issue in quantum information theory, for distributed processing and networked communications. For pure states ideal broadcasting coincides with the so-called "quantum cloning", describing an hypothetical ideal device capable of producing from a finite number N of copies of a state (drawn from a set) a larger number $M > N$ of output copies of the same state. Since such a transformation is not isometric, it cannot be achieved by any physical machine for a quantum state drawn from a non orthogonal set: this is essentially the content of the "no-cloning" theorem. For mixed states the situation is quite different, since from the point of view of each single user a local marginal mixed state is indistinguishable from the partial trace of an entangled state, and there are infinitely many joint output states that correspond to ideal broadcasting. Indeed, for sufficiently large number N of input copies, not only ideal broadcasting of noncommuting mixed states is possible, but one can even purify the state in the process. Such state purification with an increasing number of copies has been named "superbroadcasting". In this paper we will review some recent results on superbroadcasting of qubits, for two different sets of input states, corresponding to universally covariant broadcasting and to phase-covariant broadcasting of equatorial states. After illustrating the theoretical derivation of the optimal broadcasting channels, we give the maximal purity and the maximal number of output copies M for which superbroadcasting is possible. We will see that the possibility of superbroadcasting does not increase the available information about the original input state, due to detrimental correlations between the local broadcast copies, which do not allow to exploit their statistics. Thus, essentially, the superbroadcasting channel simply transfers noise from local states toward correlations. We finally propose a procedure to realize optimal superbroadcasting maps by means of optimal pure states cloners.

Keywords: quantum cloning, quantum broadcasting, Wedderburn decomposition, Schur-Weyl duality

1 Introduction

"Information" is by its nature *broadcastable*. What about when information is quantum? Do we need to distribute it among many users? Indeed, this may be useful in all situations in which quantum information is required in sharable form, e. g. in distributed quantum computation, for quantum shared secrecy, and, generally, in quantum game-theoretical contexts. However, contrarily to the case of classical information, which can be distributed at will, broadcasting quantum information can be done only in a limited fashion. Indeed, for pure states ideal broadcasting is equivalent to the so-called "quantum cloning", which is impossible due to the well-known "no-cloning" theorem [1, 2, 3]. The situation is more involved when the input states are mixed, since broadcasting can be achieved with an output

joint state that is indistinguishable from the tensor product of local mixed states from the point of view of individual receivers. Therefore, the no-cloning theorem cannot logically exclude the possibility of ideal broadcasting for sufficiently mixed states.

In Ref. [4] it has been proved that perfect broadcasting is impossible from $N = 1$ input copy to $M = 2$ output copies, and for a set of non mutually commuting density operators. This result was then considered (see Refs. [4] and [5]) as evidence of the general impossibility of broadcasting mixed states in the more general case in which $N > 1$ input copies are broadcasted to $M > N$ users, for states drawn from a non commuting set. However, in Ref. [6] some of the present authors have shown that for sufficiently many input copies N and sufficiently mixed states the no-broadcasting theorem doesn't generally hold, and it is possible to generate

$M > N$ output local mixed states which are identical to the input ones, and with the input mixed state drawn from a noncommuting set. Actually, as proved in Ref. [6], it is even possible to partially purify the local state in the broadcasting process, for sufficiently mixed input state. Such simultaneous purification and broadcasting was then named "superbroadcasting".

The possibility of superbroadcasting does not increase the available information about the original input state, due to detrimental correlations between the local broadcast copies (see Ref. [7]), which do not allow to exploit their statistics (a similar phenomenon was already noticed in Ref. [8]). Essentially, the superbroadcasting transfers noise from local states toward correlations. From the point of view of single users, however, the protocol is a purification in all respects, and this opens new interesting perspectives in the ability of distributing quantum information in a noisy environment.

This paper reviews the universal and phase-covariant optimal superbroadcasting maps. The two maps are derived in a unified theoretical framework that is thoroughly presented in Section 2. In Sections 3 and 4 we collect the main results concerning universal and phase-covariant superbroadcasting. In Section 5 we describe a scheme to achieve optimal superbroadcasting maps of mixed states by means of optimal cloners of pure states. Finally, Section 6 discusses the role of correlations in the output states.

2 Symmetric qubits broadcasting

In deriving optimal maps, we shall extensively use the formalism of Choi-Jamiolkowski isomorphism [9, 10] of CP maps \mathcal{E} from states on the Hilbert space \mathcal{H} to states on the Hilbert space \mathcal{K} , and positive bipartite operators R on $\mathcal{K} \otimes \mathcal{H}$

$$\begin{aligned} R_{\mathcal{E}} &= (\mathcal{E} \otimes \mathcal{I}) |\Psi^+\rangle\langle\Psi^+|, \\ \mathcal{E}(\rho) &= \text{Tr}_{\mathcal{K}} [(I \otimes \rho^T) R_{\mathcal{E}}], \end{aligned} \quad (1)$$

where Ψ^+ is the non normalized maximally entangled state $\sum_m |m\rangle \otimes |m\rangle$ in $\mathcal{H} \otimes \mathcal{H}$, and X^T denotes the transposition with respect to the basis $|m\rangle$ used in the definition of Ψ^+ . In term of $R_{\mathcal{E}}$, the trace-preserving condition for \mathcal{E} reads

$$\text{Tr}_{\mathcal{K}}[R_{\mathcal{E}}] = I_{\mathcal{H}}, \quad (2)$$

and covariance under the action of a group \mathbf{G} is equivalent to

$$\mathcal{E}(U_g \rho U_g^\dagger) = V_g \mathcal{E}(\rho) V_g^\dagger \Leftrightarrow [V_g \otimes U_g^*, R_{\mathcal{E}}] = 0, \quad (3)$$

where U_g and V_g are the unitary representations of $\mathbf{G} \ni g$ on the input and output spaces, respectively, whereas $X^* \doteq (X^\dagger)^T$ denotes the complex conjugated of the operator X . In terms of the operator

operator $R_{\mathcal{E}}$ the group-invariance properties from the map \mathcal{E} read as follows

$$\mathcal{E}(U_g \rho U_g^\dagger) = \mathcal{E}(\rho) \Leftrightarrow [I \otimes U_g^*, R_{\mathcal{E}}] = 0, \quad (4)$$

and

$$\mathcal{E}(\rho) = V_g \mathcal{E}(\rho) V_g^\dagger \Leftrightarrow [V_g \otimes I, R_{\mathcal{E}}] = 0. \quad (5)$$

We will consider CP maps \mathcal{B} from N -qubits states to M -qubits states, i. e. $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ and $\mathcal{K} = (\mathbb{C}^2)^{\otimes M}$. The first requirement for a broadcasting map is that all receivers get the same reduced state, a requirement that is achieved by a map whose output is permutation invariant.¹ Moreover, there is no loss of generality in requiring that it is also invariant under permutations of the input copies. This two simple properties, according to Eqs. (4) and (5), can be recast as follows

$$[\Pi_\sigma^M \otimes \Pi_\tau^N, R] = 0, \quad (6)$$

where Π_σ^M and Π_τ^N are representations of the output and input copies permutations σ and τ , respectively. Notice that permutations representations are all real, whence $\Pi_\sigma^* = \Pi_\sigma$.

A useful tool to deal with unitary group representations U_g of a group \mathbf{G} on a Hilbert space \mathcal{H} is the Wedderburn decomposition of \mathcal{H}

$$\mathcal{H} \simeq \bigoplus_{\mu} \mathcal{H}_{\mu} \otimes \mathbb{C}^{d_{\mu}}, \quad (7)$$

where the index μ labels equivalence classes of irreducible representations which appear in the decomposition of U_g . The spaces \mathcal{H}_{μ} support the irreducible representations, and $\mathbb{C}^{d_{\mu}}$ are the multiplicity spaces, with dimension d_{μ} equal to the degeneracy of the μ -th irrep. Correspondingly the representation U_g decomposes as

$$U_g = \bigoplus_{\mu} U_g^{\mu} \otimes I_{d_{\mu}}. \quad (8)$$

By Schur's Lemma, every operator X commuting with the representation U_g in turn decomposes as

$$X = \bigoplus_{\mu} I_{\mathcal{H}_{\mu}} \otimes X_{d_{\mu}}. \quad (9)$$

In the case of permutation invariance, the so-called Schur-Weyl [11] duality holds, namely the spaces $\mathbb{C}^{d_{\mu}}$ for permutations of M qubits coincide with the spaces \mathcal{H}_{μ} for the representation $U_g^{\otimes M}$ of $\mathbb{S}\mathbb{U}(2)$ where U_g is the defining representation. In other

¹ Actually, this is not strictly needed, since a joint output state having identical local partial traces is not necessarily permutation invariant. However, most figures of merit used for judging broadcasting maps enjoy this invariance, in particular the one that we consider in the present paper. Hence permutation invariance of the output can be required without loss of generality.

words, permutation invariant operators Y can act non trivially only on the spaces \mathcal{H}_μ

$$Y = \bigoplus_{\mu} Y_{\mu} \otimes I_{d_{\mu}}. \quad (10)$$

The Clebsch-Gordan series for the defining representation of $\mathbb{S}\mathbb{U}(2)$ is well-known in literature [11, 12, 13], its Wedderburn decomposition being the following

$$\mathcal{H} \simeq \bigoplus_{j=j_0}^{M/2} \mathcal{H}_j \otimes \mathbb{C}^{d_j}, \quad (11)$$

where $\mathcal{H}_j = \mathbb{C}^{2j+1}$, j_0 equals 0 for M even, $1/2$ for M odd, and

$$d_j = \frac{2j+1}{M/2+j+1} \binom{M}{M/2-j}. \quad (12)$$

Now, the Hilbert space $\mathcal{K} \otimes \mathcal{H}$ on which the operator R acts, supports the two permutations representations corresponding to the output and input qubits permutations, consequently

$$\mathcal{K} \otimes \mathcal{H} \simeq \left(\bigoplus_{j=j_0}^{M/2} \mathcal{H}_j \otimes \mathbb{C}^{d_j} \right) \otimes \left(\bigoplus_{l=l_0}^{N/2} \mathcal{H}_l \otimes \mathbb{C}^{d_l} \right). \quad (13)$$

Upon rearranging the factors in the last equation, we can recast the decomposition in a more suitable way, namely

$$\mathcal{K} \otimes \mathcal{H} \simeq \bigoplus_{j=j_0}^{M/2} \bigoplus_{l=l_0}^{N/2} (\mathcal{H}_j \otimes \mathcal{H}_l) \otimes (\mathbb{C}^{d_j} \otimes \mathbb{C}^{d_l}), \quad (14)$$

and to satisfy Eq. (6) we have the following form for R , according to Eq. (10),

$$R = \bigoplus_{j=j_0}^{M/2} \bigoplus_{l=l_0}^{N/2} R_{jl} \otimes (I_{d_j} \otimes I_{d_l}), \quad (15)$$

where R_{jl} acts on $\mathcal{H}_j \otimes \mathcal{H}_l$. In order to have trace preservation and complete positivity, the operators R_{jl} must satisfy the constraints

$$R_{jl} \geq 0, \quad \text{Tr}_j[R_{jl}] = \frac{I_{2l+1}}{d_j}, \quad (16)$$

where Tr_j denotes the partial trace over the space \mathcal{H}_j . This is the starting point for the analysis of symmetric qubits cloning devices. Requiring further conditions such as covariance under representations $V_g^{\otimes N}$, $V_g^{\otimes M}$ of a group \mathbf{G} , namely

$$\left[V_g^{\otimes M} \otimes V_g^{*\otimes N}, R \right] = 0, \quad (17)$$

will give a further constraint on the operators R_{jl} . In the following we will consider the two cases $\mathbf{G} = \mathbb{S}\mathbb{U}(2)$ (universal covariance) and $\mathbf{G} = \mathbb{U}(1)$ (phase-covariance).

Besides Wedderburn decomposition and the related Schur-Weyl duality, another useful tool we will extensively use is the decomposition of tensor-power states $\rho^{\otimes N}$

$$\rho^{\otimes N} = (r_+ r_-)^{N/2} \bigoplus_{j=j_0}^{N/2} \left(\frac{r_+}{r_-} \right)^{J_z^{(j)}} \otimes I_{d_j}, \quad (18)$$

where $\rho = \frac{1}{2}(I + r\sigma_z)$, and $J_z^{(j)} = \sum_{m=-j}^j m |jm\rangle \langle jm|$ (for a derivation of identity (18) see Refs. [14] and [7]). Notice that the *total* angular momentum component J_z of N qubits is clearly permutation invariant and can be written as

$$J_z = \frac{1}{2} \sum_{k=1}^N \sigma_z^{(k)} = \bigoplus_{j=j_0}^{N/2} J_z^{(j)} \otimes I_{d_j}, \quad (19)$$

where $\sigma_z^{(k)}$ denotes the operator acting as σ_z on the k -th qubit, and as the identity on all remaining qubits.

A simple but effective way of judging the quality of single-site output $\rho' = \text{Tr}_{M-1} [\mathcal{B}(\rho^{\otimes N})]$ is to evaluate the projection r' of the output Bloch vector over the input one

$$\text{Tr}[\sigma_z \rho'] = r'. \quad (20)$$

As we will see the single-site output copy ρ' of a covariant broadcasting map commutes with the input ρ , whence r' is indeed the length of the output Bloch vector. The trace in Eq. (20) can be evaluated by considering that the global output state $\Sigma = \mathcal{B}(\rho^{\otimes N})$ is by construction invariant under permutations, hence

$$\Sigma = \bigoplus_{j=j_0}^{M/2} \Sigma_j \otimes I_{d_j}, \quad (21)$$

and (see Ref. [7])

$$r' = \frac{2}{M} \sum_{j=j_0}^{M/2} d_j \text{Tr}[J_z^{(j)} \Sigma_j]. \quad (22)$$

In the phase-covariant case, according to the usual convention, we more conveniently take ρ diagonal on the σ_x eigenstates, and the previous formula is just substituted by

$$r' = \frac{2}{M} \sum_{j=j_0}^{M/2} d_j \text{Tr}[J_x^{(j)} \Sigma_j]. \quad (23)$$

In the following we will use as figure of merit the length r' of the output Bloch vector. This is actually a linear criterion, which restricts the search of optimal maps among just the extremal ones. We emphasize that for evaluating broadcasting maps for qubits the length of the output Bloch vector is a figure of merit more meaningful than the single-site

fidelity. Indeed, the case $r' = r$ corresponds to fidelity one, whereas superbroadcasting is achieved for $r' > r$ with fidelity actually lower than 1. Moreover, for $r' < r$ maximization of r' still corresponds to maximizing fidelity, whereas for $r' > r$, one has indeed clones that are purer than the original copies, which in applications can be more useful than perfect broadcasting—and, moreover perfect broadcasting can always be achieved by suitably mixing the output states, e. g. via a depolarizing channel. We will see that while the map maximizing r' is unique independently of the input r , the perfect broadcasting one (i. e. with unit single-site fidelity) is not unique and generally depends on the input purity (the mixing probabilities vary with r). Results will be reported in terms of the scaling factor for N inputs and M outputs $p^{N,M}(r) \doteq r'/r$, which is usually referred to as *shrinking factor* or *stretching factor*, depending whether it is smaller or greater than 1, respectively.

3 Universal covariant broadcasting

Let us consider now the universal broadcasting, namely the case in which we impose on the map the further constraint

$$\left[U_g^{\otimes M} \otimes U_g^{*\otimes N}, R \right] = 0, \quad (24)$$

where U_g is the defining representation of the group $\mathbb{S}\mathbb{U}(2)$. In Ref. [7], extremal broadcasting maps are singled out, and the one maximizing the figure of merit (22) is explicitly calculated. The optimal universal map achieves the scaling factor

$$p^{N,M}(r) = -\frac{M+2}{Mr}(r_+r_-)^{N/2} \sum_{l=0}^{N/2} \frac{d_l}{l+1} \sum_{n=-l}^l n \left(\frac{r_-}{r_+} \right)^n. \quad (25)$$

The function $p^{N,N+1}(r)$ is plotted in Fig. 1 for N from 10 to 100 in steps of 10. In a range of values of r one has a scaling factor $p^{N,N+1}(r) > 1$, corresponding to superbroadcasting. This happens for $N \geq 4$. The maximum value of r for which superbroadcasting is possible will be denoted as $r_*(N, M)$ and it is solution of the equation

$$p^{N,M}(r_*) = 1. \quad (26)$$

The maximum M for which superbroadcasting is possible for N input copies will be denoted as $M_*(N)$. It turns out that $M_*(N) = \infty$ for $N > 5$, whereas $M_*(4) = 7$ and $M_*(5) = 21$. The values of $r_*(N, N+1)$ and $r_*(N, M_*(N))$ versus N are also reported in Fig. 1. The corresponding asymptotic behaviors evaluated numerically are $2N^{-2}$ and N^{-1} , respectively.

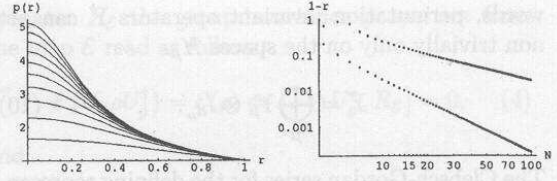


Figure 1: Universally covariant broadcasting. **Left:** the behaviour of the optimal scaling factor $p^{N,N+1}(r) = r'/r$ in Eq. (25) versus r , for N ranging from 10 to 100 in steps of 10. Notice that there is a wide range of values of r such that $p^{N,N+1}(r) > 1$, corresponding to superbroadcasting. **Right:** logarithmic plot of $1 - r_*(N, N+1)$ (lower line) and $1 - r_*(N, M_*(N))$ (upper line) for $4 \leq N \leq 100$. The corresponding asymptotic behaviors are $2N^{-2}$ and N^{-1} , respectively.

4 Phase-covariant broadcasting

Phase-covariant broadcasting corresponds to the constraint

$$\left[V_\phi^{\otimes M} \otimes V_\phi^{*\otimes N}, R \right] = 0, \quad (27)$$

where $V_\phi = e^{i\phi\sigma_z/2}$ is a representation of the group $\mathbb{U}(1)$ of rotations along the z -axis. Similarly to the case of universal covariance, the optimal map is obtained by maximizing r' among the extremal maps [7]. The structure of the optimal map depends only on the parity of $M - N$, similarly to the optimal phase-covariant cloning of pure states [15, 16]. The scaling factor is given by

$$p_e^{N,M}(r) = \frac{4}{Mr}(r_+r_-)^{N/2} \sum_{l=0}^{N/2} d_l \times \sum_{n=-l}^{l-1} \left[\exp \left(J_x^{(l)} \log \frac{1+r}{1-r} \right) \right]_{n,n+1} \left[J_x^{(j)} \right]_{n,n+1}, \quad (28)$$

for $M - N$ even, and

$$p_o^{N,M}(r) = \frac{4}{Mr}(r_+r_-)^{N/2} \sum_{l=0}^{N/2} d_l \times \sum_{n=-l}^{l-1} \left[\exp \left(J_x^{(l)} \log \frac{1+r}{1-r} \right) \right]_{n,n+1} \left[J_x^{(j)} \right]_{n+\frac{1}{2},n+\frac{3}{2}}, \quad (29)$$

for $N - M$ odd (we use the matrix notation $[X^{(j)}]_{nm} \doteq \langle jn | X^{(j)} | jm \rangle$ for the operator $X^{(j)}$ acting on $\mathbb{C}^{2^{j+1}}$). The function $p^{N,N+1}(r)$ is plotted in Fig. 2 for N from 4 to 100 in steps of 8. One has superbroadcasting for $N \geq 3$, with $M_*(3) = 12$ and $M_*(N) = \infty$ for $N > 3$.

In Fig. 2 we also report the plots of the values of $r_*(N, N+1)$ and $r_*(N, M_*(N))$, as for the universally covariant case, with asymptotic behaviors

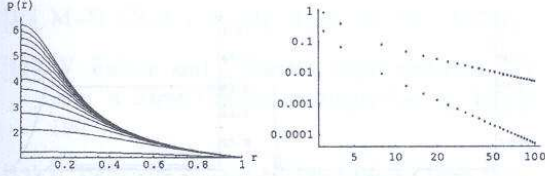


Figure 2: Phase-covariant cloning of equatorial qubits. **Left:** optimal scaling factor $p^{N,N+1}(r) = r'/r$ for N ranging from 4 to 100 in steps of 8. **Right:** logarithmic plot of $1 - r_*(N, N+1)$ (lower line) and $1 - r_*(N, M_*(N))$ (upper line) for $3 \leq N \leq 100$. The corresponding asymptotic behaviors are $\frac{2}{3}N^{-2}$ and $\frac{1}{2}N^{-1}$, respectively.

$\frac{2}{3}N^{-2}$ and $\frac{1}{2}N^{-1}$, respectively. As expected, the phase-covariant superbroadcaster is always more efficient than the universally covariant, since the set of broadcasted input states is smaller.

5 Realization scheme

We propose here a scheme to achieve the optimal $N \rightarrow M$ superbroadcasting channels, for both universal covariance and phase-covariance, using optimal pure state cloners. The method exploits a procedure similar to that presented in Ref. [14], based on the decomposition (18).

The first step is a joint measurement on $\rho^{\otimes N}$ of the observable described by the orthogonal projectors

$$\Pi_{(j,\alpha(j))} = I_{2j+1} \otimes |\alpha(j)\rangle\langle\alpha(j)|, \quad (30)$$

where $j_0 \leq j \leq N/2$ labels representation spaces, and $\{|\alpha(j)\rangle\}$ is an orthonormal basis spanning the multiplicity space \mathbb{C}^{d_j} , $0 \leq \alpha(j) \leq d_j$. For outcome (l, χ) , the (non normalized) output state after the measurement is

$$\rho_{(l,\chi)} = (r_+ r_-)^{N/2} \left(\frac{r_+}{r_-} \right)^{J_z^{(l)}} \otimes |\chi\rangle\langle\chi|, \quad (31)$$

which belongs to the *abstract* subspace $\mathbb{C}^{2l+1} \otimes \mathbb{C}^{d_l} \subseteq (\mathbb{C}^2)^{\otimes N}$. By applying a suitable unitary transformation to the collapsed state (31) it is always possible to rotate it as follows

$$U_{(l,\chi)} \rho_{(l,\chi)} U_{(l,\chi)}^\dagger = (r_+ r_-)^{N/2} \left(\frac{r_+}{r_-} \right)^{J_z^{(l)}} \otimes |\Psi^-\rangle\langle\Psi^-|^{\otimes \frac{N-2l}{2}}, \quad (32)$$

where now the first $2l$ qubits are in the (non normalized) state $(r_+ r_-)^{N/2} \left(\frac{r_+}{r_-} \right)^{J_z^{(l)}}$, whilst the remaining $N - 2l$ qubits are coupled in singlets $|\Psi^-\rangle$. Finally, once collected the outcome (l, χ) and rotated the state to the form (32), one discards the last $(N - 2l)$ qubits and applies the universal (resp. phase-covariant) optimal $2l \rightarrow M$ cloning machine for *pure*

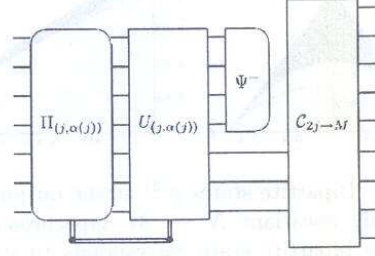


Figure 3: Sketch of the scheme proposed to realize the optimal $N \rightarrow M$ superbroadcaster. On the input state $\rho^{\otimes N}$ the measurement of $\Pi_{(j,\alpha(j))}$ in Eq. (30) is performed. Depending on the measurement outcome, the rotation $U_{j,\alpha(j)}$ is applied to the collapsed state. The $(N - 2j)$ qubits coupled in singlets are discarded. The remaining $2j$ qubits pass through the optimal $2j \rightarrow M$ cloner of pure states. At the output we get the M qubits broadcast state.

states [15, 17] to the remaining $2l$ qubits. One can prove [18] that using this scheme the optimal $N \rightarrow M$ universally covariant (resp. phase-covariant) broadcasting map is achieved in average, for universally covariant (resp. phase-covariant) cloner. The whole procedure is sketched in Fig. 3.

6 Role of correlations

The optimal superbroadcasting channel allows to obtain a large number of individually good copies of the same state, starting from fewer—and even more noisy—copies. Indeed, this is possible without violating the data processing theorem, since the *total* amount of information about the single-site input state ρ is not greater at the output than at the input. The apparently paradoxical reduction of information on ρ in the presence of purification is due to the fact that the output copies are not independent, and the total information is not simply the sum of local contributions. In other words, the phenomenon of superbroadcasting relies on the presence of correlations at the output, and the superbroadcasting channel can then be regarded as a tool that moves noise from local states into correlations between them.

It is then natural to ask which kind of correlations occur at the output state: are they classical or quantum? In order to answer this question, we analyzed the bipartite correlations at the output of the superbroadcasting channels, both for the universally covariant and the phase-covariant cases (the bipartite state corresponds to trace out $M - 2$ systems in the global output state Σ). For both types of covariance, the bipartite state is supported in the symmetric subspace of $(\mathbb{C}^2)^{\otimes 2}$ corresponding to the rep-

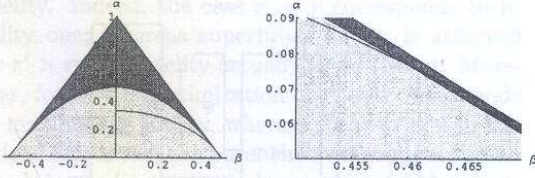


Figure 4: Bipartite states $\rho^{(2)}$ at the output of the universally covariant $N \rightarrow M$ superbroadcasting map (the bipartite state corresponds to trace out $M - 2$ systems in the global output state). **Left:** The triangle in the β, α plane contains the symmetric states $\rho^{(2)}$ of two qubits commuting with $J_z^{(1)}$ given in Eq. (33). The light grey region represents separable states with $\alpha \leq (1 - 4\beta^2)/2$, whereas the dark grey region represents entangled states. The curve represents the parametric plot $\beta(r), \alpha(r)$ of superbroadcast states for $N = 4, M = 5$. **Right:** magnification showing the point in which the bipartite output state becomes entangled.

resentation $j = 1$. In the universally covariant case, starting from $\rho = \frac{1}{2}(I + r\sigma_z)$ one gets a state commuting with $J_z^{(1)}$, which can then be parametrized by two real coefficients α and β as follows

$$\rho^{(2)} = \alpha I^{(1)} + \beta J_z^{(1)} + \frac{1 - 3\alpha}{2} J_z^{(1)2}, \quad (33)$$

where $I^{(1)}$ is the projection on the representation $j = 1$. The condition for positivity is simply $0 \leq \alpha \leq 1 - 2|\beta|$, corresponding to a triangle in the β, α plane, with vertices $(-1/2, 0)$, $(1/2, 0)$ and $(0, 1)$. The set of separable states is easily characterized in terms of α and β , since the concurrence [19] of $\rho^{(2)}$ is nonzero iff

$$\alpha \leq \frac{1 - 4\beta^2}{2}, \quad (34)$$

namely the couple (β, α) lies under a parabola which intersects the vertices of the triangle. In Fig 4 we plot the states $\rho^{(2)}$ in Eq. (33) at the output of the universally covariant $N \rightarrow M$ superbroadcasting map. One can see that mostly the bipartite correlations are classical, and only for small values of N a certain amount of entanglement is needed for r approaching purity, whereas for increasing N and M the output states exhibits essentially classical bipartite correlations. This can be seen also in Fig. 5, where the concurrence C is plotted as a function of r for different values of N , and $M = N + 1$. Entanglement is present only for high input purities and low N , and vanishes for increasing N .

The analysis of correlations in the phase-covariant case does not provide an easy geometrical visualization, and some insight is given only by the plots of concurrence as a function of r , shown in Fig. 5. Also in this case quantum correlations are very small and vanish for increasing N . However, contrarily to

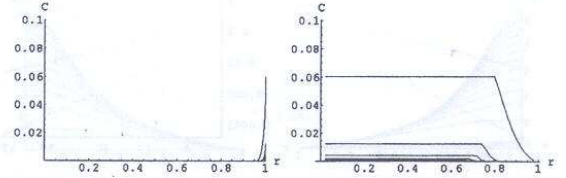


Figure 5: Concurrence C versus input purity r of the reduced bipartite states $\rho^{(2)}$ at the output of the optimal $N \rightarrow N + 1$ superbroadcasting channel. **Left:** universally covariant case. **Right:** phase-covariant case. The top curves correspond to $N = 2$, while the lower plots correspond to even N up to 20.

the universally covariant case, here concurrence decreases for r approaching 1.

The above results seem to indicate that quantum correlations—at least the bipartite ones—do not play a crucial role in superbroadcasting. The natural question is then whether superbroadcasting is a semi-classical or truly quantum in nature, namely if it can be achieved by measurement and re-preparation, or if it has a nonvanishing quantum capacity. There are two clues supporting the hypothesis that the map is truly quantum. The first is that by measurement and re-preparation it is possible to achieve superbroadcasting, but only sub-optimally, and with scaling factor independent on M (equal to the optimal factor in the limit $M \rightarrow \infty$) [20]. The second is that the last stage of the scheme of the optimal superbroadcasting channel is an optimal pure-state cloner, which is a purely quantum process [21, 22, 23].

References

- [1] W. K. Wootters, W. H. Zurek, *Nature* **299**, 802 (1982).
- [2] D. Dieks, *Phys. Lett. A*, **92**, 271 (1982).
- [3] H. P. Yuen, *Phys. Lett. A* **113**, 405 (1986).
- [4] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, *Phys. Rev. Lett.* **76**, 2818 (1996).
- [5] R. Clifton, J. Bub, and H. Halvorson, *Found. of Phys.* **33**, 1561 (2003).
- [6] G. M. D’Ariano, C. Macchiavello, and P. Perinotti, *Phys. Rev. Lett.* **95**, 060503 (2005).
- [7] F. Buscemi, G. M. D’Ariano, C. Macchiavello, and P. Perinotti, in preparation.
- [8] M. Keyl and R. F. Werner, *Ann. H. Poincaré*, **2**, 1 (2001).
- [9] A. Jamiolkowski, *Rep. Math. Phys.* **3**, 275 (1972).

- [10] M.-D. Choi, *Lin. Alg. Appl.* **10**, 285 (1975).
- [11] W. Fulton and J. Harris, *Representation Theory: a First Course* (Springer-Verlag, Berlin, 1991).
- [12] A. Messiah, *Quantum Mechanics* (John Wiley and Sons, New York, 1958).
- [13] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1960).
- [14] J. I. Cirac, A. K. Ekert, and C. Macchiavello, *Phys. Rev. Lett.* **82**, 4344-4347 (1999).
- [15] G. M. D'Ariano and C. Macchiavello, *Phys. Rev. A* **67**, 042306 (2003).
- [16] F. Buscemi, G. M. D'Ariano, and C. Macchiavello, *Phys. Rev. A* **71**, 042327 (2005).
- [17] R. F. Werner, *Phys. Rev. A* **58**, 1827 (1998).
- [18] F. Buscemi, G. M. D'Ariano, P. Perinotti, unpublished.
- [19] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
- [20] F. Buscemi, G. Chiribella, G. M. D'Ariano, C. Macchiavello, and P. Perinotti, in preparation.
- [21] V. Bužek and M. Hillery, *Phys. Rev. A* **54**, 1844, (1996).
- [22] N. Gisin and S. Massar, *Phys. Rev. Lett.* **79**, 2153 (1997).
- [23] G. M. D'Ariano and C. Macchiavello, *Phys. Rev. A* **67**, 042306, (2003).