Superbroadcasting of Harmonic Oscillators Mixed States[1]

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Abstract—We consider the problem of broadcasting quantum information encoded in the displacement parameter for an harmonic oscillator, from N to $M > N$ copies of a thermal state. We show the Weyl–Heisenberg covariant broadcasting map that optimally reduces the thermal photon number, and we prove that it minimizes the noise in conjugate quadratures at the output for general input states. We find that from two input copies broadcasting is feasible, with the possibility of simultaneous purification (*superbroadcasting*).

1. INTRODUCTION

The no-cloning theorem [1] states that one cannot produce a number of independent physical systems prepared in identical states out of a smaller number of systems prepared in the same state. This fact challenged the scientific community to find transformations which can approximate the cloning transformation with the highest possible fidelity. A whole branch in the literature is devoted to this problem, and optimal cloners have been found, for qubits [2–4], for general finite-dimensional systems [5], for restricted sets of input states [6, 7], and for infinite-dimensional systems such as harmonic oscillators—the so-called continuous variables cloners [8]. In all these cases pure states have been considered. On the other hand, the problem of cloning mixed states is somehow more interesting, since a more general type of cloning transformation can be considered—the so-called *broadcasting*—in which the output copies are in a globally correlated state whose local "reduced" states are identical to the input states. This possibility has been considered in [9], where it has been shown that broadcasting a single copy from a noncommuting set of density matrices is always impossible. The no broadcasting theorem was the only interesting result for mixed states for many years, and this led to the belief that a generalization of the no-cloning theorem to mixed states should hold. However, more recently, for qubits an effect called *superbroadcasting* [10] has been discovered, which consists in the possibility of broadcasting the state while even increasing the purity of the local state, for at least $N \geq 4$ input copies, and for sufficiently short input Bloch vector (and even for $N = 3$ input copies for phase-covariant

broadcasting instead of universal covariance [11]), and these superbroadcasting maps can be mixed with depolarizing maps in order to get exact broadcasting.

In the present paper, we analyze the broadcasting of mixed states of an harmonic oscillator by a signal-preserving map. More precisely, this means that we consider a set of input states obtained by displacing a fixed mixed state by a complex amplitude in the harmonic oscillator phase space, while the broadcasting map is covariant with respect to the (Weyl-Heisenberg) group of complex displacements. We will focus first on displaced thermal states (which are equivalent to coherent states that have suffered Gaussian noise), and then we will show that all results hold for any covariant set of mixed states in terms of noise of conjugated quadratures.

As we will see, superbroadcasting is possible for harmonic oscillator mixed states [12, 13], namely one can produce a larger number of copies, which are locally purified on each output oscillator, and with the same signal as the input. For displaced thermal states, for example, superbroadcasting can be achieved for at least $N = 2$ input copies, with thermal photon number $\bar{n}_{in} \geq 1/3$, whereas, for sufficiently large $\bar{n}_{in}$ at the input, one can broadcast to an unbounded number M of output copies. For purification (i.e., $M \leq N$), quite surprisingly the purification rate is $\bar{n}_{out}/\bar{n}_{in} = N^{-1}$, independently on M. The particular case of 2 to 1 for noisy coherent states has been reported in [14]. We will prove also that perfect broadcasting is possible, provided that one knows the input thermal photon number.

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[1] The text was submitted by the authors in English.

2. COVARIANT BROADCASTING FOR THE WEYL–HEISENBERG GROUP

We consider the problem of broadcasting N input copies of displaced (generally) mixed states of harmonic oscillators (with annihilation operators denoted by $a_0, a_1, \ldots, a_{N-1}$) to M output copies (with annihilation operators $b_0, b_1, \ldots, b_{M-1}$). In order to preserve the signal, the broadcasting map $\mathcal{B}$ must be covariant, i.e., in formula

$$
\begin{aligned}
&\mathcal{B}(D(\alpha)^{\otimes N} \Xi D(\alpha)^{\dagger \otimes N}) \\
&= D(\alpha)^{\otimes M} \mathcal{B}(\Xi) D(\alpha)^{\dagger \otimes N},
\end{aligned}
\tag{1}
$$

where $D_c(\alpha) = \exp(\alpha c^\dagger - \alpha^* c)$ denotes the displacement operator, and Ξ represents an arbitrary N-partite state. It is useful to consider the Choi-Jamiołkowski bijective correspondence of completely positive (CP) maps $\mathcal{B}$ from $\mathcal{H}_{\text{in}}$ to $\mathcal{H}_{\text{out}}$ and positive operators $R_{\mathcal{B}}$ acting on $\mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}}$, which is given by the following expressions

$$
\begin{aligned}
R_{\mathcal{B}} &= \mathcal{B} \otimes \mathcal{I}(|\Omega\rangle\langle\Omega|), \\
\mathcal{B}(\rho) &= \text{Tr}_{\text{in}}[(I_{\text{out}} \otimes \rho^\tau) R_{\mathcal{B}}],
\end{aligned}
\tag{2}
$$

where $|\Omega\rangle = \sum_{n=0}^{\infty} |\psi_n\rangle|\psi_n\rangle$ is a maximally entangled vector of $\mathcal{H}_{\text{in}}^{\otimes 2}$, and X^τ denotes transposition of X in the basis $|\psi_n\rangle$. In terms of the operator $R_{\mathcal{B}}$ the covariance property (1) can be written as

$$
[R_{\mathcal{B}}, D(\alpha)^{\otimes M} \otimes D(\alpha^*)^{\otimes N}] = 0, \quad \forall \alpha \in \mathbb{C}.
\tag{3}
$$

In order to deal with this constraint we introduce the multisplitter operators U_a and U_b, that perform the unitary transformations

$$
\begin{aligned}
U_a a_k U_a^\dagger &= 1/\sqrt{N} \sum_{l=0}^{N-1} e^{\frac{2\pi i k l}{N}} a_l, \\
U_b b_k U_b^\dagger &= 1/\sqrt{M} \sum_{l=0}^{M-1} e^{\frac{2\pi i k l}{M}} b_l.
\end{aligned}
\tag{4}
$$

Notice that such transformations perform a Fourier transform over all input and output oscillators. Moreover, we will make use of the squeezing transformation $S_{a_0 b_0}$ defined as follows

$$
\begin{aligned}
[S_{a_0 b_0}, a_n] &= [S_{a_0 b_0}, b_n] = 0, \quad n > 0, \\
S_{a_0 b_0} a_0^\dagger S_{a_0 b_0}^\dagger &= \quad a_0^\dagger - \nu b_0, \\
S_{a_0 b_0} b_0 S_{a_0 b_0}^\dagger &= \quad b_0 - \nu a_0^\dagger,
\end{aligned}
\tag{5}
$$

with $\quad = \sqrt{M/(M-N)}$ and $\nu = \sqrt{N/(M-N)}$. The squeezing transformation here acts as an hyperbolic transformation for just oscillators a_0 and b_0, by leaving

all other oscillators unaffected. In terms of such operators, condition (3) becomes

$$
\begin{aligned}
&[S_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger) R_{\mathcal{B}} (U_b \otimes U_a) S_{a_0 b_0}, \\
&\qquad D_{b_0}(\sqrt{M-N}\alpha)] = 0.
\end{aligned}
\tag{6}
$$

Hence, upon introducing an operator B on oscillators $b_1, \ldots, b_{M-1}, a_0, \ldots, a_{N-1}$, the operator $R_{\mathcal{B}}$ can be written in the form

$$
R_{\mathcal{B}} = (U_b \otimes U_a) S_{a_0 b_0} (I_{b_0} \otimes B) S_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger).
\tag{7}
$$

Notice that $R_{\mathcal{B}} \geq 0$ is equivalent to $B \geq 0$. The further condition that $\mathcal{B}$ is trace-preserving in terms of $R_{\mathcal{B}}$ becomes $\text{Tr}_b[R_{\mathcal{B}}] = I_a$, b and a collectively denoting all output and input oscillators, respectively. From the trace and completeness relations for the set of displacement operators, namely $\int d^2\alpha D(\alpha) A D^\dagger(\alpha) = \text{Tr}[A]I$, and $A = \int d^2\alpha \, \text{Tr}[D^\dagger(\alpha)A]D(\alpha)$, (see, e.g., [15]), the condition $\text{Tr}_b[R_{\mathcal{B}}] = I_a$ is verified if

$$
\begin{aligned}
&\left(\prod_{i=0}^{M-1} \int d^2\beta_i\right)\left(\bigotimes_{i=0}^{M-1} D_{b_i}(\beta_i)\right) S_{a_0 b_0}(I_{b_0} \otimes B) \\
&\times S_{a_0 b_0}^\dagger \left(\bigotimes_{i=0}^{M-1} D_{b_i}^\dagger(\beta_i)\right) = I.
\end{aligned}
\tag{8}
$$

From the relation $D_{b_0}(\beta_0) S_{a_0 b_0} = S_{a_0 b_0} D_{b_0}(\ \beta_0) \otimes D_{a_0}^\dagger(\nu\beta_0)$, one obtains the condition

$$
\text{Tr}_{b/b_0, a_0}[B] = \nu^2 I_{a/a_0},
\tag{9}
$$

where a/a_i denote all the input oscillators apart from a_i, and similarly for b/b_i.

We will now consider the map corresponding to

$$
B = \nu^2 |0\rangle\langle 0|_{b/b_0} \otimes |0\rangle\langle 0|_{a_0} \otimes I_{a/a_0}.
\tag{10}
$$

Applying the corresponding map $\mathcal{B}$ to a generic N-partite state Ξ we get

$$
\begin{aligned}
\mathcal{B}(\Xi) &= \text{Tr}_a[(I_b \otimes \Xi^\tau)(U_b \otimes U_a) \\
&\times S_{a_0 b_0}(I_{b_0} \otimes B) S_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger)],
\end{aligned}
\tag{11}
$$

which is equivalent to

$$
\begin{aligned}
\mathcal{B}(\Xi) &= \text{Tr}_a[(I_b \otimes U_a^\dagger \Xi^\tau U_a)(U_b \otimes I_a) \\
&\times S_{a_0 b_0}(I_{b_0} \otimes B) S_{a_0 b_0}^\dagger (U_b^\dagger \otimes I_a)].
\end{aligned}
\tag{12}
$$

Using the expression in Eq. (10) we obtain

$$\mathcal{B}(\Xi) = U_b \{ \mathrm{Tr}_{a_0}[(I_{b_0} \otimes \xi^{\tau}_{a_0})$$
$$\times S_{a_0 b_0}(I_{b_0} \otimes |0\rangle\langle 0|_{a_0}) S^{\dagger}_{a_0 b_0}] \otimes |0\rangle\langle 0|_{b/b_0} \} U^{\dagger}_b, \quad (13)$$

where $\xi^{\tau} = \mathrm{Tr}_{a/a_0}[U^{\dagger}_a \Xi^{\tau} U_a]$. Notice that

$$\xi = \int d^2\gamma/\pi D(\gamma)^{\tau} \mathrm{Tr}[(D_{a_0}(\gamma)^{\dagger} \otimes I_{a/a_0}) U^{\dagger}_a \Xi^{\tau} U_a]$$
$$= \int d^2\gamma/\pi D(\gamma)^{\tau} \mathrm{Tr}[U^*_a(D_{a_0}(\gamma)^* \otimes I_{a/a_0}) U^{\tau}_a \Xi], \quad (14)$$

and using the complex conjugate of Eq. (4) we have

$$\xi = \int d^2\gamma/\pi D(\gamma)^{\tau} \mathrm{Tr}[(D(\gamma^*/\sqrt{N})^{\otimes N} \Xi)]$$
$$= \int d^2\gamma/\pi D(\gamma)^{\tau} \mathrm{Tr}[(D_{a_0}(\gamma)^* \otimes I_{a/a_0}) U^{\dagger}_a \Xi U_a] \quad (15)$$
$$= \mathrm{Tr}_{a/a_0}[U^{\dagger}_a \Xi U_a].$$

Now, we can easily evaluate $S_{a_0 b_0}(I_{b_0} \otimes |0\rangle\langle 0|_{a_0}) S^{\dagger}_{a_0 b_0}$, by expanding the vacuum state as

$$|0\rangle\langle 0|_{a_0} = \int d^2\gamma/\pi e^{-\frac{|\gamma|^2}{2}} D_{a_0}(\gamma), \quad (16)$$

obtaining

$$S_{a_0 b_0}(I_{b_0} \otimes |0\rangle\langle 0|_{a_0}) S^{\dagger}_{a_0 b_0}$$
$$= \int \nu^2 d^2\gamma/\pi e^{-\frac{|\gamma|^2}{2}} D_{b_0}(\nu\gamma^*) \otimes D_{a_0}(\ \gamma). \quad (17)$$

Hence, Eq. (13) can be rewritten as

$$\mathcal{B}(\Xi) = \int \frac{d^2\gamma}{\pi} U_b(D_{b_0}(\gamma^*) \otimes |0\rangle\langle 0|_{b/b_0}) U^{\dagger}_b$$
$$\times e^{-\frac{|\gamma|^2}{2\nu^2}} \mathrm{Tr}[D_{a_0}(\ \gamma/\nu)\xi^{\tau}]. \quad (18)$$

As an example, we will now consider N displaced thermal states

$$\rho_{\alpha} \doteq \frac{1}{\bar{n}+1} D(\alpha) \left(\frac{\bar{n}}{\bar{n}+1} \right)^{a^{\dagger}a} D(\alpha)^{\dagger}, \quad (19)$$

from which we want to obtain M states, the purest as possible. Thanks to the covariance property, it is sufficient to focus attention on the output of $\rho_0^{\otimes N}$. For a tensor product of thermal input states $\Xi = \rho_0^{\otimes N}$, exploiting the fact that $U^{\dagger}_a (\sum_{j=0}^{N-1} a^{\dagger}_j a_j) U_a = \sum_{j=0}^{N-1} a^{\dagger}_j a_j$, we have

$$\xi = \xi^{\tau} = \rho_0, \quad (20)$$

and recalling the following expression for the thermal states

$$\frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1} \right)^{a^{\dagger}a} = \int \frac{d^2\beta}{\pi} e^{-\frac{|\beta|^2}{2}(2\bar{n}+1)} D(\beta), \quad (21)$$

we obtain

$$\mathcal{B}(\rho_0^{\otimes N}) = \int d^2\gamma/\pi U_b(D_{b_0}(-\gamma^*) \otimes |0\rangle\langle 0|_{b/b_0}) U^{\dagger}_b e^{-|\gamma|^2/2\nu^2[\ ^2(2\bar{n}+1)+1]} = \int d^2\gamma/\pi\bar{n}' U_b(|\gamma\rangle\langle\gamma|_{b_0} \otimes |0\rangle\langle 0|_{b/b_0}) U^{\dagger}_b e^{-|\gamma|^2/\bar{n}'}$$

$$= \int d^2\gamma/\pi\bar{n}' |\gamma/\sqrt{M}\rangle\langle\gamma/\sqrt{M}|^{\otimes M} e^{-|\gamma|^2/2\bar{n}'} = \int \frac{M d^2\gamma}{\pi\bar{n}'} |\gamma\rangle\langle\gamma|^{\otimes M} e^{-\frac{M|\gamma|^2}{\bar{n}'}}, \quad (22)$$

where

$$2\bar{n}' + 1 = 1/\nu^2[\ ^2(2\bar{n}+1)+1] = \frac{2M\bar{n}+2M-N}{N}. \quad (23)$$

The above state is permutation-invariant and separable, with a thermal local state at each oscillator with average thermal photon

$$\bar{n}'' = \bar{n}'/M = \frac{M\bar{n}+M-N}{MN}. \quad (24)$$

More generally, for any state Ξ, the choice (10) gives M identical clones whose state can be written as

$$\rho' = \int d^2\alpha/\pi e^{-|\alpha|^2/2(1/N - 2/M + 1)}$$
$$\times \{ \mathrm{Tr}[\Xi D^{\dagger}(\alpha/N)^{\otimes N}] \} D(\alpha). \quad (25)$$

Since for any oscillator c one has

$$\Delta x_c^2 + \Delta y_c^2 = 1/2 + \langle c^{\dagger}c \rangle - |\langle c \rangle|^2, \quad (26)$$

it is easy to verify that the superbroadcasting condition (output total noise in conjugate quadratures smaller than the input one), is equivalent to require a smaller number of thermal photons at the output than at the input, namely

$$\bar{n} \geq \frac{M\bar{n}+M-N}{MN} \Leftrightarrow \bar{n} \geq \frac{M-N}{M(N-1)}. \quad (27)$$

This can be true for any $N > 1$, and to any $M \leq \infty$, since

$$\lim_{M \to \infty} \frac{M-N}{M(N-1)} = \frac{1}{N-1} > 0. \quad (28)$$

3. PROOF OF OPTIMALITY
FOR THE CHANNEL IN EQ. (11)

Actually, the solution given in Eq. (24) is optimal. To prove this, in the following we will show that the expectation of the total number of photons $\text{Tr}[\sum_{l=0}^{M-1} b_l^\dagger b_l \mathscr{B}(\rho_0^{\otimes N})]$ of the M clones of ρ cannot be smaller than $M\bar{n}''$. Since the multisplitter preserves the total number of photons we have to consider the trace

$$W \doteq \text{Tr}\left[\left(\sum_{l=0}^{M-1} b_l^\dagger b_l \otimes (U_a^\dagger \rho_0^{\tau \otimes N} U_a)\right) S_{a_0 b_0} (I_{b_0} \otimes B) S_{a_0 b_0}^\dagger\right]. \tag{29}$$

We can write $W = W_0 + \sum_{l=1}^{M-1} W_l$, with

$$\begin{aligned}
W_0 &\doteq \text{Tr}[S_{a_0 b_0}^\dagger ((b_0^\dagger b_0 \otimes I_{b/b_0}) \otimes (U_a^\dagger \rho_0^{\tau \otimes N} U_a)) \\
&\quad \times S_{a_0 b_0}(I_{b_0} \otimes B)], \\
W_l &\doteq \text{Tr}[S_{a_0 b_0}^\dagger ((I_{b/b_l} \otimes b_l^\dagger b_l) \otimes (U_a^\dagger \rho_0^{\tau \otimes N} U_a)) \\
&\quad \times S_{a_0 b_0}(I_{b_0} \otimes B)],
\end{aligned} \tag{30}$$

for $1 \le l \le M-1$. Now, since $W_l \ge 0$, $W \ge W_0$. Moreover, using the identity $c^\dagger c = -\partial_{\alpha\alpha^*} e^{|\alpha|^2/2} D_c(\alpha)|_{\alpha = \alpha^* = 0}$, one obtains

$$\begin{aligned}
\text{Tr}_{b_0}[S_{a_0 b_0}^\dagger(b_0^\dagger b_0 \otimes \sigma) S_{a_0 b_0}] &= -\partial_{\alpha\alpha^*} \int d^2\gamma/\pi \\
&\quad \times \text{Tr}_{b_0}[D_{b_0}(\alpha - \nu\gamma^*) \otimes D_{a_0}(\gamma - \nu\alpha^*)] \\
&\quad \times \text{Tr}[D(\gamma)^\dagger \sigma] e^{|\alpha|^2/2}\big|_{\alpha = \alpha^* = 0} \\
&= -1/\nu^2 \partial_{\alpha\alpha^*} e^{-|\alpha|^2/\nu^2} e^{\alpha^*/\nu a_0^\dagger} e^{-\alpha/\nu a_0} \\
&\quad \times \text{Tr}[e^{\alpha^*/\nu a_0^\dagger} e^{-\alpha/\nu a_0} \sigma]\big|_{\alpha = \alpha^* = 0} \\
&= \frac{a_0^\dagger a_0 + {}^2\text{Tr}[a_0^\dagger a_0 \sigma] + 1}{\nu^4},
\end{aligned} \tag{31}$$

then, from Eq. (9) and positivity of B, one has

$$\begin{aligned}
W_0 &= \frac{\text{Tr}[(I_{b/b_0} \otimes a_0^\dagger a_0 \otimes I_{a/a_0})\{I_{b/b_0} \otimes (U_a \rho_0^{\otimes N} U_a^\dagger)^\tau\} B]}{\nu^4} \\
&\quad + \frac{{}^2\text{Tr}[(I_{b/b_0} \otimes I_{a_0} \otimes \text{Tr}_{a_0}[a_0^\dagger a_0 (U_a \rho_0^{\otimes N} U_a^\dagger)^\tau]) B] + \nu^2}{\nu^4} \ge \frac{{}^2\bar{n} + 1}{\nu^2} = \frac{N}{M}\bar{n} + \frac{M-N}{N} = M\bar{n}''.
\end{aligned} \tag{32}$$

In fact, one can easily check that the choice of B in Eq. (10) saturates the bound (32). Notice that for $\bar{n} = 0$ one has N coherent states at the input, and $\bar{n}'' = (M-N)/(MN)$, namely one recovers the optimal cloning for coherent states of [16].

Also the more general solution given in Eq. (25) for arbitrary input state is optimal, in the sense that it represents the state of M identical clones with minimal photon number, which is given by

$$\text{Tr}[b^\dagger b \rho'] = \text{Tr}[a^\dagger a \rho_0]/N + 1/N - 1/M. \tag{33}$$

4. EXACT BROADCASTING

In the previous section we proved that it is possible to superbroadcast thermal states provided that $\bar{n} \ge \frac{M-N}{M(N-1)}$. Here we will prove that in this case it is also possible to broadcast perfectly, namely with local fidelity equal to 1. Suppose indeed that the Choi-Jamiołkowski operator is given by $R_{\mathscr{B}} = (U_b \otimes U_a) S_{a_0 b_0}(I_b \otimes \otimes B') S_{a_0 b_0}^\dagger (U_b^\dagger \otimes U_a^\dagger)$, with

$$B' = \nu^2(\rho_{\bar{m}}^{\otimes M-1})_{b/b_0} \otimes |0\rangle\langle 0|_{a_0} \otimes I_{a/a_0}. \tag{34}$$

Then by using Eq. (21) we replace the first line of Eq. (22) by

$$\begin{aligned}
\mathscr{B}(\rho_0^{\otimes N}) &= \int \frac{d^2\gamma_0 \quad d^2\gamma_{M-1}}{\pi^M} \\
&\quad \times U_b[D_{b_0}(\gamma_0) \otimes D(\gamma_1) \otimes \quad \otimes D(\gamma_{M-1})] U_b^\dagger \\
&\quad \times e^{-|\gamma_0|^2/2(2\bar{n}'+1) - \frac{\sum_{k=1}^{M-1}|\gamma_k|^2}{2}(2\bar{m}+1)} \\
&= \int \frac{d^2\tilde{\gamma}_0 \quad d^2\tilde{\gamma}_{M-1}}{\pi^M}[D_{b_0}(\tilde{\gamma}_0) \otimes D(\tilde{\gamma}_1) \otimes \quad \otimes D(\tilde{\gamma}_{M-1})]
\end{aligned} \tag{35}$$

$$\times e^{-\frac{\left|1/\sqrt{M}\sum_{i=0}^{M-1}\tilde{\gamma}_i\right|^2}{2}(2\bar{n}'-2\bar{m})-\frac{\sum_{k=0}^{M-1}|\gamma_k|^2}{2}(2\bar{m}+1)},$$

where $\tilde{\gamma}_k = 1/\sqrt{M}\sum_{j=0}^{M-1}e^{\frac{2\pi ijk}{M}}\gamma_j$, and the Jacobian for the change of variables is clearly 1. Taking the partial trace over all output spaces but one, we get the following local state on the kth oscillator

$$\rho' = \int d^2\gamma/\pi D(\gamma)e^{-|\gamma|^2/2\left(2\bar{m}+1+\frac{2\bar{n}'-2\bar{m}}{M}\right)}, \qquad (36)$$

which is a thermal state with thermal photon number

$$\bar{n}'' = \frac{\bar{n}' + \bar{m}(M-1)}{M} = \frac{\bar{n}}{N} + \frac{1}{N} - \frac{1}{M} + \bar{m}\frac{M-1}{M}. \qquad (37)$$

In order to have $\bar{n}'' = \bar{n}$ it is sufficient to choose a suitable thermal number $\bar{m}$, which is the solution of the following equation

$$\bar{n} = \frac{\bar{n}}{N} + \frac{1}{N} - \frac{1}{M} + \bar{m}\frac{M-1}{M}, \qquad (38)$$

namely,

$$\bar{m} = \frac{M(N-1)\bar{n}}{N(M-1)} - \frac{(M-N)}{N(M-1)}, \qquad (39)$$

which is positive if

$$\bar{n} \geq \frac{M-N}{M(N-1)}. \qquad (40)$$

This definitely proves that perfect broadcasting is possible provided that superbroadcasting is. On the other hand, since superbroadcasting provides the maximum output purity, it is impossible to have perfect broadcasting when $\bar{n} < \frac{M-N}{M(N-1)}$, namely either superbroadcasting and perfect broadcasting are both possible, or they are both impossible.

5. NOISE IN CONJUGATE QUADRATURES

We will consider now an alternative derivation, which involves radiation modes and makes use of a theorem for linear amplifiers. We are interested in a transformation that provides M (generally correlated) modes $b_0, b_1, \ldots, b_{M-1}$ from N uncorrelated modes $a_0, a_1, \ldots, a_{N-1}$, such that the unknown complex amplitude is preserved and the output has minimal phase-insensitive noise. In formula, we have input uncorrelated modes

$$\langle a_i\rangle = \alpha,$$
$$\Delta x_{a_i}^2 + \Delta y_{a_i}^2 = \gamma_i \geq 1/2, \qquad (41)$$

for all $i = 0, 1, \ldots, N-1$, where Heisenberg uncertainty relation is taken into account. The output modes should satisfy

$$\langle b_i\rangle = \alpha,$$
$$\Delta x_{b_i}^2 + \Delta x_{b_i}^2 = \Gamma \geq 1/2, \qquad (42)$$

and we look for the minimal Γ. The minimal Γ can be obtained by applying a fundamental theorem for phase-insensitive linear amplifiers [17]: the sum of the uncertainties of conjugated quadratures of a phase-insensitive amplified oscillator with (power) gain G is bounded as follows:

$$\Delta X_B^2 + \Delta Y_B^2 \geq G(\Delta X_A^2 + \Delta Y_A^2) + (G-1)/2, \qquad (43)$$

where A and B denotes the input and the amplified mode, respectively. Our transformation can be seen as a phase-insensitive amplification from the mode $A = 1/\sqrt{N}\sum_{i=0}^{N-1}a_i$ to the mode $B = 1/\sqrt{M}\sum_{i=0}^{M-1}b_i$ with gain $G = M/N$, and hence Eq. (43) should hold. Notice that generally for any mode c one has

$$\Delta x_c^2 + \Delta y_c^2 = 1/2 + \langle c^\dagger c\rangle - |\langle c\rangle|^2. \qquad (44)$$

Hence, the bound can be rewritten as

$$\langle B^\dagger B\rangle - |\langle B\rangle|^2 \geq G(\langle A^\dagger A\rangle + 1 - |\langle A\rangle|^2) - 1. \qquad (45)$$

In the present case, since modes a_i are uncorrelated, one has

$$\langle A^\dagger A\rangle = 1/N\sum_{i,j=0}^{N-1}\langle a_i^\dagger a_j\rangle$$
$$= 1/N\left(\sum_{i=0}^{N-1}\langle a_i^\dagger a_i\rangle + \sum_{i\neq j}\langle a_i^\dagger a_j\rangle\right) \qquad (46)$$

$$= (\gamma + |\alpha|^2 - 1/2) + (N-1)|\alpha|^2 = \gamma + N|\alpha|^2 - 1/2,$$

where $\gamma = 1/N\sum_{i=0}^{N-1}\gamma_i$, and so the bound Eq. (43) is written as

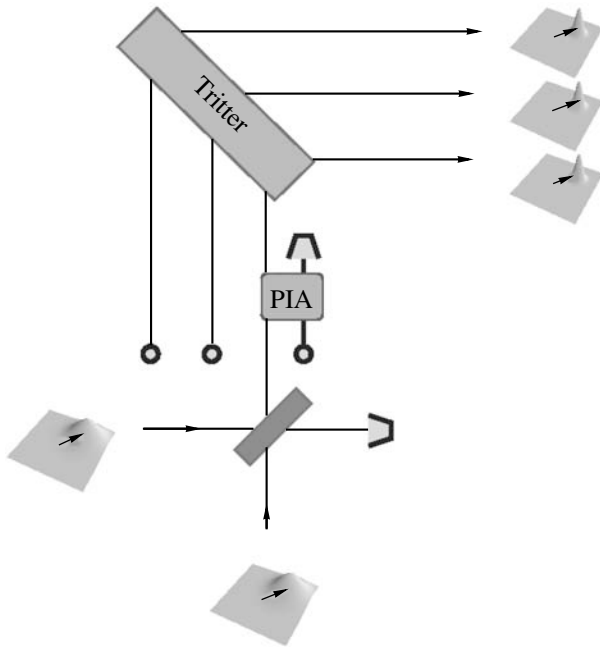$$\langle B^\dagger B\rangle \geq G(\gamma + 1/2) - 1 + M|\alpha|^2. \qquad (47)$$

On the other hand, one has

$$\langle B^\dagger B\rangle = 1/M\sum_{i,j=0}^{M-1}\langle b_i^\dagger b_j\rangle \leq 1/M\sum_{i,j=0}^{M-1}\sqrt{\langle b_i^\dagger b_i\rangle\langle b_j^\dagger b_j\rangle} \qquad (48)$$
$$= M(\Gamma + |\alpha|^2 - 1/2).$$

Equations (47) and (48) together give the bound for the minimal noise Γ

$$\Gamma - 1/2 \geq 1/N(\gamma - 1/2) + 1/N - 1/M. \qquad (49)$$

The example in the previous sections corresponds to $\gamma = \bar{n} + 1/2$ and $\Gamma = \bar{n}'' + 1/2$. A similar derivation gives a

Experimental scheme to achieve optimal superbroadcasting from 2 to 3 copies. This setup involves just a beam splitter, a phase-insensitive amplifier and a tritter, which in turn can be implemented by two suitably balanced beam splitters. The phase-insensitive amplifier can be implemented by a beam splitter and heterodyne-assisted feed-forward. The output copies carrying the same signal as the input ones are locally more pure, the noise being shifted to classical correlations between them.

bound for purification, where $N > M$. In such a case $G < 1$, and Eq. (43) is replaced with

$$\Delta X_B^2 + \Delta Y_B^2 \geq G(\Delta X_A^2 + \Delta Y_A^2) + (1 - G)/2, \quad (50)$$

and one obtains the bound

$$\Gamma - 1/2 \geq 1/N(\gamma - 1/2). \quad (51)$$

We would like to stress that the derivation of all bounds in the present section relies on the theorem of the added noise in *linear* amplifiers, namely only linear transformations of modes are considered. Hence, in principle, these bounds might be violated by more exotic and non-linear transformations. Therefore, the derivation of Eq. (32) is stronger, since it has general validity.

By a similar derivation, using the bound for phase-conjugated amplifiers $\Delta X_B^2 + \Delta Y_B^2 \geq G(\Delta X_A^2 + \Delta Y_A^2) + (G - 1)$, one can obtain the bound for phase-conjugation broadcasting

$$\Gamma - 1/2 \geq 1/N(\gamma + 1/2). \quad (52)$$

6. EXPERIMENTAL IMPLEMENTATION

The optimal broadcasting can be easily implemented on radiation modes, by means of an inverse N-splitter which concentrates the signal in one mode and

discards the other $N - 1$ modes. The mode is then amplified by a phase-insensitive amplifier with power gain $G = M/N$. Finally, the amplified mode is distributed by mixing it in an M-splitter with $M - 1$ vacuum modes. Each mode is then found in the state of Eq. (25). In the concentration stage the N modes with amplitude $\langle a_i \rangle = \alpha$ and noise $\Delta x_i^2 + \Delta y_i^2 = \gamma_i$ are reduced to a single mode with amplitude $\sqrt{N}\alpha$ and noise γ. The amplification stage gives a mode with amplitude $\sqrt{M}\alpha$ and noise $\gamma' = \gamma M/N + M/(2N) - 1/2$. Finally, the distribution stage gives M modes, with amplitude α and noise $\Gamma = 1/M(\gamma' + (M - 1)/2)$ each. In the figure we sketch the scheme for 2 to 3 superbroadcasting.

In [18] it was shown experimentally that phase insensitive amplification can be obtained by a setup consisting of a beam-splitter, a heterodyne detector and a conditional displacement. In the following we give an algebraic derivation of this result. Consider a mode in a state $\rho = \int d^2\gamma/\pi f(\gamma) D(\gamma)$ coupled to another mode in the vacuum through a beam-splitter with transmissivity τ. The output is given by the bipartite state σ

$$\sigma = \int d^2\beta d^2\gamma/\pi^2 e^{-\left|\tau\beta - \sqrt{1 - \tau^2}\gamma\right|^2/2}$$
$$\times f(\tau\gamma + \sqrt{1 - \tau^2}\beta) D(\gamma) \otimes D(\beta), \quad (53)$$

where we performed the change of variables $\beta \longrightarrow \tau\beta + \sqrt{1 - \tau^2}\gamma$, $\gamma \longrightarrow \tau\gamma - \sqrt{1 - \tau^2}\beta$. Now, the reflected mode is measured by heterodyne detection, and conditionally on the measurement outcome α, a displacement $D(k\alpha)$ is performed on the transmitted mode, whose state is then given by

$$\rho' = \int d^2\alpha d^2\beta d^2\gamma/\pi^3 e^{-\frac{\left|\tau\beta - \sqrt{1 - \tau^2}\gamma\right|^2}{2}} f(\tau\gamma + \sqrt{1 - \tau^2}\beta)$$
$$\times D(k\alpha) D(\gamma) D(k\alpha)^\dagger \langle 0|D(\alpha)^\dagger D(\beta) D(\alpha)|0\rangle$$
$$= \int d^2\alpha d^2\beta d^2\gamma/\pi^3 e^{-\frac{\left|\tau\beta - \sqrt{1 - \tau^2}\gamma\right|^2}{2}} f(\tau\gamma + \sqrt{1 - \tau^2}\beta)$$
$$\times e^{\alpha(k\gamma^* - \beta^*) - \text{c.c.}} D(\gamma) e^{-|\beta|^2/2} \quad (54)$$
$$= \int d^2\beta d^2\gamma/\pi^2 \delta^{(2)}(k\gamma - \beta) e^{-\frac{\left|\tau\beta - \sqrt{1 - \tau^2}\gamma\right|^2}{2}}$$
$$\times f(\tau\gamma + \sqrt{1 - \tau^2}\beta) e^{-|\beta|^2/2} D(\gamma)$$
$$= \int d^2\gamma/\pi f((\tau + k\sqrt{1 - \tau^2})) e^{-|\gamma|^2[k^2 + (k\tau - \sqrt{1 - \tau^2})^2]/2} D(\gamma).$$

On the other hand, the action of a phase-insensitive amplifier on ρ can be easily calculated and produces the partial output state

$$\rho'' = \int d^2\gamma/\pi e^{-|\gamma|^2 v^2/2} f(\;\gamma) D(\gamma). \qquad (55)$$

The following conditions

$$= \tau + k\sqrt{1-\tau^2}, \quad v^2 = k^2 + (k\tau - \sqrt{1-\tau^2})^2,$$
$$^2 - v^2 = 1, \qquad (56)$$

which equivalent to

$$k = v, \quad \tau = 1/\;, \qquad (57)$$

imply that ρ' = ρ''. Hence, by tuning the beam splitter transmissivity and the parameter of the conditional displacement k, one can then simulate the amplifier by a linear device assisted by heterodyne and feed-forward.

7. CONCLUSIONS

In conclusion, we proved that broadcasting of M copies of a mixed radiation state starting from $N < M$ copies is possible, even along with lowering the total noise in conjugate quadratures. Since the noise cannot be removed without violating the quantum data processing theorem, the price to pay for having higher purity at the output is that the output copies are correlated. Essentially noise is moved from local states to their correlations, and our superbroadcasting channel does this optimally. We obtained similar results also for purification (i.e., $M \leq N$), along with the case of simultaneous broadcasting and phase-conjugation, with the output copies carrying a signal which is complex-conjugated of the input one. Despite the role that correlations play in this effect, no entanglement is present in the output (as long as the single input copy has a positive P-function), as it can be seen by the analytical expression of the output states. Moreover, a practical and very simple scheme for experimental achievement of the maps has been shown, involving mainly passive media and only one parametric amplifier. The superbroadcasting effect has a relevance form the fundamental point of view, opening new perspectives in the understanding of correlations and their interplay with noise, but may be also promising from a practical point of view, for communication tasks in the presence of noise.

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