EUROPHYSICS LETTERS 15 January 2004

Europhys. Lett., **65** (2), pp. 165–171 (2004)
DOI: 10.1209/epl/i2003-10079-1

Quantum universal detectors

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(received 5 September 2003; accepted 18 November 2003)

PACS. 03.65.Ta – Foundations of quantum mechanics; measurement theory.
PACS. 03.65.Wj – State reconstruction, quantum tomography.

Abstract. – We address the problem of achieving a universal measuring apparatus that allows to estimate the expectation value of any arbitrary operator by changing only the data processing of the outcomes. The "universal detector" performs a joint measurement on the system and on a suitably prepared ancilla. We characterize such universal detectors, and show how they can be obtained either via Bell measurements or via local measurements and classical communication between system and ancilla.

Quantum technology is nowadays turning from the stage of experimental setup design to that of quantum system engineering. The aim is to produce tools for communication, information processing, and storage, which rely on the principles of quantum mechanics, with the chance of achieving much higher speeds and capacities than those of classical devices. In this scenario, a new kind of quantum lab can be devised, in which universality and programmability are crucial features, with different tasks achieved by a basic set of devices.

A "universal detector" would allow the estimation of ensemble averages of arbitrary operators using a single measuring apparatus, and by changing only the data processing of the outcomes, according to which ensemble average is estimated. Such a device would be very useful for many kinds of quantum information processing tasks, such as in quantum computation [1, 2], teleportation [3, 4], entanglement detection [5], and entanglement distillation protocols [6]. In some way the universal detector is similar to a quantum tomographic apparatus [7]: however, the latter would typically require a *quorum* of observables —corresponding to a set of devices or a single tunable device— whereas the universal detector would just measure a fixed single observable on an extended Hilbert space including an ancilla.

In this letter, we introduce the general concept of universal detector, and characterize universal detectors via a necessary and sufficient condition written in terms of spanning sets of operators. We then show how such universal detectors can be achieved via Bell measurements —*i.e.* measurement that are described by projectors on maximally entangled states. The usefulness of Bell measurements is not surprising. In fact, quantum teleportation, dense coding, entanglement swapping [3, 4], high-sensitivity measurements [8], tomography of quantum operations [9], some types of quantum cryptography [10], and many other applications [11–13] require preparation of entangled states and/or Bell measurements. However, entanglement is not an essential ingredient to build a universal detector, since, as we will show in the following, the universal observables of ref. [14] also enter the present framework as a type of universal detector described by a POVM that is based on local measurements and classical communication between system and ancilla.

Let us start by defining the concept of universal detector, or, more abstractly, of universal POVM. We are considering a quantum system in a Hilbert space $\mathcal{H}$, coupled to an ancilla with Hilbert space $\mathcal{K}$. A POVM $\{\Pi_i\}$, $\Pi_i \geq 0$ and $\sum_i \Pi_i = I_{\mathcal{H}} \otimes I_{\mathcal{K}}$ on the Hilbert space $\mathcal{H} \otimes \mathcal{K}$ is *universal* for the system iff there exists a state of the ancilla ν such that for any operator O one has

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O)\,\mathrm{Tr}[(\rho \otimes \nu)\Pi_i], \tag{1}$$

where $f_i(\nu, O)$ is a suitable function of the outcome i and the operator O, which we will refer to as the *data processing*. The detector will be named *universal* when it is described by a universal POVM. In order to give a necessary and sufficient condition for universality, we need to introduce some notation, and the concept of spanning set of operators. We will use the following symbols for bipartite pure states in $\mathcal{H} \otimes \mathcal{K}$:

$$|A\rangle\rangle = \sum_{n=1}^{\dim \mathcal{H}} \sum_{m=1}^{\dim \mathcal{K}} A_{nm} |n\rangle \otimes |m\rangle, \tag{2}$$

where $|n\rangle$ and $|m\rangle$ are fixed orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$, respectively. Equation (2) exploits the isomorphism [15] between the Hilbert space of the Hilbert-Schmidt operators A, B from $\mathcal{K}$ to $\mathcal{H}$, with scalar product $\langle A, B\rangle = \mathrm{Tr}[A^\dagger B]$, and the Hilbert space of bipartite vectors $|A\rangle\rangle, |B\rangle\rangle \in \mathcal{H} \otimes \mathcal{K}$, with $\langle\langle A \mid B\rangle\rangle \equiv \langle A, B\rangle$. It is easy to show the following identities [15]:

$$\begin{aligned} A \otimes B |\, C\rangle\rangle &= |\, A C B^\tau\rangle\rangle, \\ \mathrm{Tr}_{\mathcal{K}}[|\, A\rangle\rangle\langle\langle B\,|] &= A B^\dagger, \\ \mathrm{Tr}_{\mathcal{H}}[|\, A\rangle\rangle\langle\langle B\,|] &= A^\tau B^*, \end{aligned} \tag{3}$$

where τ and $*$ denote transposition and complex conjugation with respect to the given bases, respectively.

A spanning set for operators A from $\mathcal{K}$ to $\mathcal{H}$ [16] is a set $\{\Xi_i\}$ that, along with a dual set $\{\Theta_i\}$, provides expansions for A in the form

$$A = \sum_i \mathrm{Tr}\left[\Theta_i^\dagger A\right] \Xi_i. \tag{4}$$

The completeness relation of the spanning set and its dual reads

$$\sum_i \langle\psi|\Xi_i|\phi\rangle\langle\varphi|\Theta_i^\dagger|\eta\rangle = \langle\psi \mid \eta\rangle\langle\varphi \mid \phi\rangle, \tag{5}$$

for any $\phi, \varphi \in \mathcal{H}$ and $\psi, \eta \in \mathcal{K}$. For continuous sets, the sums in eqs. (4) and (5) are replaced by integrals.

Let us now consider a universal POVM on $\mathcal{H} \otimes \mathcal{K}$. The elements $\{\Pi_i\}$ can be diagonalized as follows:

$$\Pi_i = \sum_{j=1}^{r_i} \big|\Psi_j^{(i)}\rangle\rangle\langle\langle\Psi_j^{(i)}\big|, \tag{6}$$

where the vectors $|\,\Psi_j^{(i)}\rangle\!\rangle$ have norm equal to the j-th eigenvalue of Π_i, and r_i is the rank of Π_i. From the normalization condition $\sum_i \Pi_i = I_{\mathcal{H}} \otimes I_{\mathcal{K}}$, it follows that the set of operators $\{\Psi_j^{(i)}\}$ from $\mathcal{K}$ to $\mathcal{H}$ must be a spanning set itself.

The characterization of universal POVMs is then given by the condition that there exists a density operator ν for the ancilla such that the following operators:

$$\Xi_i[\nu] \equiv \sum_{j=1}^{r_i} \Psi_j^{(i)} \nu^\tau \Psi_j^{(i)\dagger} \tag{7}$$

are a spanning set for operators on $\mathcal{H}$. In fact, using eq. (6), eq. (1) rewrites

$$\mathrm{Tr}[\rho O] = \sum_i f_i(\nu, O)\, \mathrm{Tr}\left[\rho \sum_{j=1}^{r_i} \Psi_j^{(i)} \nu^\tau \Psi_j^{(i)\dagger}\right], \tag{8}$$

and this is true independently of ρ iff

$$O = \sum_i f_i(\nu, O) \Xi_i[\nu]. \tag{9}$$

From linearity one has

$$f_i(\nu, O) = \mathrm{Tr}\left[\Theta_i^\dagger[\nu] O\right], \tag{10}$$

where $\Theta_i[\nu]$ is a dual set of $\Xi_i[\nu]$. Hence, after finding a dual set for $\Xi_i[\nu]$, the data processing function is easily evaluated via eq. (10).

We will now focus attention on Bell POVMs on $\mathcal{H}\otimes\mathcal{H}$. In the notation of eq. (2), maximally entangled vectors correspond to unitary operators [15], and thus a Bell POVM has elements of the form

$$\Pi_i = \frac{\alpha_i}{d} |\, U_i \rangle\!\rangle \langle\!\langle U_i \,|, \tag{11}$$

where d is the dimension of $\mathcal{H}$, α_i are suitable positive constants and U_i are unitaries. When the POVM is orthogonal, one has $\alpha_i = 1$ and $\mathrm{Tr}[U_i^\dagger U_j] = d\delta_{ij}$. Particular cases of Bell POVMs are those in which U_i are a unitary irreducible representation (UIR) of some group $\boldsymbol{G}$. As an example, consider a projective UIR of an Abelian group, which therefore satisfies the relation

$$U_\alpha U_\beta U_\alpha^\dagger = e^{ic(\alpha,\beta)} U_\beta. \tag{12}$$

In this case, the Bell POVM is orthogonal, with number of elements equal to the cardinality of the group d^2. One can show that a suitable ν always exists such that the set of $\Xi_\alpha[\nu] = \frac{1}{d} U_\alpha \nu^\tau U_\alpha^\dagger$ is a spanning set. In fact, for any ν such that $\mathrm{Tr}[U_\alpha^\dagger \nu^\tau] \neq 0$ for all α, using the identity

$$\sum_{\alpha=1}^{d^2} e^{ic(\alpha,\gamma)} e^{ic(\beta,\alpha)} = d^2 \delta_{\gamma\beta}, \tag{13}$$

one has

$$\begin{aligned} O &= \frac{1}{d} \sum_{\beta=1}^{d^2} \mathrm{Tr}\left[U_\beta^\dagger O\right] U_\beta = \frac{1}{d^3} \sum_{\beta,\alpha,\gamma=1}^{d^2} \frac{\mathrm{Tr}\left[U_\beta^\dagger O\right]}{\mathrm{Tr}\left[U_\beta^\dagger \nu^\tau\right]} \mathrm{Tr}\left[U_\gamma^\dagger \nu^\tau\right] U_\gamma e^{ic(\alpha,\gamma)} e^{ic(\beta,\alpha)} \\ &= \frac{1}{d^2} \sum_{\alpha,\beta=1}^{d^2} \frac{\mathrm{Tr}\left[U_\beta^\dagger O\right]}{\mathrm{Tr}\left[U_\beta^\dagger \nu^\tau\right]} e^{ic(\beta,\alpha)} U_\alpha \nu^\tau U_\alpha^\dagger. \end{aligned} \tag{14}$$

The dual set is then given by

$$\Theta_\alpha[\nu] = \frac{1}{d} \sum_{\beta=1}^{d^2} \frac{U_\beta}{\mathrm{Tr}\left[U_\beta \nu^*\right]} e^{-ic(\beta,\alpha)}, \tag{15}$$

and it is unique since the unitaries U_β are linearly independent. By identifying $U_1 \equiv I$, a possible choice of the ancilla state is

$$\nu = \frac{1}{d} I + \frac{1}{d(d^2-1)} \sum_{\alpha>1} U_\alpha. \tag{16}$$

For an explicit example, consider the group $\mathbb{Z}_d \times \mathbb{Z}_d$ and its d-dimensional projective UIR,

$$U_{m,n} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d} km} |k\rangle\langle k \oplus n|, \quad m,n \in [0, d-1], \tag{17}$$

which gives the Bell measurement used in the teleportation schemes of ref. [3]. The composition and orthogonality relations of the set are given by

$$U_{m,n} U_{p,q} U_{m,n}^\dagger = e^{\frac{2\pi i}{d}(np-mq)} U_{p,q}, \tag{18}$$

$$\mathrm{Tr}\left[U_{p,q}^\dagger U_{m,n}\right] = d\delta_{mp}\delta_{nq}. \tag{19}$$

Using eqs. (10), (15), (18) and (19), one easily evaluates the data processing function for any operator O in the form

$$f_{m,n}(\nu, O) = e^{\frac{2\pi i}{d}(mq-np)} \sum_{p,q} \frac{\mathrm{Tr}\left[U_{p,q}^\dagger O\right]}{\mathrm{Tr}\left[U_{p,q}^\dagger \nu^\tau\right]}. \tag{20}$$

The ancilla can also be prepared in the following pure state:

$$|\psi\rangle = \sqrt{\frac{1-|\alpha|^2}{1-|\alpha|^{2d}}} \sum_{n=0}^{d-1} \alpha^n |n\rangle, \tag{21}$$

for any α with $0 < |\alpha| < 1$.

As an example for the infinite-dimensional case, consider the displacement operators for a bosonic mode a, namely $D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a]$, with $\alpha \in \mathbb{C}$. Such operators are the elements of a projective UIR of the Weyl-Heisenberg group, and generate the Bell measurement corresponding to the continuous variables teleportation schemes of refs. [4, 17]. The composition and orthogonality relations write

$$D(\alpha) D(\beta) D^\dagger(\alpha) = e^{2i\,\mathrm{Im}(\alpha\beta^*)} D(\beta), \tag{22}$$

$$\mathrm{Tr}\left[D^\dagger(\alpha) D(\beta)\right] = \pi \delta^{(2)}(\alpha - \beta), \tag{23}$$

where $\delta^{(2)}(\alpha) \equiv (1/\pi^2) \int_{\mathbb{C}} \mathrm{d}^2\gamma e^{\alpha\gamma^* - \alpha^*\gamma}$ denotes the Dirac-delta on the complex plane. The processing function is given by

$$f_\alpha(\nu, O) = \int_{\mathbb{C}} \frac{\mathrm{d}^2\beta}{\pi} e^{\alpha^*\beta - \alpha\beta^*} \frac{\mathrm{Tr}\left[D^\dagger(\beta) O\right]}{\mathrm{Tr}\left[D^\dagger(\beta) \nu^\tau\right]}. \tag{24}$$

The Dirac vectors $|D(z)\rangle\!\rangle$ are the eigenvectors with eigenvalue z of the "current" $Z = a - b^\dagger$, which in the case of e.m. radiation is the Bell observable of heterodyne [18], eight-port homodyne [19, 20] or six-port homodyne detectors [21], whereas for atoms coupled with two light fields the observable is achieved by measuring the corresponding phase-shifts [22]. The present infinite-dimensional case, however, needs care in checking convergence of the integral in eq. (24). For example, if we take the vacuum state $\nu = |0\rangle\langle 0|$, the universal measurement will be the phase-space averaging with the so-called Q-function $Q(z) = \frac{1}{\pi}\langle z|\rho|z\rangle$ ($|z\rangle$ coherent state), and we know that this gives expectations only for operators admitting anti-normal ordered field expansion [23]. In particular, the matrix elements of the density operator cannot be recovered in this way [7]. Therefore, in infinite dimensions the universality can be limited by convergence.

There are universal Bell POVMs also from non-Abelian groups. For example, the $SU(2)$ group corresponds to a non-orthogonal POVM [17]. Consider the j-dimensional UIR of the $SU(2)$ group [24], parameterized as $U(\psi,\vec{n}) = \exp[i\psi\vec{J}\cdot\vec{n}]$, where $\psi \in [0, 2\pi)$, $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a unit vector on a sphere S^2, and J_α are customary angular momentum operators. The projectors on maximally entangled states

$$\Pi(\psi,\vec{n}) = |U(\psi,\vec{n})\rangle\!\rangle\langle\!\langle U(\psi,\vec{n})| \tag{25}$$

provide the resolution of the identity [17]

$$\frac{2j+1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\psi\,\sin^2\frac{\psi}{2}\int_{S^2}\mathrm{d}\vec{n}\,\Pi(\psi,\vec{n}) = I_{\mathcal{H}}\otimes I_{\mathcal{H}}. \tag{26}$$

Notice, however, that the states $|U(\psi,\vec{n})\rangle\!\rangle$ are not orthogonal, namely the set is overcomplete for $\mathcal{H}\otimes\mathcal{H}$. The universality of this POVM can be proved by invoking the fact that the projectors on spin coherent states $|\psi,\varphi;m\rangle$ are also an operator spanning set. In fact, spin coherent states are obtained by applying the unitary operators

$$D(\psi,\varphi) = e^{i\frac{\psi}{2}(J_+e^{-i\varphi}+J_-e^{i\varphi})} \tag{27}$$

on a fixed eigenstate $|m\rangle$ of J_z, namely $|\psi,\varphi;m\rangle \doteq D(\psi,\varphi)|m\rangle$. Notice that the operators in eq. (27) *do not* form a group, nevertheless one has the completeness relation

$$\frac{2j+1}{4\pi}\int_0^{2\pi}\mathrm{d}\psi\int_0^{\pi}\mathrm{d}\varphi\,\sin(\varphi)|\psi,\varphi;m\rangle\langle\psi,\varphi;m| = I_{\mathcal{H}}, \tag{28}$$

and the P-representation for any operator O,

$$O = \frac{2j+1}{4\pi}\int_0^{2\pi}\mathrm{d}\psi\int_0^{\pi}\mathrm{d}\varphi\,\sin(\varphi)P_O(\psi,\varphi)|\psi,\varphi;m\rangle\langle\psi,\varphi;m|. \tag{29}$$

Explicit constructions for the function $P_O(\psi,\varphi)$ can be found in ref. [24]. The above expansion is similar to the P-function representation of quantum optics; however, here the function $P_O(\psi,\varphi)$ is always well defined, due to the finite dimensionality of the Hilbert space. The universality of the Bell POVM in eq. (25) is now proved by showing that there exists a state ν such that $U(\psi,\vec{n})\nu^\tau U^\dagger(\psi,\vec{n})$ is a spanning set. In fact, using the following $SU(2)$ change of parameterization:

$$U\big(\psi,\vec{n}\big) = D\big(\psi',\varphi'\big)e^{2i\theta' J_z}, \tag{30}$$

according to eq. (29), for any $\nu^\tau = |m\rangle\langle m|$, the operators

$$U\big(\psi,\vec{n}\big)\nu^\tau U^\dagger\big(\psi,\vec{n}\big) = D\big(\psi',\varphi'\big)\nu^\tau D^\dagger\big(\psi',\varphi'\big) \tag{31}$$

make a spanning set [25].

Another interesting example is represented by the group $SU(d)$. In this case, the universality of the corresponding Bell POVM is proved by constructing the dual set of $\Xi_\alpha[\nu] = U_\alpha \nu^\tau U_\alpha^\dagger$. From eq. (5), one can check that a dual set for $\Xi_\alpha[\nu]$, for any state ν on $\mathcal{H}$, is given by

$$\Theta_\alpha[\nu] = U_\alpha \xi U_\alpha^\dagger, \tag{32}$$

for arbitrary ξ with $\mathrm{Tr}[\xi] = 1$ and $\mathrm{Tr}[\nu^\tau \xi^\dagger] = d$. For example, for pure $\nu^\tau = |\phi\rangle\langle\phi|$, one can take

$$\xi = \frac{1}{1-F}\big[(d-F)|\phi\rangle\langle\phi| - (d-1)|\psi\rangle\langle\psi|\big], \tag{33}$$

with any state $|\psi\rangle$ with $F \equiv |\langle\psi \mid \phi\rangle|^2 < 1$. The corresponding data processing function is

$$f_\alpha(|\phi\rangle\langle\phi|, O) = \frac{1}{1-F}\big[(d-F)\langle\phi|U_\alpha^\dagger O U_\alpha|\phi\rangle - (d-1)\langle\psi|U_\alpha^\dagger O U_\alpha|\psi\rangle\big]. \tag{34}$$

All previous examples presented universal POVMs which are Bell measurements. However, by enlarging the ancillary Hilbert space, one can obtain separable POVMs that are universal. The following example was first introduced in ref. [14]. Let us consider a spanning set $\{C(l), l = 1, 2, \ldots, L\}$, for operators on $\mathcal{H}$ such that all $C(l)$ are normal, namely they have orthogonal eigenvectors $|c_k(l)\rangle$, and

$$C(l) = \sum_k c_k(l)|c_k(l)\rangle\langle c_k(l)|. \tag{35}$$

Notice that necessarily one has $L \geq (\dim(\mathcal{H}))^2$. By taking an ancillary Hilbert space $\mathcal{K}$, with $\dim(\mathcal{K}) = L$, and an orthonormal basis $\{|l\rangle\}$, one can write the following orthogonal POVM for $\mathcal{H} \otimes \mathcal{K}$:

$$\Pi_{k,l} = |c_k(l)\rangle\langle c_k(l)| \otimes |l\rangle\langle l|, \tag{36}$$

which can be achieved by local measurement and classical communication between system and ancilla. The universality of $\Pi_{k,l}$ easily follows by using eq. (1) with the processing function

$$f_{k,l}(\nu, O) = \frac{\mathrm{Tr}\left[C^\dagger(l)0\right]}{\langle l|\nu|l\rangle} c_k(l), \tag{37}$$

with the condition $\langle l|\nu|l\rangle \neq 0$ for all l.

The form of the above separable universal POVMs opens some questions on the general structure of universal POVMs. For example, it is possible that also universal Bell POVMs could be constructed using general unitary spanning sets that are not a group representation, and the role of such symmetry is probably not essential. Also, it is likely that when the ancilla space has the same dimension of the system, then the universal POVM must be Bell. Moreover, a general classification of universal POVMs along with the pertaining ancilla states and data processing functions is needed for performance optimization. Finally, the possibility of *weakly universal* POVMs, in which the ancilla state depends on the operator to be estimated, sets a link with the related problem of "programmable" detectors [26].

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This work has been sponsored by INFM through the project PRA-2002-CLON, and by EEC through the ATESIT project IST-2000-29681. GMD also acknowledges partial support from Department of Defense Multidisciplinary University Research Initiative (MURI) program administered by the Army Research Office under Grant No. DAAD19-00-1-0177.

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[25] Notice that the processing function $\tilde{P}_O(\psi,\vec{n})$ is related to the customary P-function for the operator O. However, one has a nontrivial relation, because the factorization formula in eq. (30) is nonlinear between $(\psi,\vec{n})$ and (ψ',φ',θ') (D'Ariano G. M., Perinotti P. and Sacchi M. F., unpublished).
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