PHYSICAL REVIEW A **80**, 022339 (2009)

Theoretical framework for quantum networks

Giulio Chiribella,* Giacomo Mauro D'Ariano,† and Paolo Perinotti‡

Dipartimento di Fisica "A. Volta," QUIT Group, and INFN, via Bassi 6, 27100 Pavia, Italy

(Received 28 April 2009; published 31 August 2009)

We present a framework to treat quantum networks and all possible transformations thereof, including as special cases all possible manipulations of quantum states, measurements, and channels, such as, e.g., cloning, discrimination, estimation, and tomography. Our framework is based on the concepts of *quantum comb*—which describes all transformations achievable by a given quantum network—and *link product*—the operation of connecting two quantum networks. Quantum networks are treated both from a constructive point of view—based on connections of elementary circuits—and from an axiomatic one—based on a hierarchy of admissible quantum maps. In the axiomatic context a fundamental property is shown, which we call *universality of quantum memory channels:* any admissible transformation of quantum networks can be realized by a suitable sequence of memory channels. The open problem whether this property fails for some nonquantum theory, e.g., for no-signaling boxes, is posed.

DOI: 10.1103/PhysRevA.80.022339 PACS number(s): 03.67.Ac, 03.65.Ta

I. INTRODUCTION

In the last decade the general description of quantum states, measurements, and transformations in terms of density matrices, positive-operator-valued measures (POVMs), and channels [1–3] has been widely exploited in quantum information, with many applications in high-precision measurements, quantum cryptography, optimal cloning, quantum communication, and many others. The success of such general description comes from the fact that it allows one to optimize the design of quantum devices over all possibilities admitted by quantum mechanics, thus finding the ultimate performances in the realization of desired tasks. Although a quantum channel can be always thought of as the result of a unitary interaction of the system with an environment [4,5], and a POVM as a joint von Neumann measurement on system and environment [6], the neat advantage of using channels and POVMs is that they simplify optimization, by getting rid of all those details that pertain specific realizations but are irrelevant for the final purpose.

Channels and POVMs provide an efficient description of elementary circuits that transform or measure quantum states. When elementary circuits are combined in a larger quantum network, however, the variety of possible tasks one can perform grows exponentially. For example, a quantum computing network can be used as a programmable machine, which implements different transformations on input data depending on the quantum state of the program. In some cases the program itself can be a quantum channel, rather than a state: during computation, for instance, the network can call a variable channel as a subroutine so that the overall transformation of the input data is programmed by it. Even more generally, the action of the network can be programmed by a sequence of variable states and channels that are called at different times, that is, at different steps of the computation. A similar situation arises in multiple-rounds quantum games [7], where the overall outcome of the game is determined by the sequence of moves (state preparations, measurements, and channels) performed by different players. For example, in a two-party game Alice's strategy can be seen as a particular quantum network in which Bob's moves act as variable subroutines. Of course, the subroutines corresponding to Bob's moves are in turn parts of Bob's network, so that the whole protocol can be seen as the interlinking of two networks corresponding to Alice's and Bob's strategies.

A quantum network can be used in a number of different ways, each way corresponding to a different kind of transformation achievable with it, e.g., transformations from states to channels, from channels to channels, and from sequences of states and/or channels to channels, as discussed above. In fact, if we consider networks of arbitrary size, there is an infinite number of different transformations that we can implement. This fact suggests to find new notions that generalize those of channels and POVMs in the case of quantum networks: apparently one would have to introduce a new mathematical object for any possible transformation. In addition, since a quantum network can contain random circuits performing measurements and quantum operations, for any transformation one would have to take into account also its probabilistic version. Clearly, defining a new kind of quantum map for any possible use of a network is not a viable approach. On the other hand, using the current framework based solely on states as inputs and outputs to describe a quantum network, one is presently forced to specify all elementary channels and measuring devices in it, and if one needs to optimize the network for some desired task, then one has to face cumbersome optimization of all its elements. Optimizing a quantum network without suitable tools is indeed comparable to treating tasks such as quantum error correction and quantum state estimation without the notions of channel and POVM.

Luckily enough, an efficient treatment of quantum networks is possible, despite the infinity of different transformations associated to them. In this paper we will provide a complete toolbox for the description and the optimization of quantum networks by answering the following two ques-

*chiribella@fisicavolta.unipv.it
†dariano@unipv.it
‡perinotti@fisicavolta.unipv.it

tions: (i) which are the possible tasks that a given network can accomplish? and (ii) which are the transformations that a given network can undergo? Both questions will be tackled in Secs. III and IV from two different complementary points of view. On the one hand, in Sec. III we will consider quantum states, POVMs, and channels, as elementary building blocks to construct quantum networks. The main focus will be the description of actual networks by means of Choi-Jamiołkowski operators, and the description of connections among networks by means of a suitably defined composition of Choi-Jamiołkowski operators. On the other hand, in Sec. IV we will derive quantum networks and their transformations on a purely axiomatic basis, by defining a hierarchy of admissible quantum maps. The physical realizability of these general transformations is proved, in a way that is similar to the unitary realization of quantum channels: we will prove that any deterministic admissible map can be physically obtained by a suitable sequence of memory channels. We call this property *universality of memory channels*, as it implies that, under mild assumptions, any deterministic transformation that is conceivable in quantum mechanics can be always realized by some sequence of memory channels. The case of probabilistic transformations is also considered, showing that any probabilistic transformation can be realized by a sequence of memory channels followed by a von Neumann measurement on some output subsystem.

II. PRELIMINARIES AND NOTATION

In this section we list a set of elementary facts about linear maps and Choi-Jamiołkowski operators. The product of Choi-Jamiołkowski operators induced by the composition of the corresponding linear maps is defined and analyzed.

A. Linear operators and linear maps

In the following we denote with $\mathcal{L}(\mathcal{H})$ the set of linear operators on the finite dimensional Hilbert space $\mathcal{H}$. The set of linear operators from $\mathcal{H}_0$ to $\mathcal{H}_1$ is denoted by $\mathcal{L}(\mathcal{H}_0,\mathcal{H}_1)$. Operators X in $\mathcal{L}(\mathcal{H}_0,\mathcal{H}_1)$ are in one-to-one correspondence with vectors $|X\rangle\rangle$ in $\mathcal{H}_1\otimes\mathcal{H}_0$ as follows:

$$|X\rangle\rangle = (X\otimes I_{\mathcal{H}_0})|I_{\mathcal{H}_0}\rangle\rangle,$$

$$=(I_{\mathcal{H}_1}\otimes X^T)|I_{\mathcal{H}_1}\rangle\rangle, \tag{1}$$

where $I_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$, $|I_{\mathcal{H}}\rangle\rangle\in\mathcal{H}^{\otimes 2}$ is the maximally entangled vector $|I_{\mathcal{H}}\rangle\rangle=\Sigma_n|n\rangle|n\rangle$ (with $\{|n\rangle\}$ a fixed orthonormal basis for $\mathcal{H}$), and $X^T\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_0)$ is the transpose of A with respect to the two fixed bases chosen in $\mathcal{H}_0$ and $\mathcal{H}_1$.

The set of linear maps from $\mathcal{L}(\mathcal{H}_0)$ to $\mathcal{L}(\mathcal{H}_1)$ is denoted by $\mathcal{L}(\mathcal{L}(\mathcal{H}_0),\mathcal{L}(\mathcal{H}_1))$. Linear maps $\mathcal{M}$ in $\mathcal{L}(\mathcal{L}(\mathcal{H}_0),\mathcal{L}(\mathcal{H}_1))$ are in one to one correspondence with linear operators on $\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_0)$ as follows:

$$M=\mathfrak{C}(\mathcal{M}) := \mathcal{M}\otimes\mathcal{I}_{\mathcal{L}(\mathcal{H}_0)}(|I_{\mathcal{H}_0}\rangle\rangle\langle\langle I_{\mathcal{H}_0}|), \tag{2}$$

where $\mathcal{I}_{\mathcal{L}(\mathcal{H}_0)}$ is the identity map on $\mathcal{L}(\mathcal{H}_0)$. The inverse map $\mathfrak{C}^{-1}$ transforms $M\in\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_0)$ into a map in $\mathcal{L}(\mathcal{L}(\mathcal{H}_0),\mathcal{L}(\mathcal{H}_1))$ that acts on an operator $X\in\mathcal{L}(\mathcal{H}_0)$ as follows:

$$[\mathfrak{C}^{-1}(M)](X) = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1}\otimes X^T)M], \tag{3}$$

$\mathrm{Tr}_{\mathcal{H}}$ denoting the partial trace over $\mathcal{H}$.

Definition 1 (Choi-Jamiołkowski isomorphism). The bijective correspondence $\mathfrak{C}:\mathcal{M}\to M$ defined through Eq. (2) is called Choi-Jamiołkowski isomorphism. Its inverse $\mathfrak{C}^{-1}:M\to\mathcal{M}$ is defined through Eq. (3).

For conciseness, we will use the notation M for $\mathfrak{C}(\mathcal{M})$ throughout the paper. The operator M corresponding to the map $\mathcal{M}$ is called *Choi-Jamiołkowski operator of* $\mathcal{M}$.

Lemma 1. A linear map $\mathcal{M}$ is trace preserving if and only if its Choi-Jamiołkowski operator enjoys the following property:

$$\mathrm{Tr}_{\mathcal{H}_1}[M] = I_{\mathcal{H}_0}. \tag{4}$$

Proof. The trace-preserving condition writes $\mathrm{Tr}[\mathcal{M}(X)] = \mathrm{Tr}[X]$. Since

$$\mathrm{Tr}[\mathcal{M}(X)] = \mathrm{Tr}\{(I_{\mathcal{H}_1}\otimes X^T)M\} = \mathrm{Tr}_{\mathcal{H}_0}\{X^T\mathrm{Tr}_{\mathcal{H}_1}[M]\}, \tag{5}$$

and $\mathrm{Tr}[X]=\mathrm{Tr}[X^T]$, the condition is satisfied for arbitrary X if and only if $\mathrm{Tr}_{\mathcal{H}_1}[M]=I_{\mathcal{H}_0}$. ■

Lemma 2. A linear map $\mathcal{M}$ is Hermitian preserving if and only if its Choi-Jamiołkowski operator M is Hermitian.

Proof. A map $\mathcal{M}$ is Hermitian preserving if $\mathcal{M}(H)^\dagger = \mathcal{M}(H)$ for any Hermitian operator H, or equivalently, if $\mathcal{M}(X)^\dagger=\mathcal{M}(X^\dagger)$ for any operator X. The adjoint of $\mathcal{M}(X)$ is expressed as

$$\mathcal{M}(X)^\dagger = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1}\otimes X^*)M^\dagger] = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1}\otimes X^{\dagger T})M^\dagger]. \tag{6}$$

Clearly, if $M^\dagger=M$ one has $\mathcal{M}(X)^\dagger=\mathcal{M}(X^\dagger)$. On the other hand, if

$$\mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1}\otimes X^{\dagger T})M^\dagger] = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1}\otimes X^{\dagger T})M] \tag{7}$$

for all X, then $M^\dagger=M$, due to the Choi-Jamiołkowski isomorphism. ■

Lemma 3. A linear map $\mathcal{M}$ is completely positive (CP) if and only if its Choi-Jamiołkowski operator M is positive semidefinite.

Proof. Clearly, if $\mathcal{M}$ is CP, by Eq. (2) $M\geqslant 0$. On the other hand, if $M\geqslant 0$, it can be diagonalized as follows:

$$M=\sum_j |K_j\rangle\rangle\langle\langle K_j|, \tag{8}$$

and consequently, exploiting Eqs. (1) and (3), we can write its action in the Kraus form [1]

$$\mathcal{M}(X)=\sum_j K_j X K_j^\dagger. \tag{9}$$

The Kraus form coming from diagonalization of M is called *canonical*. On the other hand, since the same reasoning holds for any decomposition $M=\Sigma_k|F_k\rangle\rangle\langle\langle F_k|$, there exist infinitely many possible Kraus forms. The Kraus form implies complete positivity: indeed, the extended map $\mathcal{M}\otimes\mathcal{I}_{\mathcal{L}(\mathcal{H}_A)}$ transforms any positive operator $P \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_A)$ into a positive operator, as follows:

$$\mathcal{M} \otimes \mathcal{I}_{\mathcal{L}(\mathcal{H}_A)}(P) = \sum_j (K_j \otimes I_{\mathcal{H}_A}) P (K_j^\dagger \otimes I_{\mathcal{H}_A}) \geq 0. \quad (10)$$

■

B. Link product

The Choi-Jamiołkowski isomorphism poses the natural question on how the composition of linear maps is translated to a corresponding composition between the respective Choi-Jamiołkowski operators.

Consider two linear maps $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_1))$ and $\mathcal{N} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_1), \mathcal{L}(\mathcal{H}_2))$ with Choi-Jamiołkowski operators $M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ and $N \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1)$, respectively. The two maps are composed to give the linear map $\mathcal{C} = \mathcal{N} \circ \mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_2))$. This can be easily obtained upon considering the action of $\mathcal{C}$ on an operator $X \in \mathcal{L}(\mathcal{H}_0)$ written in terms of the Choi-Jamiołkowski operators of the composing maps

$$\begin{aligned} \mathcal{C}(X) &= \mathrm{Tr}_{\mathcal{H}_1}[(I_{\mathcal{H}_2} \otimes \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_1} \otimes X^T)M]^T)N], \\ &= \mathrm{Tr}_{\mathcal{H}_1,\mathcal{H}_0}[(I_{\mathcal{H}_2} \otimes I_{\mathcal{H}_1} \otimes X^T)(I_{\mathcal{H}_2} \otimes M^{T_1})(N \otimes I_{\mathcal{H}_0})]. \end{aligned} \quad (11)$$

Upon comparing the above identity with the Eq. (3) for the map $\mathcal{C}$, namely, $\mathcal{C}(X) = \mathrm{Tr}_{\mathcal{H}_0}[(I_{\mathcal{H}_2} \otimes X^T)C]$, one obtains

$$C = \mathrm{Tr}_{\mathcal{H}_1}[(I_{\mathcal{H}_2} \otimes M^{T_1})(N \otimes I_{\mathcal{H}_0})], \quad (12)$$

where M^{T_i} denotes the partial transpose of M on the space $\mathcal{H}_i$. The above result can be expressed in a compendious way by introducing the notation

$$N * M := \mathrm{Tr}_{\mathcal{H}_1}[(I_{\mathcal{H}_2} \otimes M^{T_1})(N \otimes I_{\mathcal{H}_0})], \quad (13)$$

which we call *link product* of the operators $M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ and $N \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1)$. The above result can be synthesized in the following statement.

Theorem 1 (Composition rules). Consider two linear maps $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_1))$ and $\mathcal{N} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_1), \mathcal{L}(\mathcal{H}_2))$ with Choi-Jamiołkowski operators $M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ and $N \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_1)$, respectively. Then, the Choi-Jamiołkowski operator $M \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_0)$ of the composition $\mathcal{C} = \mathcal{N} \circ \mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_0), \mathcal{L}(\mathcal{H}_2))$ is given by the link product of the Choi-Jamiołkowski operators $C = N * M$.

In the following we will consider more generally maps with input and output spaces that are tensor products of Hilbert spaces, and which will be composed only through some of these spaces, e.g., for quantum circuits which are composed only through some wires. For describing these compositions of maps we will need a more general definition of link product. For such purpose, consider now a couple of operators $M \in \mathcal{L}(\otimes_{m \in \mathcal{M}} \mathcal{H}_m)$ and $N \in \mathcal{L}(\otimes_{n \in \mathcal{N}} \mathcal{H}_n)$, where $\mathcal{M}$ and $\mathcal{N}$ describe set of indices for the Hilbert spaces, which generally have nonempty intersection [8].

The general definition of link product then reads:

Definition 2 (General link product). The link product of two operators $M \in \mathcal{L}(\otimes_{m \in \mathcal{M}} \mathcal{H}_m)$ and $N \in \mathcal{L}(\otimes_{n \in \mathcal{N}} \mathcal{H}_n)$ is the operator $M * N \in \mathcal{L}(\mathcal{H}_{\mathcal{N} \setminus \mathcal{M}} \otimes \mathcal{H}_{\mathcal{M} \setminus \mathcal{N}})$ given by

$$N * M := \mathrm{Tr}_{\mathcal{M} \cap \mathcal{N}}[(I_{\mathcal{N} \setminus \mathcal{M}} \otimes M^{T_{\mathcal{M} \cap \mathcal{N}}})(N \otimes I_{\mathcal{M} \setminus \mathcal{N}})], \quad (14)$$

where the set subscript $\mathcal{X}$ is a shorthand for $\otimes_{i \in \mathcal{X}} \mathcal{H}_i$, and $\mathcal{A} \setminus \mathcal{B} := \{i \in \mathcal{A}, i \notin \mathcal{B}\}$ for two sets $\mathcal{A}$ and $\mathcal{B}$.

Examples. For $\mathcal{M} \cap \mathcal{N} = \emptyset$, e.g., for two operators M and N acting on different Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_0$, respectively, their link product is the tensor product

$$N * M = N \otimes M \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0). \quad (15)$$

For $\mathcal{N} = \mathcal{M}$, i.e., when the two operators M and N act on the same Hilbert space, the link product becomes the trace

$$A * B = \mathrm{Tr}[A^T B]. \quad (16)$$

Theorem 2 (Properties of the link product). The operation of link product has the following properties:

(1) $M * N = E(N * M)E$, where E is the unitary swap on $\mathcal{H}_{\mathcal{N} \setminus \mathcal{M}} \otimes \mathcal{H}_{\mathcal{M} \setminus \mathcal{N}}$.

(2) If M_1, M_2, and M_3 act on Hilbert spaces labeled by the sets $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, respectively, and $\mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3 = \emptyset$, then $M_1 * (M_2 * M_3) = (M_1 * M_2) * M_3$.

(3) If M and N are Hermitian, then $M * N$ is Hermitian.

(4) If M and N are positive semidefinite, then $M * N$ is positive semidefinite.

Proof. Properties 1, 2, and 3 are immediate from the definition. For property 4, consider the two maps $\mathcal{M} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathcal{M} \setminus \mathcal{N}}), \mathcal{L}(\mathcal{H}_{\mathcal{M} \cap \mathcal{N}}))$ and $\mathcal{N} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_{\mathcal{M} \cap \mathcal{N}}), \mathcal{L}(\mathcal{H}_{\mathcal{N} \setminus \mathcal{M}}))$, associated to M and N by Eq. (3). Due to Lemma 3, the maps $\mathcal{M}$, $\mathcal{N}$ are both CP. Moreover, due to Theorem 1 the link product $C = N * M$ is the Choi-Jamiołkowski operator of the composition $\mathcal{C} = \mathcal{N} \circ \mathcal{M}$. Since the composition of two CP maps is CP, the Choi-Jamiołkowski operator $C = N * M$ must be positive semidefinite. ■

Remark. As it should be clear to the reader, the advantage in using multipartite operators instead of maps is that we can associate many different kinds of maps to the same operator $M \in \mathcal{L}(\otimes_{i \in I} \mathcal{H}_i)$, depending on how we group the Hilbert spaces in the tensor product. Indeed, any partition of the set I into two disjoint sets I_0 and I_1 defines a different linear map from $\mathcal{L}(\otimes_{i \in I_0} \mathcal{H}_i)$ to $\mathcal{L}(\otimes_{i \in I_1} \mathcal{H}_i)$ via Eq. (3). We will see in the next section that dealing with operators and link products allows one to efficiently treat all possible maps associated to quantum networks.

III. QUANTUM NETWORKS: CONSTRUCTIVE APPROACH

A. Channels and states: deterministic Choi-Jamiołkowski operators

In the general description of quantum mechanics, quantum states are density matrices on Hilbert space $\mathcal{H}$ of the system, i.e., positive semidefinite operators $\rho \in \mathcal{L}(\mathcal{H})$ with $\mathrm{Tr}[\rho] = 1$. Deterministic transformations of quantum states are the so-called quantum channels, a quantum channel $\mathcal{C}$ from states on $\mathcal{H}_0$ to states on $\mathcal{H}_1$ being a trace preserving completely positive map. According to Lemmas 1, 2, and 3, the Choi-Jamiołkowski operator corresponding to $\mathcal{C}$ is a positive semidefinite operator $C \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ satisfying $\mathrm{Tr}_{\mathcal{H}_1}[C]=I_{\mathcal{H}_0}$.

It is immediate to see that a density matrix is a particular case of Choi-Jamiołkowski operator of a channel, namely, a Choi-Jamiołkowski operator with one-dimensional input space $\mathcal{H}_0$: in this case the condition $\mathrm{Tr}_{\mathcal{H}_1}[C]=I_{\mathcal{H}_0}$ becomes indeed $\mathrm{Tr}[C]=1$. This reflects the fact that having a quantum state is equivalent to having at disposal one use of a suitable preparation device. The application of the channel $\mathcal{C}$ to the state ρ is equivalent to the composition of two channels, and is indeed given by the link product of the corresponding Choi-Jamiołkowski operators

$$\mathcal{C}(\rho) = C * \rho, \tag{17}$$

which agrees both with Eq. (3) and Theorem 1.

The opposite example is the completely demolishing "trace channel" $\mathcal{T}(\rho)=\mathrm{Tr}[\rho]$, which transforms quantum states into their probabilities (of course, normalized density matrices give unit probabilities): this channel has one-dimensional output space $\mathcal{H}_1$, and, accordingly its Choi-Jamiołkowski operator is $T=I_{\mathcal{H}_0}$. Notice that the normalization of the Choi-Jamiołkowski operator $C \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_0)$ of a channel $\mathcal{C}$ can be also written in terms of concatenation with the trace channel as

$$C * I_{\mathcal{H}_1} = I_{\mathcal{H}_0}. \tag{18}$$

B. Instruments, random sources, and POVMs: Probabilistic Choi-Jamiołkowski operators

In addition to the Choi-Jamiołkowski operators of deterministic quantum devices, one can consider their probabilistic versions. A complete family of probabilistic transformations from states on $\mathcal{H}_0$ to states on $\mathcal{H}_1$, known as *quantum instrument*, is a set of CP maps $\{\mathcal{C}_i | i \in I\}$ summing up to a trace-preserving CP map $\mathcal{C}=\sum_{i\in I}\mathcal{C}_i$. The corresponding Choi-Jamiołkowski operators $\{C_i | i \in I\}$ are positive semidefinite operators summing up to a deterministic Choi-Jamiołkowski operator $C=\sum_{i\in I} C_i$ with $C * I_{\mathcal{H}_1}=I_{\mathcal{H}_0}$. For families of probabilistic transformations, the index i has always to be intended as a classical outcome, which is known to the experimenter, and heralds the occurrence of different random transformations.

For one-dimensional input space $\mathcal{H}_0$, a complete family of probabilistic Choi-Jamiołkowski operators $\{\rho_i | i \in I\}$ with $\sum_i \rho_i=\rho, \mathrm{Tr}[\rho]=1$ describes a *random source of quantum states*. Applying the trace channel $\mathcal{T}$ after the source gives the probability of the source emitting the ith state: $p_i=\mathrm{Tr}[\rho_i] =\rho_i * I_{\mathcal{H}_1}$ (of course $p_i \geqslant 0$ and $\sum_i p_i=1$).

For one-dimensional output space $\mathcal{H}_1$, a complete family of probabilistic Choi-Jamiołkowski operators is instead a POVM $\{P_i | i \in I\}$, $\sum_i P_i=I_{\mathcal{H}_1}$. Measuring the POVM on the state ρ is equivalent to applying the random device described by $\{P_i\}$ after the preparation device for state ρ, producing as the outcome the probabilities

$$p(i|\rho) = \rho * P_i = \mathrm{Tr}[\rho P_i^T]. \tag{19}$$

Apart from the transpose, which can be absorbed in the definition of the POVM, this is nothing but the Born rule for probabilities, obtained here from the composition of a preparation channel with a random transformation with one-dimensional output space.

In conclusion, states, channels, random sources, instruments, and POVMs can be treated on the same footing as deterministic and probabilistic transformations, which in turn can be described using only Choi-Jamiołkowski operators and link product.

C. Quantum networks and memory channels

In the previous subsections we have shown that all elementary quantum circuits can be described in terms of Choi-Jamiołkowski operators and their link products. Here this approach is exploited to describe quantum networks, as a result of the composition of such elementary circuits. This is the approach outlined in Ref. [9].

1. Topology, causal ordering, and sequential ordering

A quantum network is obtained by assembling a number of elementary circuits, each of them represented by its Choi-Jamiołkowski operator. In the remainder of the paper we adopt the following convention, which appears to be very convenient for the description of quantum networks: if an elementary circuit is run more than once, i.e., at different steps of the computation, we attach to each different use a different label so that different uses of the same circuit are actually considered as different circuits.

To build up a particular quantum network one needs to have at disposal the whole list of elementary circuits and a list of instructions about how to connect them. In connecting circuits there are clearly two restrictions: (i) one can only connect the output of a circuit with the input of another circuit, and (ii) there cannot be cycles [10]. These restrictions ensure causality, namely, the fact that quantum information in the network flows from input to output without loops. This implies that the connections in the quantum network can be represented in a directed acyclic graph (DAG), where each vertex represents a quantum circuit, and each arrow represents a quantum system traveling from one circuit to another, as in Fig. 1(a). Notice that such a graph represents only the internal connections of the networks, while to have a complete graphical representation one should also append to the vertices a number of free incoming and outgoing arrows representing quantum systems that enter or exit the network. In other words, the graphical representation of a quantum network is provided by a DAG where some sources (vertices without incoming arrows) and some sinks (vertices without outgoing arrows) have been removed, as in Fig. 1(b). The free arrows remaining after removing a source represent input systems entering the network, while the free arrows remaining after removing a sink represent output systems exiting the network.

The flow of quantum information along the arrows of the graph induces a partial ordering of the vertices: we say that

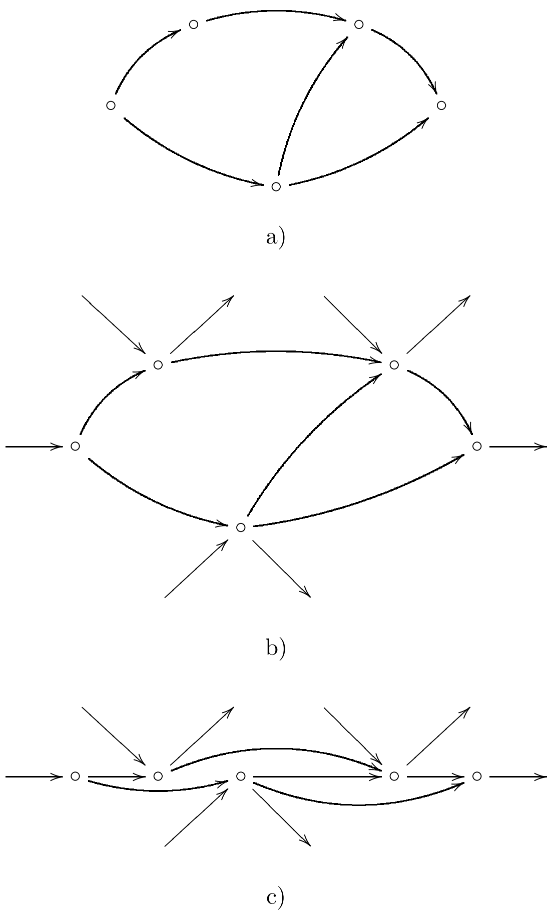

FIG. 1. (a) Graphical representation of internal connections in a quantum network: vertices represent quantum operations, incoming and outgoing arrows represent input and output systems. The resulting diagram is a direct acyclic graph. (b) Graphical representation of a quantum network: free incoming (outgoing) arrows have been added to the diagram in (a) in order to represent input (output) systems entering (exiting) the network. (c) Totally ordered quantum network. The vertices in diagram (b) have been ordered from left to right according to a sequential ordering compatible with the causal ordering fixed by input-output relations.

the circuit in vertex v_1 *causally precedes* the circuit in vertex $v_2 (v_1 \preccurlyeq v_2)$ if there is a directed path from v_1 to v_2. A well-known theorem in graph theory states that for a directed acyclic graph there always exists a way to extend the partial ordering $\preccurlyeq$ to a total ordering $\leqslant$ of the vertices. Intuitively speaking, the relation $\leqslant$ fixes a schedule for the order in which the circuits in the network can be run, compatibly with the causal ordering of input-output relations. In general, the total ordering $\leqslant$ is not uniquely determined by the partial ordering $\preccurlyeq$: the same quantum network can be used in different ways, corresponding to different orders in which the elementary circuits are run.

A quantum network with a given sequential ordering of the vertices becomes a compound quantum circuit, in which different operations are performed according to a precise schedule. Totally ordered quantum networks have a large number of applications in quantum information, and, accordingly, they have been given different names, depending on the context. For example, they are referred to as *quantum strategies* in quantum game theoretical and cryptographic applications [7]. Moreover, a totally ordered quantum network is equivalent to a sequence of channels with memory, as illustrated in Fig. 2(a). Currently, the most studied case in the literature on memory channels is that in which all channels of the sequence are identical, as represented in Fig. 2(b): here the memory must be first initialized in some fixed state $|0\rangle$, and eventually traced out. Clearly, the network in Fig. 2(b) is the particular case of that in Fig. 2(a) corresponding to $\mathcal{C}_0(\rho)=\mathcal{C}(\rho\otimes|0\rangle\langle 0|)$, $\mathcal{C}_2=\mathcal{C}_3=\mathcal{C}_{N-2}=\mathcal{C}$, and $\mathcal{C}_{N-1}(\rho)=\mathrm{Tr}_M[\mathcal{C}(\rho)]$, Tr_M being the partial trace over the memory system.

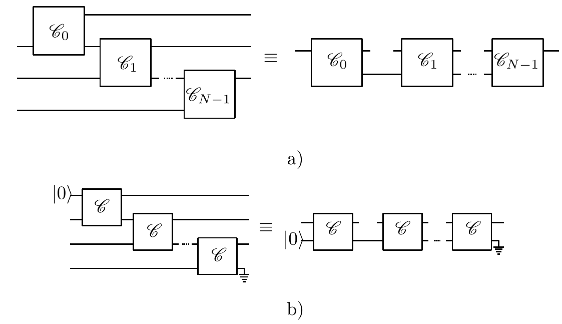

FIG. 2. (a) Equivalence between an arbitrary sequence of memory channels and a totally ordered quantum network: a sequence of quantum memory channels from Alice (left side) to Bob (right side) is equivalent modulo stretching and reshuffling of the quantum wires to an array of channels connected by internal ancillae, i.e., a totally ordered quantum network. (b) A sequence consisting of identical memory channels, with the memory initialized in state $|0\rangle$ before the first use, and traced out after the last use.

In the following we will be always interested in quantum networks equipped with a total ordering of the vertexes, and, accordingly, the expressions "quantum network," "quantum strategy," and "sequence of memory channels" will be used as synonymous.

2. Deterministic quantum networks

We start here by considering deterministic quantum networks, i.e., networks which do not produce random transformations. A deterministic quantum network is composed by deterministic quantum circuits, i.e., quantum channels. Let $\{\mathcal{C}_j | j\in V\}$ be the channels corresponding to the vertices of the graph, and $\{C_j | j\in V\}$ their Choi-Jamiołkowski operators.

Let us consider a network with a finite number of vertices $N=|V|<\infty$, and let us label the vertices with numbers from 0 to $N-1$, according to the sequential ordering of the network. The Hilbert spaces of each Choi-Jamiołkowski operator C_j are labeled by indices in the sets $A_j^-(A_j^+)$ of incoming (outgoing) arrows at vertex j, the elements in $A_j^-(A_j^+)$ corresponding to input (output) systems of the quantum channel $\mathcal{C}_j$. Let $A_j=A_j^-\cup A_j^+$ be the set of all arrows at vertex j, and let $\mathcal{H}_{A_j^-}$, $\mathcal{H}_{A_j^+}$, and $\mathcal{H}_{A_j}$ be the tensor products of all Hilbert spaces associated to the sets A_j^-, A_j^+, and A_j, respectively. Then, the normalization of the channel C_j reads

$$I_{A_j^+} * C_j = I_{A_j^-}, \tag{20}$$

which comes from Eq. (18).

Since $A_i \cap A_j \cap A_k = \emptyset$ for any $i,j,k=0,1,\ldots,N$, we can always define the link product $C_i * C_j * C_k$ (the link product is associative due to Theorem 2). Accordingly, we can define the *Choi-Jamiołkowski operator of the network* as

$$R^{(N)} = C_0 * C_1 * \cdots * C_{N-1} = \mathop{*}_{j\in V} C_j. \qquad (21)$$

Let us denote by $\mathcal{H}_{2j}$ and $\mathcal{H}_{2j+1}$ the Hilbert spaces of all free (i.e., not connected) input and output systems at vertex j, respectively. Since the Hilbert spaces of the connected systems are traced out in the link product, it is immediate to see that the Choi-Jamiołkowski operator of the network is an operator $R^{(N)}$ on $\otimes_{j=0}^{2N-1}\mathcal{H}_j$.

The normalization of the Choi-Jamiołkowski operator of the network is given by the following condition:

Lemma 4 (Normalization condition). Let $R^{(N+1)} \in \mathcal{L}(\otimes_{j=0}^{2N+1}\mathcal{H}_j)$ be the Choi-Jamiołkowski operator of a deterministic quantum network with $N+1$ vertices, ordered from 0 to N. Then, $R^{(N+1)}$ is positive semidefinite and satisfies the relation

$$I_{2N+1} * R^{(N+1)} = I_{2N} * R^{(N)}, \qquad (22)$$

where $R^{(N)} \in \mathcal{L}(\otimes_{j=0}^{2N-1}\mathcal{H}_j)$ is the Choi-Jamiołkowski operator of a network with N vertices ordered from 0 to $N-1$.

Notice that in terms of partial traces and tensor products the normalization of the Choi-Jamiołkowski operator $R^{(N+1)}$ can be equivalently written in the (less symmetric) form

$$\mathrm{Tr}_{2N+1}[R^{(N+1)}] = I_{2N} \otimes R^{(N)}. \qquad (23)$$

Proof. Denote by $\mathcal{H}_{\overline{2N}}$ the Hilbert spaces of all incoming internal connections at vertex N so that $\mathcal{H}_{A_N^-} = \mathcal{H}_{2N} \otimes \mathcal{H}_{\overline{2N}}$. We have $I_{2N+1} * R^{(N)} = C_0 * \cdots * C_{N-1} * (I_{2N+1} * C_N)$. Since N is the last vertex, all outgoing arrows are free, i.e., $\mathcal{H}_{A_N^+} = \mathcal{H}_{2N+1}$. Therefore the normalization of the channel $\mathcal{C}_N$ [Eq. (20)] gives $I_{2N+1} * C_N = I_{A_N^-} = I_{2N} \otimes I_{\overline{2N}} = I_{2N} * I_{\overline{2N}}$ [see Eq. (15) for the last equality]. We then obtain $I_{2N+1} * R^{(N)} = I_{2N} * C_0 * \cdots * C_{N-2} * C'_{N-1}$, where $C'_{N-1} = I_{\overline{2N}} * C_{N-1}$ is the Choi-Jamiołkowski operator of the channel $\mathcal{C}_{N-1}$ followed by the partial trace over the space $\mathcal{H}_{\overline{2N}}$. Clearly, $R^{(N-1)} = C_1 * \cdots * C_{N-2} * C'_{N-1}$ is the Choi-Jamiołkowski operator of a network with N vertices. ■

Iterating the above result we then have the following:

Corollary 1. Let $R^{(N)} \in \mathcal{L}(\otimes_{j=0}^{2N-1}\mathcal{H}_j)$ be the Choi-Jamiołkowski operator of a quantum network with N vertices. Then, $R^{(N)} \geqslant 0$ and the following relations hold:

$$\mathrm{Tr}_{2j-1}[R^{(j)}] = I_{2j-2} \otimes R^{(j-1)}, \quad 2 \leqslant j \leqslant N$$

$$\mathrm{Tr}_1[R^{(1)}] = I_0, \qquad (24)$$

each $R^{(j)}$ being a suitable positive operator on $\otimes_{k=0}^{2j-1}\mathcal{H}_k$.

We conclude the paragraph by noting that also the converse of Lemma 4 can be proved. The proof is essentially based on the same argument as in Ref. [11] (uniqueness of the minimal Stinespring dilation).

Theorem 3. Let $R^{(N)} \in \mathcal{L}(\otimes_{j=0}^{2N-1}\mathcal{H}_j)$ be a positive operator satisfying the relations

$$\mathrm{Tr}_{2j-1}[R^{(j)}] = I_{2j-2} \otimes R^{(j-1)}, \quad 2 \leqslant j \leqslant N$$

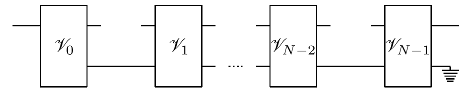

FIG. 3. Quantum network resulting from a concatenation of N (generally different) isometric channels $\mathcal{V}_j(\rho) \coloneqq V_j \rho V_j^\dagger$, with the last channel followed by partial trace over the ancillary degrees of freedom. Any positive operator satisfying Eq. (25) is the Choi-Jamiołkowski operator of a network of this form.

$$\mathrm{Tr}_1[R^{(1)}] = I_0. \qquad (25)$$

where $R^{(j)}, 1 \leqslant j \leqslant n-1$ are suitable positive operators. Then $R^{(N)}$ is the Choi-Jamiołkowski operator of a quantum network.

Proof. First, notice that each $R^{(j)}$ is the Choi-Jamiołkowski operator of a channel $\mathcal{R}^{(j)}$ from states on the even Hilbert spaces $\otimes_{k=0}^{j-1}\mathcal{H}_{2k}$ to states on the odd Hilbert spaces $\otimes_{k=0}^{j-1}\mathcal{H}_{2k+1}$. Indeed, Eq. (25) implies that

$$\mathrm{Tr}_{1,3,\ldots,2j-1}[R^{(j)}] = I_{0,2,\ldots,2j-2}, \qquad (26)$$

whence $\mathcal{R}^{(j)}$ is trace preserving due to Lemma 1. The problem is then to show that the multipartite channel $\mathcal{R}^{(N)}$ arises from the concatenation of N channels as in Fig. 3. In particular, we show that $\mathcal{R}^{(N)}$ can be obtained as a concatenation of N isometries. The proof is by induction. For $N=1$ the statement is equivalent to Stinespring's dilation of channels [4]: the Kraus operators of the channel $\mathcal{R}^{(1)}$ define an isometry $W^{(1)} = \sum_i |i\rangle_A \otimes K_i^{(1)}$, where $\{|i\rangle_A\}$ are orthonormal states for an ancilla A. As the induction hypothesis, we suppose now that the isometry $W^{(N)} \coloneqq \sum_i |i\rangle_A \otimes K_i^{(N)}$, defined by the canonical Kraus operators of $\mathcal{R}^{(N)}$, arises from the concatenation of N isometries, as in Fig. 3. Using such hypothesis, we then prove that also the isometry $W^{(N+1)} = \sum_i |i\rangle_B \otimes K_i^{(N+1)}$ is the concatenation of $N+1$ isometries as in Fig. 3. Indeed, using Eq. (3), it is immediate to see that the condition

$$\mathrm{Tr}_{2N+1}[R^{(N+1)}] = I_{2N} \otimes R^{(N)} \qquad (27)$$

implies that

$$\mathrm{Tr}_{2N+1}[\mathcal{R}^{(N+1)}(\rho)] = \mathcal{R}^{(N)}(\mathrm{Tr}_{2N}[\rho]), \qquad (28)$$

for any state ρ on $\otimes_{j=0}^{N}\mathcal{H}_{2j}$. Therefore $\{\langle m|K_i^{(N+1)}\}$ and $\{K_j^{(N)} \otimes \langle n|\}$ are two Kraus representations of the same channel, the latter being canonical, as $\mathrm{Tr}[K_i^{(N)\dagger}K_{i'}^{(N)} \otimes |n\rangle\langle n'|] = \delta_{nn'}\delta_{ii'}$. Since any Kraus representation is connected to the canonical one by the matrix elements of an isometry, we have

$$\langle m|K_i^{(N+1)} = \sum_{nj} V_{mi,nj} K_j^{(N)} \otimes \langle n|, \qquad (29)$$

or, equivalently,

$$K_i^{(N+1)} = {}_B\langle i|(V \otimes I_{2N+1,\ldots,1})(I_{2N} \otimes W^{(N)}), \qquad (30)$$

where $V = \sum_{mi,nj} V_{mi,nj}|m\rangle\langle n| \otimes_B |i\rangle\langle j|_A$ is an isometry from $\mathcal{H}_{2N} \otimes \mathcal{H}_A$ to $\mathcal{H}_{2N+1} \otimes \mathcal{H}_B$. Therefore we have

$$W^{(N+1)} = \sum_i |i\rangle_B \otimes K_i^{(N+1)},$$

$$=(V \otimes I_{2N+1,\ldots,1})(I_{2N} \otimes W^{(N)}). \tag{31}$$

Accordingly, the map $\mathcal{R}^{(N+1)}$ can be expressed as

$$\mathcal{R}^{(N+1)}(\rho) = \mathrm{Tr}_B[(V \otimes I)(I \otimes W^{(N)})\rho(I \otimes W^{(N)\dagger})(V^\dagger \otimes I)], \tag{32}$$

where V maps the $(2N)$th system and ancilla A to the $(2N+1)$th system and ancilla B. Along with the induction hypothesis, this proves the theorem. ∎

3. Network complexity

In theorem 3 we proved that quantum networks are in one to one correspondence with Choi-Jamiołkowski operators satisfying the conditions in Eq. (25). In particular, the proof involves the minimal Stinespring isometry of the channel $\mathcal{R}^{(N)}$ from states on $\mathcal{H}_{\mathrm{in}} \coloneqq \otimes_{k=0}^{j-1}\mathcal{H}_{2k}$ to states on $\mathcal{H}_{\mathrm{out}} \coloneqq \otimes_{k=0}^{j-1}\mathcal{H}_{2k+1}$. In Ref. [15], the expression of the minimal Stinespring isometry in terms of the Choi-Jamiołkowski operator was derived,

$$W^{(N)} = (I \otimes \sqrt{R^{(N)*}})|I\rangle\rangle_{\mathrm{out,out}} \otimes I_{\mathrm{in}}, \tag{33}$$

where the ancillary Hilbert space is isomorphic to a subspace of $\mathcal{H}_{\mathrm{out}} \otimes \mathcal{H}_{\mathrm{in}}$ with dimension equal to the rank of the Choi-Jamiołkowski operator $R^{(N)}$. Repeating the same argument for each $R^{(j)}$ with $1 \leqslant j \leqslant N-1$, one obtains an isometric extension of the channel $\mathcal{R}^{(N)}$ with N ancillae, each one appearing at a vertex $j-1$ and disappearing at the subsequent vertex j (apart from the last one, appearing at vertex $N-1$ and purifying the output system). The maximum $d_{\max} \coloneqq \max_{1\leqslant j \leqslant N}\mathrm{rank}(R^{(j)})$ denotes the maximum dimension of the ancilla required by the network described by $R^{(N)}$. Moreover, if one defines $r_j \coloneqq \mathrm{rank}(R^{(j)})\max\{d_{2j+1}, d_{2j+2}\}$, for $0 \leqslant j \leqslant N-2$, and $r_{N-1} \coloneqq \mathrm{rank}(R^{(N)})d_{2N-1}$, the number

$$r(R^{(N)}) \coloneqq \max_{0\leqslant j \leqslant N-1} r_j \tag{34}$$

is the maximal dimension that must be coherently controlled in order to implement the network. We can say that the quantity $d_{\max}$ describes the complexity of the network corresponding to the Choi operator $\mathcal{R}^{(N)}$ in terms of quantum memory needed, while $r(R^{(N)})$ describes the complexity of the network in terms of coherent control. However, $d_{\max}$ and $r(R^{(N)})$ provide only upper bounds on the actual memory and coherence control complexity. Indeed, in the Stinespring isometric extension coherence of ancillary systems is preserved up to the last step. However, it can often happen that some ancillary subsystem interacts with the systems only at vertex j. In this case, one could trace out such subsystem just after the interaction at vertex j.

On the other hand, the analysis of complexity in terms of number of elementary gates needed requires a detailed description of all the unitaries that one must use to implement the isometries $W^{(j)}$.

4. Probabilistic quantum networks

A probabilistic quantum network is a network in which the channels $\{\mathcal{C}_n | n \in V\}$ are replaced by quantum instruments $\{\mathcal{C}_{n,i_n} | n \in V\}$, where i_n is the label of the random transformation taking place at vertex n (in practical terms, the outcome of the nth measurement). Defining the set I of polyindices $i=(i_0, i_1, \ldots, i_N)$ corresponding to measurement outcomes, we have a family $\{R_i^{(N)}\}$ of Choi-Jamiołkowski operators of the probabilistic network, given by

$$R_i^{(N)} = C_{0,i_0} * \cdots * C_{N,i_N}, \quad i \in I. \tag{35}$$

Clearly, the sum of the operators $R_i^{(N)}$ over i gives the Choi-Jamiołkowski operator of a deterministic quantum network. Moreover, also the converse statement is true:

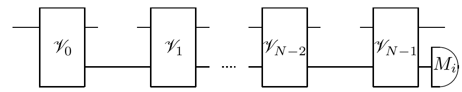

FIG. 4. Quantum network resulting from a concatenation of N isometric channels, with the last channel followed by a von Neumann measurement $\{M_i\}$ over the ancillary degrees of freedom. Any collection of positive operators $\{R_i^{(N)}\}$ summing up the Choi-Jamiołkowski operator of a deterministic quantum network describes a probabilistic quantum network of this form.

Theorem 4. Let $\{R_i^{(N)} \in \mathcal{L}(\otimes_{j=0}^{2N-1}\mathcal{H}_j) | i=1,\ldots,k\}$ be a collection of operators with the properties (i) $R_i^{(N)} \geqslant 0$ and $\sum_{i=1}^k R_i^{(N)} = R^{(N)}$, with $R^{(N)}$ satisfying the relations of Eq. (22). Then, each $R_i^{(N)}$ is the Choi-Jamiołkowski operator of the probabilistic quantum network, consisting of N isometric interactions, followed by a von Neumann measurement on a k-dimensional ancilla giving outcome i, as in Fig. 4.

Proof. Let us consider the following Choi-Jamiołkowski operator:

$$\tilde{R}^{(N)} \cdots \sum_{i \in \mathcal{I}} R_i^{(N)} \otimes |i\rangle\langle i|_A, \tag{36}$$

where $|i\rangle_A, i=1,\ldots,k$ is an orthonormal basis for an ancillary Hilbert space $\mathcal{H}_A$. Using Eq. (22) and Theorem 3, it is immediate to see that $R^{(N)}$ is the Choi-Jamiołkowski operator of a deterministic quantum network with N vertices, the last vertex having the output space $\tilde{\mathcal{H}}_{2N-1} \coloneqq \mathcal{H}_{2N-1} \otimes \mathcal{H}_A$. In particular, we know that $R^{(N)}$ can be realized by a sequence of isometric channels. Now apply the von Neumann measurement given by $\{M_i = |i\rangle\langle i|\}$ on the ancilla $\mathcal{H}_A$. Conditionally to outcome i, the Choi-Jamiołkowski operator of the network will be $\tilde{R}^{(N)} * M_i = \langle i|\tilde{R}^{(N)}|i\rangle_A = R_i^{(N)}$, where we used Eq. (16). ∎

5. Transformations achievable with a given quantum network

Given a network of quantum circuits, we can perform a number of different tasks. We can use the network as a programmable device, by feeding into it some quantum systems acting as the program, or we can connect some outputs with some inputs through a set of external circuits. Alternatively, we can make measurements on some outputs and decide accordingly which states to send to the next inputs, or we simply can use the network as a single multipartite channel. Any different use of a quantum network, however, will be always equivalent to the connection of the network with another quantum network, as in Fig. 5.

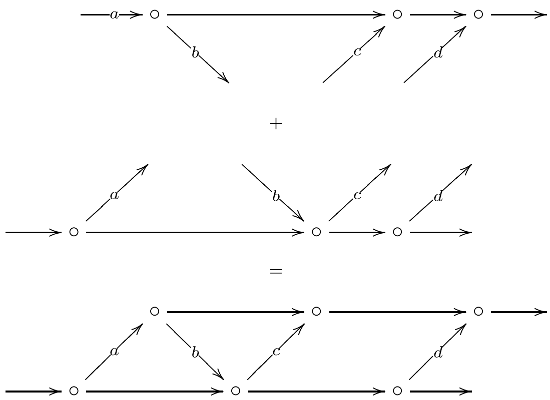

FIG. 5. The scheme represents the connection of two networks, in which junction of two arrows means identification of the corresponding quantum systems. The final network is still a direct acyclic graph, with the set of vertices coinciding with the union of sets of vertices of the component sub-networks, and with some free input and output arrows.

Connecting two networks with vertices V and W, respectively, means composing the corresponding graphs by joining some of the free outgoing arrows of a network with the free incoming arrows of the other, in such a way that the new graph is still a directed acyclic graph, with vertices $U = V \cup W$. Again, the requirement that the graph of connections in the composite network is acyclic is crucial in order to have a network where quantum information flows from input to outputs without loops. We adopt the convention that if two vertices $v \in V$ and $w \in W$ are connected by joining two arrows, the two quantum systems corresponding to such arrows are identified (see Fig. 5).

Let us proceed to determine the Choi-Jamiołkowski operator resulting from the composition of two networks, with $|V| = N$ and $|W| = M$ vertices, respectively, and with a given ordering of the vertices. Notice that, although the order of vertices within each network is fixed, *a priori* there is no relative ordering between vertices of one network and vertices in the other. However, once we fix a legitimate way of connecting the two networks we can also define a total ordering of the vertices which is compatible with the causal flow of quantum information in the composite network. In other words, we can order the vertices $U = V \cup W$ of the composite network by labeling them with numbers from 0 to $N + M - 1$. With this labeling, V and W become two disjoint partitions of the set $\{0, 1, \ldots, N+M-1\}$. We then have the following:

Corollary 2. Let R and S be the Choi-Jamiołkowski operators of two quantum networks. The Choi-Jamiołkowski operator of the network resulting from their composition (output of R fed into the input of S) is given by

$$T = S * R. \tag{37}$$

Proof. The proof is an immediate consequence of associativity of the link product. ∎

A possible way of transforming a given network is to connect it with another network containing state preparations and measurements, so that the resulting network has neither incoming nor outgoing quantum systems. In this case, any measurement outcome corresponds to a probabilistic transformation, which turns the input network into a probability. Corollary 2 shows that the probabilities in such an experiment will be given by the generalized Born rule

$$p(i|R) = R * S_i = \mathrm{Tr}[R S_i^T]. \tag{38}$$

This means that two networks with the same Choi-Jamiołkowski operator R are experimentally indistinguishable. More precisely, as long as one is not interested in the internal functioning of the network and is only concerned only with its input/output relations, two networks with the same Choi-Jamiołkowski operator are indistinguishable.

In conclusion, the action of a quantum network can be completely identified by its Choi-Jamiołkowski operator. Notice that, moreover, the Choi-Jamiołkowski operator provides a much simpler description of a quantum network than the list of all channels and all connections among them. Indeed, the operator R acts only on the Hilbert spaces of the quantum systems that actually enter and exit the network, and *not* of the quantum systems that are internal to the network.

As we will see in the following section, the Choi-Jamiołkowski operator of a quantum network coincides with the quantum comb, an abstract object that can be derived on a purely axiomatic basis.

IV. AXIOMATIC APPROACH: THE HIERARCHY OF ADMISSIBLE QUANTUM MAPS

While in the previous section we focused on the description of transformations that can be achieved by assembling elementary circuits into networks and by connecting networks with each other, in the following we take an axiomatic point of view, aimed to classify the transformations that are admissible *in principle* according to quantum mechanics. With "admissible transformations" we mean here general input-output transformations that (i) are compatible with the probabilistic structure of the theory, and (ii) produce a legitimate output when applied locally on one side of a bipartite input. Such transformations are defined recursively, by starting from channels and quantum operations, and progressively generating an infinite hierarchy of quantum maps. Despite the hierarchy of transformations being unbounded, we will show that a dramatic simplification arises in quantum mechanics: the inputs and outputs of every admissible transformation will turn out to be a concatenation of memory channels, and every admissible transformation will be itself realized by a suitable concatenation of memory channels. Notice that in this approach memory channels are not assumed from the beginning, but are derived on the basis of purely *a priori* considerations on the admissibility of quantum maps.

A. Quantum combs and admissible N-maps

Quantum channels and operations are the most general transformations of quantum states that satisfy the two minimal requirements of linearity and complete positivity (see, e.g., [16]). Linearity is required by the probabilistic structure

$$\begin{tikzcd} \mathscr{T} \arrow[r, "\tilde{\mathscr{S}}"] & \mathscr{T}' \arrow[d, "\mathfrak{C}"] \\ T \arrow[u, "\mathfrak{C}^{-1}"] \arrow[r, "\mathscr{S}"'] & T' \end{tikzcd}$$

FIG. 6. The commutative diagram shows the relation between a transformation $\tilde{\mathcal{S}}$ from linear maps $\mathcal{T}$ to linear maps $\mathcal{T}'$ and its conjugate $\mathcal{S} := \mathfrak{C} \circ \tilde{\mathcal{S}} \circ \mathfrak{C}^{-1}$ through the Choi-Jamiołkowski isomorphism, transforming Choi-Jamiołkowski operators T to Choi-Jamiołkowski operators T'.

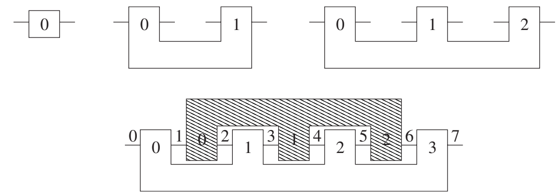

FIG. 7. In the first row we illustrate the diagrammatic representation of combs. A quantum system is represented by a line, a quantum operation (1-comb) by a box, a 2-comb by a diagram with two teeth, and a 3-comb by a diagram with three teeth. In the second row we represent the map corresponding to a 4-comb, transforming the input 3-comb in an output 1-comb.

of quantum mechanics. Indeed, if we apply the transformation $\mathcal{C}$ to the state $\rho = \Sigma_i p_i \rho_i$—corresponding to a random choice of the states $\{\rho_i\}$ with probabilities $\{p_i\}$—then the output state must be a random choice of the states $\{\mathcal{C}(\rho_i)\}$ with the same probabilities, i.e., $\mathcal{C}(\rho) = \Sigma_i p_i \mathcal{C}(\rho_i)$. For the same reason, we should also have $\mathcal{C}(p\rho) = p\mathcal{C}(\rho)$ for any $0 \leqslant p \leqslant 1$. These two conditions together imply that $\mathcal{C}$ can be extended without loss of generality to a linear map on $\mathcal{L}(\mathcal{H}_S)$, $\mathcal{H}_S$ being the system's Hilbert space. On the other hand, complete positivity is required if we want the transformation $\mathcal{C}$ to produce a legitimate output $\mathcal{C} \otimes \mathcal{I}_A(\rho_{SA})$ when acting locally on a bipartite input state ρ_{SA} on $\mathcal{H}_S \otimes \mathcal{H}_A$: in this case, this means that we want the output $\mathcal{C} \otimes \mathcal{I}_A(\rho_{SA})$ to be a positive matrix for any positive input ρ_{SA}. We now raise the level from states to channels, and ask which are the admissible transformations of channels. Again, the minimal requirements for an admissible transformation will be linearity and complete positivity. Linearity is motivated in the very same way as for transformations of states. Likewise, complete positivity is needed to ensure that the transformation can be applied locally on a bipartite channel. This investigation has been carried out in Ref. [17].

Let us consider maps $\tilde{\mathcal{S}}$ from linear maps $\mathcal{T}: \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ to linear maps $\mathcal{T}': \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_3)$. We say that $\tilde{\mathcal{S}}$ is admissible if (i) it is linear and if (ii) it preserves complete positivity, also when it is applied locally on a bipartite map $\mathcal{R}$. More explicitly, condition (ii) requires that if $\mathcal{R}$ from $\mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_2) \otimes \mathcal{L}(\mathcal{H}_B)$ is CP, then also $\mathcal{R}' = (\tilde{\mathcal{S}} \otimes \tilde{\mathcal{I}})(\mathcal{R})$ from $\mathcal{L}(\mathcal{H}_0) \otimes \mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_3) \otimes \mathcal{L}(\mathcal{H}_B)$ is CP. The admissibility properties can be mathematically characterized if we consider the conjugate map $\mathcal{S}$ of $\tilde{\mathcal{S}}$, defined as follows:

$$\mathcal{S} := \mathfrak{C} \circ \tilde{\mathcal{S}} \circ \mathfrak{C}^{-1}, \tag{39}$$

which transforms the Choi-Jamiołkowski operator T of $\mathcal{T}$ into the Choi-Jamiołkowski operator T' of the map $\mathcal{T}'$ (see Fig. 6). Linearity of $\tilde{\mathcal{S}}$ is equivalent to linearity of $\mathcal{S}$, while the second property for $\tilde{\mathcal{S}}$ is equivalent to complete positivity of $\mathcal{S}$. Since $\mathcal{S}$ is in one-to-one correspondence with $\tilde{\mathcal{S}}$, we associate the Choi-Jamiołkowski operator S of $\mathcal{S}$ to both of them. In the present section we will systematically use the map $\mathcal{S}$ instead of $\tilde{\mathcal{S}}$ for simplicity; however the whole construction that follows must be intended as dealing with transformations of transformations rather than with transformations of operators, thus, generating an infinite hierarchy of higher-rank quantum maps.

To tackle the characterization of all admissible quantum maps, we start by defining a particular family of maps along with their Choi-Jamiołkowski operators.

Definition 3. A quantum 1-comb on $(\mathcal{H}_0, \mathcal{H}_1)$ is the Choi-Jamiołkowski operator of a linear CP map from $\mathcal{L}(\mathcal{H}_0)$ to $\mathcal{L}(\mathcal{H}_1)$. For $N \geqslant 2$ a quantum N-comb on $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$ is the Choi-Jamiołkowski operator of an admissible N-map, i.e., a linear completely positive map transforming $(N-1)$-combs on $(\mathcal{H}_1, \ldots, \mathcal{H}_{2N-2})$ into 1-combs on $(\mathcal{H}_0, \mathcal{H}_{2N-1})$.

Definition 4. A deterministic 1-comb is the Choi-Jamiołkowski operator of a channel. A deterministic N-comb $S^{(N)}$ is the Choi-Jamiołkowski operator of a deterministic N-map, i.e., a map $\mathcal{S}^{(N)}$ that transforms deterministic $(N-1)$-combs into deterministic 1-combs. A probabilistic N-comb on $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$ is a positive operator $R^{(N)} \in \mathcal{L}(\otimes_{k=0}^{2N-1} \mathcal{H}_k)$ such that $R^{(N)} \leqslant S^{(N)}$ for some deterministic N-comb $S^{(N)}$ on $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$.

Definition 3 generates recursively an infinite family of maps. However, N-maps do not cover all possible maps one can define in quantum mechanics. Indeed, one might also consider maps from N-combs to M-combs, take their Choi-Jamiołkowski operators, define maps thereof, and so on, with an exponential growth of the tree of admissible quantum maps. However, we will prove in Sec. IV C that all admissible quantum maps can be reduced to N-maps.

Remark (labeling of Hilbert spaces). A quantum comb is defined as an operator acting on an ordered sequence of Hilbert spaces. Precisely, an N-comb is associated to an ordered sequence of $2N$ Hilbert spaces, which in Definition 3 are generically labeled as $\mathcal{H}_k$, $0 \leqslant k \leqslant 2N-1$ (in the following we will need to relabel spaces also with $1 \leqslant k \leqslant 2N$); we will then denote the set of deterministic N-combs on such N-tuple of spaces by $\mathrm{comb}(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$ [or by $\mathrm{comb}(\mathcal{H}_1, \ldots, \mathcal{H}_{2N})$].

The assignment of labels can be easily done by exploiting an intuitive diagrammatic representation of quantum combs. In Fig. 7 an N-comb is denoted by a comblike diagram with N teeth labeled by an ordered sequence of integers from left to right. Quantum systems are denoted by lines, and quantum operations (1-combs) by boxes. Each tooth $j(0\leqslant j\leqslant N-1)$ has an input (left) and output (right) system, which, apart from cases that will be specified, are canonically labeled $2j$ and $2j+1$, respectively.

To describe the action of an N-comb on an $(N-1)$-comb, the Hilbert spaces of the N-comb are labeled canonically as $\mathcal{H}_k$, $0\leqslant k\leqslant 2N-1$, while the spaces of the input $(N-1)$-comb are labeled as $\mathcal{H}_k$, $1\leqslant k\leqslant 2N-2$. The output is an element of $\mathrm{comb}(\mathcal{H}_0,\mathcal{H}_{2N-1})$, as in Definition 3.

In the following we characterize the convex set of quantum N-combs:

Theorem 5. A positive operator $S^{(N)}$ on $\otimes_{k=0}^{2N-1}\mathcal{H}_k$ is a deterministic N-comb if and only if the following identity holds:

$$\mathrm{Tr}_{2j-1}[S^{(j)}]=I_{2j-2}\otimes S^{(j-1)},\quad 2\leqslant j\leqslant N$$

$$\mathrm{Tr}_1[S^{(1)}]=I_0, \tag{40}$$

where $S^{(j)},1\leqslant j\leqslant N-1$ are deterministic j-combs.

Before proving the theorem, we introduce two lemmas that will make the proof simpler.

Lemma 5. The set of positive operators $R^{(N)}$ such that $R^{(N)}\leqslant S^{(N)}$ for some $S^{(N)}$ satisfying Eq. (40) generates the positive cone in $\mathcal{L}(\otimes_{k=0}^{2N-1}\mathcal{H}_k)$.

Proof. The operator $J^{(N)}:=I/(d_2\ldots d_{2k}\ldots d_{2N})$, where $d_k=\dim\mathcal{H}_k$, clearly satisfies Eq. (40). On the other hand, any positive operator $T^{(N)}$ on $\otimes_{k=1}^{2N}\mathcal{H}_k$, suitably rescaled, is smaller than $J^{(N)}$, whence it is proportional by a positive factor to a positive operator $R^{(N)}\leqslant J^{(N)}$. ■

Lemma 6. Consider two positive operators R_i^N, $i=1,2$, such that $R_i^{(N)}\leqslant S_i^{(N)}$ for some $S_i^{(N)}$ satisfying Eq. (40). If

$$\mathrm{Tr}_{2N-1}[R_1^{(N)}]=\mathrm{Tr}_{2N-1}[R_2^{(N)}], \tag{41}$$

then there exists $T^{(N)}\geqslant 0$ such that $O_i^{(N)}:=R_i^{(N)}+T^{(N)}$ satisfy Eq. (40) for $i=1,2$.

Proof. Since $R_1^{(N)}\leqslant S_1^{(N)}$ there exists $T^{(N)}\geqslant 0$ such that $O_1^{(N)}:=S_1^{(N)}=R_1^{(N)}+T^{(N)}$. Due to Eqs. (40) and (41) also the operator $O_2^{(N)}:=R_2^{(N)}+T^{(N)}$ satisfies Eq. (40). ■

Proof of theorem 5. The proof proceeds by induction. For $N=1$ the thesis is trivial: an operator $S^{(1)}\in\mathcal{L}(\mathcal{H}_0\otimes\mathcal{H}_1)$ is the Choi-Jamiołkowski operator of a channel from $\mathcal{L}(\mathcal{H}_0)$ to $\mathcal{L}(\mathcal{H}_1)$, if and only if $\mathrm{Tr}_1[S^{(1)}]=I_0$ [see Eq. (4)]. We now suppose the theorem holds for $1\leqslant M\leqslant N$, and show that it must hold also for $N+1$.

Sufficient condition. If $S^{(N+1)}$ is positive and satisfies Eq. (40), then it is a deterministic $(N+1)$-comb. Indeed, it is the Choi-Jamiołkowski operator of the CP map $\mathcal{S}^{(N+1)}$, from $\mathcal{L}(\otimes_{k=1}^{2N}\mathcal{H}_k)$ to $\mathcal{L}(\mathcal{H}_{2N+1}\otimes\mathcal{H}_0)$, defined by

$$\mathcal{S}^{(N+1)}(R^{(N)})=\mathrm{Tr}_{2N,\ldots,1}[(I_{2N+1}\otimes R^{(N)T}\otimes I_0)S^{(N+1)}]. \tag{42}$$

For deterministic $R^{(N)}$ the operator $\mathcal{S}^{(N+1)}(R^{(N)})$ is the Choi-Jamiołkowski operator of a channel, because $\mathcal{S}^{(N+1)}(R^{(N)})\geqslant 0$ and

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_{2N+1}}[\mathcal{S}^{(N+1)}(R^{(N)})]&=\mathrm{Tr}_{2N+1,\ldots,1}[(I_{2N+1}\otimes R^{(N)T}\otimes I_0)S^{(N+1)}]\\&=\mathrm{Tr}_{2N,\ldots,1}[(R^{(N)T}\otimes I_0)(I_{2N}\otimes S^{(N)})]\\&=\mathrm{Tr}_{2N-1,\ldots,1}[(I_{2N-1}\otimes R^{(N-1)T}\otimes I_0)S^{(N)}]\\&=I_0.\end{aligned} \tag{43}$$

The final equality is obtained considering that by the induction hypothesis $R^{(N-1)}$ is a deterministic $N-1$-comb, and by hypothesis $S^{(N)}$ is a deterministic N-comb.

Necessary condition. Let $S^{(N+1)}$ be an $(N+1)$-comb and $\mathcal{S}^{(N+1)}$ be the corresponding map, which transforms a deterministic N-comb $O^{(N)}\in\mathrm{comb}(\mathcal{H}_1,\ldots,\mathcal{H}_{2N})$ into a deterministic 1-comb $\mathcal{S}^{(N+1)}(O^{(N)})\in\mathrm{comb}(\mathcal{H}_0,\mathcal{H}_{2N+1})$. Then, consider a couple of probabilistic N-combs $R_1^{(N)}$, $R_2^{(N)}$ on $\otimes_{k=1}^{2N}\mathcal{H}_k$, such that

$$\mathrm{Tr}_{2N}[R_1^{(N)}]=\mathrm{Tr}_{2N}[R_2^{(N)}]. \tag{44}$$

Since $R_i^{(N)}$ is probabilistic, by Definition 4 there exists a deterministic N-comb $Q_i^{(N)}$ such that $R_i^{(N)}\leqslant Q_i^{(N)}$. By lemma 6 there exists $T^{(N)}$ such that $O_i^{(N)}:=R_i^{(N)}+T^{(N)}$ is deterministic for some $i=1,2$. Then we have

$$\mathrm{Tr}_{2N+1}[\mathcal{S}^{(N+1)}(O_1^{(N)})]=I_0=\mathrm{Tr}_{2N+1}[\mathcal{S}^{(N+1)}(O_2^{(N)})], \tag{45}$$

and consequently

$$\mathrm{Tr}_{2N+1}[\mathcal{S}^{(N+1)}(R_1^{(N)})]=\mathrm{Tr}_{2N+1}[\mathcal{S}^{(N+1)}(R_2^{(N)})]. \tag{46}$$

In particular, by taking $R_2^{(N)}=\sigma\otimes\mathrm{Tr}_{2N}[R_1^{(N)}]$ for some state σ on $\mathcal{H}_{2N}$, and using Eq. (3) one has

$$\mathrm{Tr}_{2N,\ldots,1}[(R_1^{(N)T}\otimes I_0)\mathrm{Tr}_{2N+1}[S^{(N+1)}]] \tag{47}$$

$$=\mathrm{Tr}_{2N,\ldots,1}[(R_1^{(N)T}\otimes I_0)(I_{2N}\otimes S^{(N)})], \tag{48}$$

where we defined

$$S^{(N)}:=\mathrm{Tr}_{2N+1,2N}[(I_{2N+1}\otimes\sigma^T\otimes I_{2N-1}\otimes\cdots\otimes I_0)S^{(N+1)}].$$

Comparing Eq. (47) with Eq. (48), and using the fact that probabilistic combs generate the cone of positive operators, we then obtain

$$\mathrm{Tr}_{2N+1}[S^{(N+1)}]=I_{2N}\otimes S^{(N)}. \tag{49}$$

To conclude the proof, we need to prove that $S^{(N)}$ is a deterministic N-comb. To this purpose, define the CP map $\mathcal{S}^{(N)}$ from operators on $\otimes_{k=1}^{2N-2}\mathcal{H}_k$ to operators on $\mathcal{H}_0\otimes\mathcal{H}_{2N-1}$ as

$$\mathcal{S}^{(N)}(R^{(N-1)}):=\mathrm{Tr}_{2N-2,\ldots,1}[(I_{2N-1}\otimes R^{(N-1)T}\otimes I_0)S^{(N)}]. \tag{50}$$

The map sends deterministic $N-1$-combs in deterministic 1-combs. Indeed, for any deterministic $N-1$-comb $R^{(N-1)}$ we have

$$\mathrm{Tr}_{2N-1}[\mathcal{S}^{(N)}(R^{(N-1)})]=\mathrm{Tr}_{2N-1,\ldots,1}[(I_{2N-1}\otimes R^{(N-1)T}\otimes I_0)S^{(N)}],$$

$$=\mathrm{Tr}_{2N+1,\ldots,1}[(I_{2N+1}\otimes\sigma^T\otimes I_{2N-1}\otimes R^{(N-1)T}\otimes I_0)S^{(N+1)}],$$

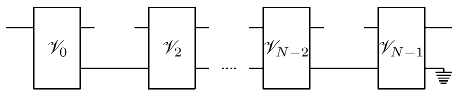

FIG. 8. Identification of a quantum N-comb with the Choi-Jamiołkowski operator of an N-partite memory channel. The teeth of the comb correspond to the isometries $\{\mathcal{V}_0,\ldots,\mathcal{V}_{N-1}\}$ in the memory channel.

$$=\mathrm{Tr}_{2N+1}[\mathcal{S}^{(N+1)}(\sigma\otimes I_{2N-1}\otimes R^{(N-1)})] \tag{51}$$

for any state σ on $\mathcal{H}_{2N}$. Using the induction hypothesis, we know that $\sigma\otimes I_{2N-1}\otimes R^{(N-1)}$ is a deterministic N-comb. Then the map $\mathcal{S}^{(N)}$ is deterministic, since

$$\mathrm{Tr}_{2N-1}[\mathcal{S}^{(N)}(R^{(N-1)})]=\mathrm{Tr}_{2N+1}[\mathcal{S}^{(N+1)}(\sigma\otimes I_{2N-1}\otimes R^{(N)})]=I_0. \tag{52}$$

This completes the proof. ■

The deterministic N-combs $S^{(N)}\in\mathrm{comb}(\mathcal{H}_0,\ldots,\mathcal{H}_{2N-1})$ form a convex set $\mathcal{K}_N$ which is the intersection of the cone of positive operators with the hyperplanes defined by Eq. (40). If we consider also the probabilistic combs, we have then the following:

Remark. The cone generated by probabilistic N-combs in $\mathcal{L}(\otimes_{k=0}^{2N-1}\mathcal{H}_k)$ is the whole cone of positive operators.

Essentially, the above result implies that the only relevant cone in quantum mechanics is the cone of positive operators. Another important consequence of Theorem 5 is the isomorphism between deterministic N-combs and Choi-Jamiołkowski operators of N-partite channels with memory:

Corollary 3. A deterministic N-comb is also the Choi-Jamiołkowski operator of an N-partite memory channel.

Proof. Immediate from Theorems 3 and 5. ■

The following theorem finally proves that any deterministic map in the hierarchy has a physical realization provided by a quantum memory channel. Notice that, as we mentioned at the beginning of the present section, the realization theorem regards the maps $\widetilde{\mathcal{S}}^{(N)}$ acting on $N-1$-maps $\widetilde{\mathcal{T}}^{(N+1)}$.

Theorem 6 (Realization of admissible N-maps). For all N, any deterministic N-map $\widetilde{\mathcal{S}}^{(N)}$ can be achieved by a physical scheme tallying with the memory channel corresponding to its deterministic N-comb $S^{(N)}$. Let $T^{(N-1)}$ be any $(N-1)$-comb in $\mathrm{comb}(\mathcal{H}_1,\ldots,\mathcal{H}_{2N-2})$. The transformation

$$\widetilde{\mathcal{S}}^{(N)}:\widetilde{\mathcal{T}}^{(N-1)}\mapsto\widetilde{\mathcal{T}}'^{(1)}=\widetilde{\mathcal{S}}^{(N)}(\widetilde{\mathcal{T}}^{(N-1)}) \tag{53}$$

can be achieved by connecting the two memory channels represented by $S^{(N)}$ and $T^{(N-1)}$ as in Fig. 9.

Proof. The statement is trivial for a deterministic 1-comb, which is a quantum channel. Now, by induction, suppose that the transformation $\widetilde{\mathcal{T}}^{(N-1)}$ corresponding to a deterministic $N-1$ comb $T^{(N-1)}$ is realized by the $N-1$-partite memory channel having Choi-Jamiołkowski operator $T^{(N-1)}$, as in Fig. 8. Let $W_0, i=1,\ldots,N-2$ be the Choi-Jamiołkowski operators of the n interactions occurring in the memory channel, then $T^{(N-1)}$ can be expressed as

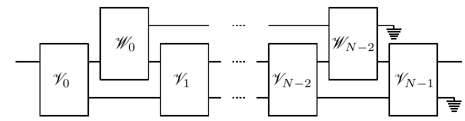

FIG. 9. Realization of admissible N-maps by connection of memory channels. The input of the map is an $(N-1)$-comb, corresponding to a sequence of $N-1$ isometric channels $\{\mathcal{W}_0,\ldots,\mathcal{W}_{N-2}\}$. The Choi-Jamiołkowski operator of the map is an N-comb, corresponding to a sequence of N isometric channels $\{\mathcal{V}_0,\ldots,\mathcal{V}_{N-1}\}$. The output of the map is obtained by connecting the free wires of the two memory channels.

$$T^{(N-1)}=\bar{W}_{N-2}*W_{N-1}*\cdots*W_0, \tag{54}$$

where the Choi-Jamiołkowski operator $\bar{X}$ denotes the interaction described by X with the final ancilla traced out. By Corollary 3 also $S^{(N)}$ is the Choi-Jamiołkowski operator of a memory channel, then $S^{(N)}$ can be expressed as

$$S^{(N)}=\bar{V}_{N-1}*V_{N-2}*\cdots*V_0, \tag{55}$$

for suitable isometries V_i, where the link connects all the spaces representing ancillae. The application of $\mathcal{S}^{(N)}=\mathfrak{C}\circ\widetilde{\mathcal{S}}^{(N)}\circ\mathfrak{C}^{-1}$ to $T^{(N-1)}=\mathfrak{C}(\widetilde{\mathcal{T}}^{(N-1)})$ provides

$$\mathcal{S}^{(N)}(T^{(N-1)})=S^{(N)}*T^{(N-1)},$$

$$=\bar{V}_{N-1}*\bar{W}_{N-2}*V_{N-2}*\cdots*W_0*V_0. \tag{56}$$

This proves that also the N-map $\widetilde{\mathcal{S}}^{(N)}$ can be physically realized by a scheme as in Fig. 8. Clearly, Eq. (56) prescribes that the action of $\widetilde{\mathcal{S}}^{(N)}$ on $\widetilde{\mathcal{T}}^{(N-1)}$ corresponds to connecting the two memory channels associated to $S^{(N)}$ and $T^{(N-1)}$ as in Fig. 9. ■

Remark (axiomatic approach to memory channels). It is worth noticing that in the present setting N-partite memory channels are derived from the recursive construction of admissible maps, rather than being assumed as a particular type of channels with additional causal structure. In this respect our approach differs with the axiomatization put forward by Kretschamnn and Werner in Ref. [11], where memory channels are derived by starting from the axiomatic definition of *causal automata*, i.e., multipartite quantum channels with the properties that (i) the output systems at former times are not influenced by input systems at later times and (ii) the action of the channel is invariant under time translations. In the present approach, instead, the quantum memory channel emerges in the Russian-dolls construction of maps on maps and the causal structure is generated by the map recursion.

In this respect, we would like to stress the interpretation of Eq. (25) as the mathematical translation of causal ordering. In technical terms, this equation reflects the semicausality property [12] for transformations occurring at teeth j and i, with $j<i$. This property is the mathematical translation of independence of the jth transformation from the ith transformation for $j<i$, namely, the fact that information can be transmitted from systems j to system $i>j$, while the converse is impossible.

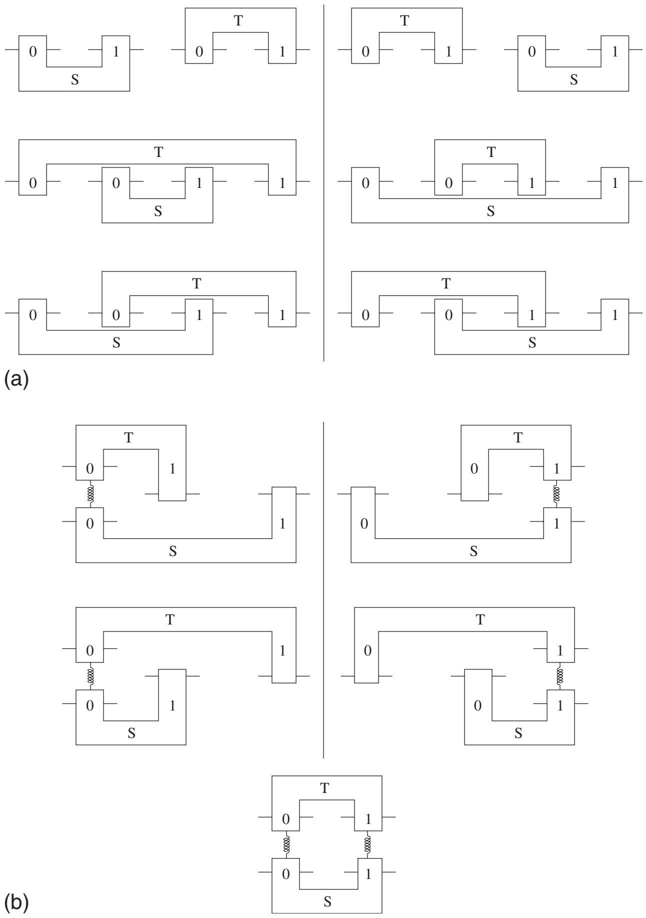

FIG. 10. Diagrammatic representation of all different quantum combs arising from the tensor product of a 2-comb $S^{(2)}$ with a 2-comb $T^{(2)}$.

B. Tensor product combs and separable combs

As defined in Sec. IV A, a quantum N-comb is a positive operator over a tensor product of Hilbert spaces labeled by elements of a totally ordered set. We now show how to combine two combs, say $S^{(N)} \in \mathrm{comb}(\mathcal{K}_0,\ldots,\mathcal{K}_{2N-1})$ and $T^{(M)} \in \mathrm{comb}(\mathcal{K}'_0,\ldots,\mathcal{K}'_{2M-1})$, in such a way to obtain a new comb whose teeth are the teeth of both $S^{(N)}$ and $T^{(M)}$, e.g., putting them in series, or in parallel, or in any other way as in Fig. 10(a). This corresponds to take the tensor product of the operators $S^{(N)}$ and $T^{(M)}$ and suitably reorder the Hilbert spaces of their teeth. Instead of counting the swap operators corresponding to such reordering, we will explicitly show how to construct the resulting comb space. We need to consider also situations as in Fig. 10(b), where two teeth, one from each comb, are identified in a single tooth. It follows that the general rule for the tensor product of combs is the following.

Let $S^{(N)} \in \mathrm{comb}(\mathcal{K}_0,\ldots,\mathcal{K}_{2N-1})$ and $T^{(M)} \in \mathrm{comb}(\mathcal{K}'_0,\ldots,\mathcal{K}'_{2M-1})$ be two quantum combs. Consider the following procedure:

(1) Merge the sets of teeth of both combs in a single ordered set, preserving the relative ordering of each subset.

(2) Consider the set $\mathcal{C}$ of all couples of neighboring teeth containing a tooth from each comb, and select a subset $\mathcal{S}\subseteq\mathcal{C}$ of pairwise disjoint couples, whose cardinality is denoted by $S:=|\mathcal{S}|$.

(3) Identify each couple in $\mathcal{S}$ in a single tooth (namely, the final tooth has input space given by the tensor product of the input spaces of the teeth in the couple, and similarly for the output space).

As a result, we obtain an ordered sequence of Hilbert spaces $(\mathcal{H}_0,\ldots,\mathcal{H}_{2L-1})$, with $L:=(N+M-S)$, which are the input and output spaces of the teeth defined and ordered trough the previous procedure.

Definition 5 Tensor product combs. A tensor product comb of $S^{(N)}$ and $T^{(M)}$ is the element of $\mathrm{comb}(\mathcal{H}_0,\ldots,\mathcal{H}_{2L-1})$ corresponding to the operator $S^{(N)}\otimes T^{(M)}$ with $(\mathcal{H}_0,\ldots,\mathcal{H}_{2L-1})$ defined through the previous procedure.

As a consequence, the tensor product of $S^{(N)}$ and $T^{(M)}$ is not unique, depending on the merging of teeth and on the choice of the set $\mathcal{S}$ of identified couples. As an example, in Fig. 10 we represent all possible tensor products of two combs in the case $N=M=2$.

Remark. We could have enclosed a more general situation in the definition of the tensor product of two combs, as follows. After dividing the couples in the set $\mathcal{S}$ into two sets $\mathcal{E}$ and $\mathcal{I}$, proceed as in step 3 of the procedure with $\mathcal{S}$ replaced by $\mathcal{E}$. As regards the remaining couples in $\mathcal{I}$, consider them to be *independent*. In this way, the final operator $S^{(N)}\otimes T^{(N)}$ is considered as an element of a subset $S_{\mathcal{I}}\subseteq \mathrm{comb}(\mathcal{H}_0,\ldots,\mathcal{H}_{2L-1})$, such that if one swaps couples in $\mathcal{I}$ the resulting operator is in $\mathrm{comb}(\mathcal{H}'_0,\ldots,\mathcal{H}'_{2L-1})$, where $(\mathcal{H}'_0,\ldots,\mathcal{H}'_{2L-1})$ is the corresponding reordering of $(\mathcal{H}_0,\ldots,\mathcal{H}_{2L-1})$. If $(i,j)\in\mathcal{I}$, then any comb in $S_{\mathcal{I}}$ satisfies the following identities:

$$\mathrm{Tr}_{2j+1}[R^{(j+1)}] = I_{2j}\otimes R'^{(j)},$$

$$\mathrm{Tr}_{2i+1}[R^{(j+1)}] = I_{2i}\otimes R''^{(j)}. \tag{57}$$

A very simple example of 2-comb in $S_{\mathcal{I}}$ with $\mathcal{I}=\{(0,1)\}$ is the comb of any convex combination of tensor product channels $R:=\Sigma_i p_i C^{(i)}_{32}\otimes D^{(i)}_{10}$, where p_i are probabilities, $\mathrm{Tr}_3[C^{(i)}]=I_2$ and $\mathrm{Tr}_1[D^{(i)}]=I_0$ for all i. Another important example of 2-comb in $S_{\mathcal{I}}$ with the same $\mathcal{I}$ as in the previous case is the following. Consider two channels, with input spaces $\mathcal{H}_0\otimes\mathcal{H}_A$ and $\mathcal{H}_2\otimes\mathcal{H}_B$, respectively. The output spaces are $\mathcal{H}_1$ and $\mathcal{H}_3$, respectively. If the channels are applied to a fixed state $|\Psi\rangle\rangle_{AB}\in\mathcal{H}_A\otimes\mathcal{H}_B$, then the resulting bipartite channel from $\mathcal{H}_0\otimes\mathcal{H}_2$ to $\mathcal{H}_1\otimes\mathcal{H}_3$ can be viewed as a 2-comb in both $\mathrm{comb}(\mathcal{H}_0,\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)$ and $\mathrm{comb}(\mathcal{H}_2,\mathcal{H}_3,\mathcal{H}_0,\mathcal{H}_1)$. One might think that all combs in $S_{\mathcal{I}}$, with $\mathcal{I}=\{(0,1)\}$, are achievable by two local channels—one from $\mathcal{H}_0\otimes\mathcal{H}_A$ to $\mathcal{H}_1$ and one from $\mathcal{H}_2\otimes\mathcal{H}_B$ to $\mathcal{H}_3$—applied to a bipartite, possibly entangled ancillary state on $\mathcal{H}_A\otimes\mathcal{H}_B$. However, there exist counterexamples to this conjecture, introduced in Refs. [13,14]. In particular, the explicit counterexample of Ref. [14] corresponds to the following comb in $\mathrm{comb}(\mathcal{H}_0,\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)$ with $\mathcal{H}_i\simeq\mathbb{C}^2$, that for sake of simplicity we write as an operator on $\mathcal{H}_1\otimes\mathcal{H}_3\otimes\mathcal{H}_0\otimes\mathcal{H}_2$,

$$R := \frac{1}{2}|I\rangle\rangle\langle\langle I|_{13} \otimes (I-P)_{02} + \frac{1}{2}|\sigma_x\rangle\rangle\langle\langle\sigma_x|_{13} \otimes P_{02}, \quad (58)$$

where $P=|1\rangle\langle 1|\otimes|1\rangle\langle 1|$. One can verify that $R\in S_{\mathcal{I}}$. However, any conceivable scheme for achieving the corresponding map requires at least one round of classical information, in addition to a shared entangled state.

C. Admissible (N,M)-maps and higher-order quantum maps

In this paragraph we give a definition of admissible quantum maps $\tilde{\mathcal{S}}^{(N,M)}$ from N-maps $\tilde{\mathcal{T}}^{(N)}$ to M-maps $\tilde{\mathcal{T}}'^{(M)}$, showing that the corresponding Choi-Jamiołkowski operators are quantum combs themselves. This will allow us to prove that the whole hierarchy of admissible quantum maps defined axiomatically can be realized in terms of quantum memory channels. While a reasonable definition of a map from N-maps to M-maps might seem to require only linearity and complete positivity, such a definition turns out to be inadequate. As we will see in the following, a consistent definition of admissible map involves an additional requirement, that is *compatibility with remote connections*. To introduce this requirement, and the correct definition, we first start from the definition involving only linearity and complete positivity, and show the need of this additional property. As in the previous subsection, we will focus attention on the conjugate maps $\mathcal{S}^{(N,M)} := \mathfrak{C}\circ\tilde{\mathcal{S}}^{(N,M)}\circ\mathfrak{C}^{-1}$, transforming N-combs into M-combs.

Definition 6. An (N,M)-map $\mathcal{S}^{(N,M)}$ is a linear completely positive map transforming $\mathrm{comb}(\mathcal{K}_0,\ldots,\mathcal{K}_{2N-1})$ into $\mathrm{comb}(\mathcal{K}'_0,\ldots,\mathcal{K}'_{2M-1})$.

Definition 7. An (N,M)-map $\mathcal{S}^{(N,M)}$ is deterministic if it sends deterministic N-combs to deterministic M-combs. An (N,M)-map $\mathcal{R}^{(N,M)}$ is probabilistic if its Choi-Jamiołkowski operator $R^{(N,M)}$ satisfies $R^{(N,M)}\leqslant S^{(N,M)}$ with $S^{(N,M)}$ the Choi-Jamiołkowski operator of some deterministic map $\mathcal{S}^{(N,M)}$.

We have then the following equivalence:

Lemma 7. Let $\mathcal{S}^{(N,M)}$ be a deterministic (N,M)-map. Then $\mathcal{S}^{(N,M)}$ is in one-to-one correspondence with a *CP*-map $\mathcal{S}^{(N\times M)}$ that transforms tensor product operators $R^{(N)}\otimes O^{(M-1)}$ of deterministic N- and $(M-1)$-combs into deterministic 1-combs.

Notice that the above statement does not involve tensor product combs, but only tensor product operators: in other words, there is no fixed total ordering of the Hilbert spaces on which the operator $R^{(N)}\otimes O^{(M-1)}$ acts.

Proof. Suppose that $\mathcal{S}^{(N,M)}$ maps an N-comb $R^{(N)}\in\mathrm{comb}(\mathcal{K}_0,\ldots,\mathcal{K}_{2N-1})$ to $R'^{(M)}=\mathcal{S}^{(N,M)}(R^{(N)})\in\mathrm{comb}(\mathcal{K}'_0,\ldots,\mathcal{K}'_{2M-1})$. In terms of Choi-Jamiołkowski operators, we have $R'^{(M)}=S^{(N,M)}*R^{(N)}$, where $S^{(N,M)}$ is the Choi-Jamiołkowski operator of $\mathcal{S}^{(N,M)}$. By definition, the output comb $R'^{(M)}$ will be in turn the Choi-Jamiołkowski operator of a map $\mathcal{R}'^{(M)}$ that transforms $\mathrm{comb}(\mathcal{K}'_1,\ldots,\mathcal{K}'_{2M-2})$ into $\mathrm{comb}(\mathcal{K}'_0,\mathcal{K}'_{2M-1})$ as follows:

$$\mathcal{R}'^{(M)}(O^{(M-1)}) = R'^{(M)} * O^{(M-1)},$$

$$= S^{(N,M)} * R^{(N)} * O^{(M-1)},$$

$$= S^{(N,M)} * (R^{(N)}\otimes O^{(M-1)}), \quad (59)$$

where the last equality exploits Eq. (15). Therefore, the map $\mathcal{S}^{(N,M)}$ induces a map sending tensor product operators $R^{(N)}\otimes O^{(M-1)}$ to 1-combs,

$$\mathcal{S}^{(N\times M)}(R^{(N)}\otimes O^{(M-1)}) := S^{(N,M)} * (R^{(N)}\otimes O^{(M-1)}). \quad (60)$$

Clearly, if $R^{(N)}$ and $O^{(M-1)}$ are deterministic then $\mathcal{S}^{(N\times M)}(R^{(N)}\otimes O^{(M-1)})$ is deterministic. Vice versa, given a CP-map $\mathcal{S}^{(N\times M)}$ with Choi-Jamiołkowski operator $S^{(N\times M)}$, we can define the map $\mathcal{S}^{(N,M)}$ as $\mathcal{S}^{(N,M)}(R^{(N)})=S^{(N\times M)}*R^{(N)}$. If $\mathcal{S}^{(N\times M)}$ sends products of deterministic combs into channels, then $\mathcal{S}^{(N,M)}$ is deterministic. ■

The (N,M)-maps defined in Definitions 6 and 7 are then identified with maps that transform tensor products of N- and $(M-1)$-combs into 1-combs. In other words, this means that if we have at disposal a device implementing an (N,M)-map $\tilde{\mathcal{S}}^{(N,M)}=\mathfrak{C}^{-1}\circ\mathcal{S}^{(N,M)}\circ\mathfrak{C}$ from transformations $\tilde{\mathcal{R}}^{(N)}$ to transformations $\tilde{\mathcal{R}}'^{(M)}$, we can use it to transform a pair of independent transformations $\tilde{\mathcal{R}}^{(N)}\otimes\tilde{\mathcal{O}}^{(M-1)}$ into a channel as follows:

$$\tilde{\mathcal{S}}^{(N\times M)}(\tilde{\mathcal{R}}^{(N)}\otimes\tilde{\mathcal{O}}^{(M-1)}) := [\tilde{\mathcal{S}}^{(N,M)}(\tilde{\mathcal{R}}^{(N)})](\tilde{\mathcal{O}}^{(M-1)}). \quad (61)$$

However, we want to be able to use this device also locally on transformations $\tilde{\mathcal{T}}^{(N)}\otimes\tilde{\mathcal{T}}'^{(M-1)}$ with multipartite input and output spaces, still producing a legitimate output. If the map $\tilde{\mathcal{S}}^{(N\times M)}$ can act locally on two multipartite maps, the conjugate map $\mathcal{S}^{(N\times M)}$ acts locally on the tensor product of two multipartite quantum combs $T^{(N)}\otimes T'^{(M-1)}$. Since the physical implementation of $T^{(N)}$ and $T'^{(M-1)}$ is provided by two memory channels, we must also admit that the two input networks can be remotely connected among themselves by some quantum memory.

Deciding which remote connections we assume to be possible is equivalent to fixing a prescription for the causal ordering of the Hilbert spaces in the tensor product, thus turning the tensor product operator $R^{(N)}\otimes O^{(M-1)}$ into a tensor product comb, in the sense of Definition 5. Moreover, the possibility of remote connections entails the need of replacing the tensor product comb $R^{(N)}\otimes O^{(M-1)}$—representing two independent quantum networks—with a general $(N+M-S-1)$-comb $R^{(N+M-S-1)}$—representing the compound network obtained by remote connections. Therefore, in order for the map $\mathcal{S}^{(N,M)}$ to represent a legitimate deterministic quantum device, it should induce a transformation of deterministic $(N+M-S-1)$-combs into channels. In other words, $\mathcal{S}^{(N\times M)}$ must be an admissible map on $(N+M-S-1)$-combs defined through the tensor product. This crucial property, however, is not guaranteed by Definitions 6 and 7.

As a consequence of the choice of one definition of tensor product, the map $\mathcal{S}^{(N\times M)}$ is then an admissible $N+M-S$-map $\mathcal{S}^{N+M-S}$ in the sense of Definition 3, with respect to the total ordering of the Hilbert spaces in the tensor product. The above discussion motivates the following definition:

Definition 8 [Admissible (N,M)-maps]. Let $(\mathcal{H}_1,\ldots\mathcal{H}_{2L})$ be a reordering of spaces $(\mathcal{K}_0,\ldots,\mathcal{K}_{2N-1})$ and $(\mathcal{K}'_0,\ldots,\mathcal{K}'_{2M-3})$ as in Definition 5, with $L=M+N-S-1$. An (N,M)-map $\mathcal{S}^{(N,M)}$ from $\mathrm{comb}(\mathcal{K}_0,\ldots,\mathcal{K}_{2N-1})$ to

$\mathrm{comb}(\mathcal{K}'_0,\ldots,\mathcal{K}'_{2M-1})$ is admissible if the associated map $\mathcal{S}^{(L)}$ is an admissible L-map, sending $\mathrm{comb}(\mathcal{H}_1,\ldots,\mathcal{H}_{2L})$ to $\mathrm{comb}(\mathcal{H}_0,\mathcal{H}_{2L+1})$.

Definition 9. An admissible (N,M)-map is deterministic (probabilistic) if the corresponding map $\mathcal{S}^{(N+M-S)}$ is deterministic (probabilistic).

As an immediate consequence of the definition, we then have the following identification:

Theorem 7. The Choi-Jamiołkowski operator of an admissible (N,M)-map is a quantum $(N+M-S-1)$-comb. The comb is deterministic if and only if the map is deterministic.

Proof. By definition the map $\mathcal{S}^{(N,M)}$ has the same Choi-Jamiołkowski operator as $\mathcal{S}^{(N+M-S-1)}$. The Choi-Jamiołkowski operator $S^{(N,M)}$ is then the Choi-Jamiołkowski operator of an admissible $(N+M-S)$-map, i.e., it is an $(N+M-S)$-comb. ■

One might continue now the recursive generation of quantum maps by defining admissible maps that transform admissible (N,M)-maps into admissible (K,L)-maps. However, it is now clear that—as long as independent teeth are excluded—such maps are in correspondence with $N+M+K+L$-combs. Similarly, further levels in the hierarchy of admissible maps are always admissible maps on combs, and hence combs themselves. In other words, the whole hierarchy of admissible quantum maps eventually collapses on N-maps $\widetilde{\mathcal{S}}^{(N)}$, corresponding to quantum combs $S^{(N)}$. The conclusion of the whole construction is the following property of universality of memory channels, holding if one neglects the possibility of tensor product combs with independent teeth

Theorem 8 (Universality of quantum memory channels). The Choi-Jamiołkowski operator of every deterministic admissible quantum map is a quantum comb $S^{(N+M)}$, and coincides with the Choi-Jamiołkowski operator of a suitable sequence of memory channels. Any deterministic admissible quantum map is realized by interconnection of the input sequence of memory channels, corresponding to the input comb $T^{(N)}$, with the sequence corresponding to $S^{(N+M)}$.

As an example, we show all possible schemes for admissible $(2,2)$-maps in Fig. 11.

Remark. If we allow for independent teeth, the (N,M)-map as a map on the tensor product $S^{(N)}\otimes T^{(M-1)}$ must be admissible in a new sense, which is less restrictive than Definition 8. Indeed, it must only map the set $S_{\mathcal{I}}$ to the set of channels. Such maps are difficult to characterize and in general they are not combs, as one can understand by the following example. We will analyze the most elementary case, namely, $S_{\mathcal{I}}\subseteq\mathrm{comb}(\mathcal{H}_1,\ldots,\mathcal{H}_4)$, with $\mathcal{I}=\{(0,1)\}$,

$$\mathrm{Tr}_4[R]=I_3\otimes C_{21},\quad \mathrm{Tr}_2[C]=I_1, \tag{62}$$

$$\mathrm{Tr}_2[R]=I_1\otimes D_{43},\quad \mathrm{Tr}_4[D]=I_3. \tag{63}$$

These combs can be interpreted both as combs on $\mathcal{H}_4\otimes\mathcal{H}_3\otimes\mathcal{H}_2\otimes\mathcal{H}_1$ and on $\mathcal{H}_2\otimes\mathcal{H}_1\otimes\mathcal{H}_4\otimes\mathcal{H}_3$, and the set of most general admissible maps transforming them into channels contains both 3-combs A on $\mathcal{H}_5\otimes\mathcal{H}_4\otimes\mathcal{H}_3\otimes\mathcal{H}_2\otimes\mathcal{H}_1\otimes\mathcal{H}_0$ and B on $\mathcal{H}_5\otimes\mathcal{H}_2\otimes\mathcal{H}_1\otimes\mathcal{H}_4\otimes\mathcal{H}_3\otimes\mathcal{H}_0$. Thus, the most general admissible maps on $S_{\mathcal{I}}$ include convex combinations of the kind $C=pA+(1-p)B$, which are not combs in any conceivable way, only satisfying $\mathrm{Tr}_{135}[C]=I_{024}$. Moreover, not all channels with the property $\mathrm{Tr}_{135}[C]=I_{024}$ do actually represent admissible maps on $S_{\mathcal{I}}$. We conclude the present paragraph with the following open questions:

(1) What is the most general realization scheme for elements of $S_{\mathcal{I}}$?

(2) How can we characterize and realize admissible maps on $S_{\mathcal{I}}$?

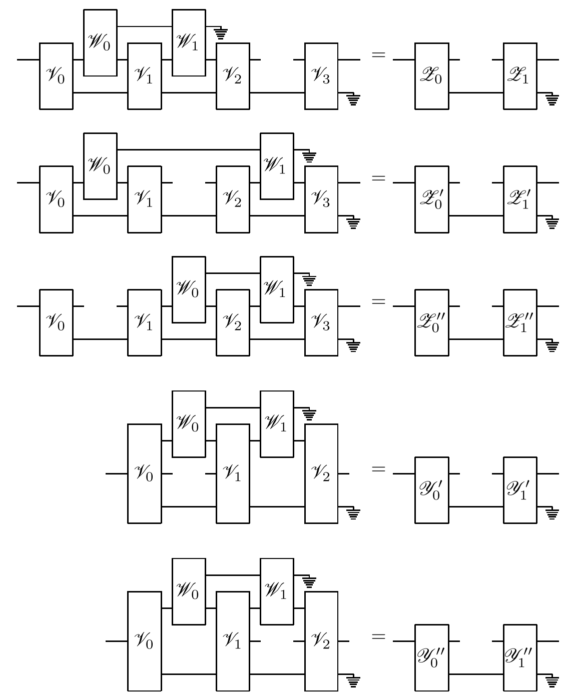

FIG. 11. The five possible realization schemes for admissible (2,2) maps, that transform a 2-comb in a 2-comb.

D. Generalized quantum instruments

Here we consider an analogue of quantum instruments that is suitable to treat a generalized measurement process where the measured object is a quantum network (described by its Choi-Jamiołkowski operator), rather than a quantum system (described by its state). Such a generalized instrument will associate to each measurement outcome the conditional Choi-Jamiołkowski operator of the quantum network. Notice that the number of input/output systems in the network can change in this generalized measurement process, so that in principle we should consider probabilistic transformations from networks with N inputs/outputs (described by N-combs) to arbitrary networks with M inputs/outputs (described by M-combs). However, since we proved in the previous paragraph that any admissible map from N-combs to M-combs is equivalent to an admissible map from $(N+M-1)$-combs to 1-combs, we can reduce without loss of generality the analysis of instruments to this simpler case.

Definition 10 Generalized N-instrument. An N-instrument $\mathcal{I}$ is a set of probabilistic N-combs $\{S_i^{(N)}\}$ such that $\Sigma_i S_i^{(N)}$ is a deterministic N-comb.

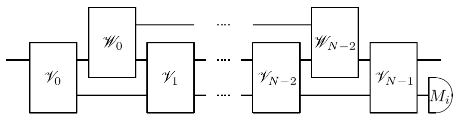

FIG. 12. Realization of an N-instrument as a sequence of N isometric channels $\{\mathcal{V}_0,\ldots,\mathcal{V}_{N-1}\}$ followed by a von Neumann measurement on the ancillary degrees of freedom. The input of the instrument is an $(N-1)$-comb, corresponding to a sequence of isometric channels $\{\mathcal{W}_0,\ldots,\mathcal{W}_{N-2}\}$. Conditionally to outcome i, the output of the instrument is a quantum operation, which represents the input-output transformation of the whole composite network.

For simplicity we have confined here our attention to the case of instruments with finite number of outcomes. The extension to the case of measurements with arbitrary outcome space Ω is obtained by defining the instrument as a *Choi-Jamiołkowski-operator valued measure* $S^{(N)}(B)$ [15], which associates to any event $B\subseteq\Omega$ a probabilistic quantum N-comb $S^{(N)}(B)$. The normalization of the measure amounts to the requirement that $S^{(N)}(\Omega)$ is a deterministic N-comb.

Theorem 9. For any probabilistic N-comb $R^{(N)}$ there exists an N-instrument $\mathcal{I}$ such that $R^{(N)}\in\mathcal{I}$.

Proof. By definition, there exists a deterministic N-comb $S^{(N)}$ such that $S^{(N)}\geqslant R^{(N)}$. Then, $\widetilde{R}^{(N)}:=S^{(N)}-R^{(N)}\geqslant 0$ and $S^{(N)}\geqslant\widetilde{R}^{(N)}$, then $\widetilde{R}^{(N)}$ is a probabilistic N-comb, and $\mathcal{I}:=\{R^{(N)},\widetilde{R}^{(N)}\}$ is a generalized N-instrument. ■

A quantum N-instrument $\{S_i^{(N)}\}$ can be used to define a family of probabilistic maps $\{\mathcal{S}_i^{(N)}\}$ from $(N-1)$-combs $R^{(N-1)}$ to Choi-Jamiołkowski operators of quantum operations $\{Q_i^{(1)}\}$, by means of

$$Q_i^{(1)}=\mathcal{S}_i^{(N)}(R^{(N-1)})=S_i^{(N)}*R^{(N-1)}. \tag{64}$$

This means that the quantum network with Choi-Jamiołkowski operator $R^{(N-1)}$ is randomly transformed in one of the quantum operations with Choi-Jamiołkowski operators $\{Q_i^{(1)}\}$. The realization of any generalized instrument is given by the following:

Theorem 10 (Realization of N -instruments). Let $\mathcal{I}=\{S_i^{(N)}|i=1,\ldots,k\}$ be an N-instrument, and let $R^{(N-1)}$ be an $(N-1)$-comb. The probabilistic transformations $\{\mathcal{S}_i^{(N)}\}$ given by

$$\mathcal{S}_i^{(N)}:R^{(N-1)}\mapsto\mathcal{S}_i^{(N)}(R^{(N-1)})=S_i^{(N)}*R^{(N-1)} \tag{65}$$

can be achieved by a physical scheme as in Fig. 12, involving isometric interactions of systems with quantum memories and a final von Neumann measurement on an ancilla with Hilbert space $\mathcal{H}_A$ of dimension $\dim\mathcal{H}_A=k$.

Proof. Consequence of Theorems 6 and 4.

E. Quantum testers and the generalized Born rule

Here we consider the particular case of admissible transformations of quantum networks in which input is a quantum N-comb and the output is just a probability. Such transformations are the analog of the customary POVMs describing measurements on quantum systems.

Definition 11 (Quantum tester). An N-tester is a set of positive operators $\{P_i|i=1,\ldots,k\}$ such that the quantities

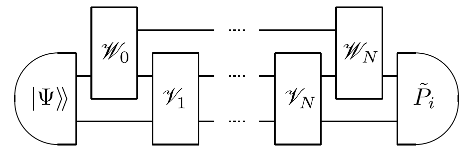

FIG. 13. Realization of a quantum N-tester as a probabilistic quantum network consisting of preparation of a pure input state $|\Psi\rangle\rangle$, isometric channels $\{\mathcal{V}_1,\ldots,\mathcal{V}_N\}$, and a final measurement with POVM $\{\widetilde{P}_i\}$. The memory channel corresponding to the sequence of isometric channels $\{\mathcal{W}_0,\ldots,\mathcal{W}_N\}$ is tested by connecting its wires with the wires of the tester and by running the resulting quantum circuit.

$$p(i|R):=\mathrm{Tr}[P_i^TR] \tag{66}$$

are probabilities for all deterministic N-combs R, i.e., $p(i|R)\geqslant 0$ and $\Sigma_i p(i|R)=1$.

Lemma 8. A set $\{P_i\}$ is an N-tester if and only if $\mathcal{I}=\{P_i\}$ is an $(N+1)$-instrument with $\dim\mathcal{H}_0=\dim\mathcal{H}_{2N+1}=1$.

Proof. The operator $T:=\Sigma_iP_i$ is a deterministic $(N+1)$-comb because it transforms any deterministic N-comb R to the c-number 1, which—regarded as a Choi-Jamiołkowski operator—represents the only deterministic channel in a one-dimensional Hilbert space ■

Since the tester is a particular case of generalized instrument, the normalization condition for the tester is given by Eq. (25), which in terms of T becomes

$$T=I_{2N+2}\otimes\Theta^{(N)},$$

$$\mathrm{Tr}_{2j+1}[\Theta^{(j)}]=I_{2j}\otimes\Theta^{(j-1)},\quad 1\leqslant j\leqslant n$$

$$\mathrm{Tr}_1[\Theta^{(0)}]=1. \tag{67}$$

The following corollary comes immediately from Theorems 10 and 8.

Theorem 11 (Realization of testers). Any N-tester $\{P_i|i=1,\ldots,k\}$ can be realized by an $(N+1)$-comb with $\dim\mathcal{H}_0=1$ and $\dim\mathcal{H}_{2N+1}=k$ and by a von Neumann measurement on $\mathcal{H}_{2N+1}$.

Proof. The scheme is the same as in Theorem 10, except the fact that the ancillary Hilbert space $\mathcal{H}_A$ is now named $\mathcal{H}_{2N+1}$. Since the space $\mathcal{H}_0$ is one dimensional, the first isometry V_1 is simply the preparation of an entangled state $|\Psi\rangle\rangle$.■

Notice that the probabilities $p(i|R)$ in generalized Born rule (66) arise as probabilities of outcomes in an experiment as in Fig. 13, where a pure entangled state $|\Psi\rangle\rangle$ is prepared, and is evolved through a sequence of interactions until the final measurement on $\mathcal{H}_{2N+1}$.

We now provide an alternative proof for the realization of testers that will be useful for later applications:

Theorem 12 (Realization scheme for testers). Let $\{P_i\}$ be an N-tester with $\Sigma_iP_i=T$. The tester can be split into a coherent part (state preparation and isometries) and a POVM, as in Fig. 13. The coherent part is described by a map $\mathcal{S}$ sending N-combs to quantum states according to

$$\mathcal{S}(R)=\sqrt{T}^TR\sqrt{T}^T. \tag{68}$$

The POVM $\{\widetilde{P}_i\}$ is given by

$$\widetilde{P}_i = \sqrt{T^{\ddagger}} P_i \sqrt{T^{\ddagger}} + Q_i, \tag{69}$$

where $T^{\ddagger}$ is the Moore-Penrose generalized inverse of T—i.e., $T^{\ddagger}T=TT^{\ddagger}=\Pi$, with Π the projector on the support of T—and $\{Q_i\}$ is any set of positive operators such that $\Sigma_i Q_i = I-\Pi$. The probabilities $p(i|R)=\mathrm{Tr}[P_i R]$ are given by

$$p(i|R) = \mathrm{Tr}[\widetilde{P}_i^T \mathcal{S}(R)]. \tag{70}$$

Proof. Clearly, the set $\{\sqrt{T^{\ddagger}}P_i\sqrt{T^{\ddagger}}\}$ is a POVM on the support of T, namely, $\Sigma_i \sqrt{T^{\ddagger}}P_i\sqrt{T^{\ddagger}}=\Pi$. One can consider a POVM $\{Q_i\}$ on the kernel of T with the same cardinality as $\{P_i\}$. It is now clear that the operators

$$\widetilde{P}_i \coloneqq \sqrt{T^{\ddagger}} P_i \sqrt{T^{\ddagger}} + Q_i \tag{71}$$

define a POVM. Notice that by definition of generalized inverse one has $P_i=\sqrt{T}\widetilde{P}_i\sqrt{T}$. The probabilities in generalized Born rule Eq. (66) are then obtained as follows:

$$p(i|R) = \mathrm{Tr}[\sqrt{T^T}\widetilde{P}_i^T\sqrt{T^T}R], \tag{72}$$

$$= \mathrm{Tr}[\widetilde{P}_i^T\sqrt{T^T}R\sqrt{T^T}], \tag{73}$$

$$= \mathrm{Tr}[\widetilde{P}_i^T \mathcal{S}(R)]. \tag{74}$$

Notice now that $\rho \coloneqq \sqrt{T^T}R\sqrt{T^T}$ is a state since $\mathrm{Tr}[\rho] = \mathrm{Tr}[T^T R]=1$ due to Eq. (67). The map $\mathcal{S}(R)=\sqrt{T^T}R\sqrt{T^T}$ is clearly completely positive, and transforms deterministic $(N-1)$-combs in states, which are Choi-Jamiołkowski operators of channels with one-dimensional input space $\mathcal{H}_0$. Hence $\mathcal{S}$ is an N-comb and can be realized by a sequence of isometries according to Theorem 6. The first isometry is necessarily a state preparation since $\mathcal{H}_0$ is one dimensional. ■

The special case of N-testers with $N=1$, corresponding to measurements on single channels, has been independently considered in Ref. [18], under the name process-POVM. In this case, the realization scheme of the previous theorem can be specialized to the following:

Corollary 4 (Realization scheme for 1-testers). Let $\{P_i\}$ be a 1-tester and C be the Choi-Jamiołkowski operator of a channel. The normalization condition of the 1-tester is

$$\sum_i P_i = I \otimes \sigma, \tag{75}$$

where σ is a state. The probabilities $p(i|C)=\mathrm{Tr}[P_i^T C]$ can be obtained by preparing a purification of the state σ, evolving it through the channel $\mathcal{C}\otimes\mathcal{I}$, and finally performing a measurement with POVM $\{\widetilde{P}_i=(I\otimes\sqrt{\sigma^{\ddagger}})P_i(I\otimes\sqrt{\sigma^{\ddagger}})+Q_i\}$, with Q_i as in Theorem 12.

Proof. The normalization $T=I\otimes\sigma$ follows immediately from Eq. (67). According to Theorem 12, the tester can be split in a coherent part and a POVM, with the coherent part producing the state ρ given by

$$\rho = \mathcal{S}(C),$$

$$= \sqrt{T^T}C\sqrt{T^T},$$

$$= (I\otimes\sqrt{\sigma^T})\mathcal{C}\otimes\mathcal{I}(|I\rangle\rangle\langle\langle I|)(I\otimes\sqrt{\sigma^T}),$$

$$= \mathcal{C}\otimes\mathcal{I}(|\sqrt{\sigma}\rangle\rangle\langle\langle\sqrt{\sigma}|). \tag{76}$$

The last expression represents exactly the action of the channel $\mathcal{C}\otimes\mathcal{I}$ on the purification $|\sqrt{\sigma}\rangle\rangle$. ■

It is worth noting the peculiarity of the case of 1-testers, where the coherent part is simply achieved by preparing an entangled state on which the variable channel $\mathcal{C}$ is applied. Typically, this is not the case for $N>1$, as the general realization scheme given by Fig. 13 also contains the isometries $\{V_i i+1,\ldots,N-1\}$. Such isometries generally play a crucial role, as they allow to exploit memory effects that are extremely relevant when the measured channel $\mathcal{C}$ is an N-partite memory channel [19]. As we will see in the following, N-testers are the proper tool to treat the discrimination of two memory channels, and to introduce a notion of distance between memory channels that is related to statistical distinguishability.

V. APPLICATION TO DISCRIMINATION AND TOMOGRAPHY OF QUANTUM NETWORKS

A. Distance and distinguishability

According to generalized Born rule (38), two quantum networks with the same quantum comb are experimentally indistinguishable. More generally, we are now in position to give a notion of distance that captures the distinguishability of quantum transformations.

Consider the problem of discriminating two N-partite memory channels, described by the quantum N-combs R_0 and R_1, respectively. In view of the discussion of the previous paragraphs, this is enough to study the discrimination of all admissible transformations in quantum mechanics. For simplicity, we discuss here the problem of minimum error discrimination, in which the two memory channels are given with prior probabilities π_0 and π_1. Since the most general transformation sending an N-comb in a set of classical probabilities is given by an N-tester, any discrimination experiment will be described by an N-tester $\{P_0,P_1\}$ with $P_0+P_1 = T$ as in Eq. (67). The average probability of error is then given by

$$p_e = \pi_0\mathrm{Tr}[P_1^T R_0] + \pi_1\mathrm{Tr}[P_0^T R_1], \tag{77}$$

$$= \pi_0\mathrm{Tr}[\widetilde{P}_1^T\mathcal{S}(R_0)] + \pi_1\mathrm{Tr}[\widetilde{P}_0^T\mathcal{S}(R_1)], \tag{78}$$

$$= \pi_0 - \mathrm{Tr}\{[\pi_0\mathcal{S}(R_0) - \pi_1\mathcal{S}(R_1)]\widetilde{P}_0^T\}, \tag{79}$$

where the map $\mathcal{S}$ and the POVM $\{\widetilde{P}_0,\widetilde{P}_1\}$ are as in Theorem 12. The discrimination of the two memory channels R_0 and R_1 is then reduced to the discrimination of the states $\mathcal{S}(R_0)$ and $\mathcal{S}(R_1)$. Using Helstrom optimal measurement [20] we get the bound

$$p_e \geqslant \frac{1-\|\pi_0\mathcal{S}(R_0)-\pi_1\mathcal{S}(R_1)\|_1}{2}, \tag{80}$$

where $\|A\|_1 \coloneqq \mathrm{Tr}|A|$ is the usual trace-norm. Recalling that $\mathcal{S}(R)=\sqrt{T^T}R\sqrt{T^T}$, and optimizing over T finally gives following bound:

$$p_e \geq \frac{1 - \max_T \|\sqrt{T^T}(\pi_0 R_0 - \pi_1 R_1)\sqrt{T^T}\|_1}{2}, \quad (81)$$

where the maximum is taken over all operators $T \geq 0$ satisfying the constraints in Eq. (67). The bound is achievable, namely, once we have the optimal operator T_{opt}, and the Helstrom POVM $\{\tilde{P}_0, \tilde{P}_1\}$ for the minimum error discrimination of the states $\rho_i = \sqrt{T_{\rm opt}}^T R_i \sqrt{T_{opt}}^T$ we can define the optimal tester $\{P_i\}$ by $P_i = \sqrt{T_{opt}} \tilde{P}_i \sqrt{T_{opt}}$. Theorem 12 then ensures that there is a suitable scheme realizing the optimal tester.

The above discussion motivates the following definition:

Definition 12. (Distance between quantum combs). Let R_0 and R_1 be two N-combs. The distance between R_0 and R_1 is given by

$$d(R_0, R_1) = \frac{1}{2} \max_T \|\sqrt{T^T}(R_0 - R_1)\sqrt{T^T}\|_1, \quad (82)$$

where T is a positive semidefinite operator that satisfies Eq. (67).

This definition provides the suitable notion of distance between two memory channels. This distance generalizes the notion of distance based on the cb-norm [21] (alternatively called *diamond norm* [22]), which is typically used for quantum channels and quantum operations. The cb-norm distance of two quantum operations $\mathcal{O}_0$ and $\mathcal{O}_1$ from states on $\mathcal{H}$ to states on $\mathcal{H}$ is given by

$$d_{cb}(\mathcal{O}_0, \mathcal{O}_1) = \frac{1}{2} \sup_n \sup_\rho \|[(\mathcal{O}_0 - \mathcal{O}_1) \otimes \mathcal{I}_n](\rho)\|_1, \quad (83)$$

where $\mathcal{I}_n$ is the identity map on $\mathcal{L}(\mathbb{C}^n)$, and σ is a state on the extended Hilbert space $\mathcal{H} \otimes \mathbb{C}^n$. Using convexity of the trace distance and the finite dimensionality of the input space $\mathcal{H}$, the above expression can be rewritten as [23]

$$d_{cb}(\mathcal{O}_0, \mathcal{O}_1) = \frac{1}{2} \max_\sigma \|(I_{\mathcal{H}} \otimes \sqrt{\sigma^T})\Delta(I_{\mathcal{H}} \otimes \sqrt{\sigma^T})\|_1, \quad (84)$$

where σ is a state on $\mathcal{H}$, $\Delta := O_0 - O_1$, and O_0, O_1 are the Choi-Jamiołkowski operators of the quantum operations $\mathcal{O}_0, \mathcal{O}_1$, respectively. Recalling that the Choi-Jamiołkowski operator of a quantum operation is a quantum comb with $N=1$, and that for $N=1$ Eq. (67) gives $T = I \otimes \sigma$, we obtain

$$d_{cb}(\mathcal{O}_0, \mathcal{O}_1) = d(O_0, O_1) \quad \text{for } N = 1, \quad (85)$$

namely, for $N=1$ the cb-norm distance is a special case of distance between two quantum combs.

Note that for N-partite memory channels $\mathcal{C}_0$ and $\mathcal{C}_1$ with Choi-Jamiołkowski operators C_0 and C_1, respectively, the operational distance introduced here is typically larger than the cb-norm distance, i.e.,

$$d(C_0, C_1) \geq d_{cb}(\mathcal{C}_0, \mathcal{C}_1). \quad (86)$$

Indeed, Eq. (82) involves maximization over all operators $T \geq 0$ satisfying constraints (67), while Eq. (84) involves maximization over operators of the special form $T = I_{\mathcal{H}} \otimes \sigma$, where now $\mathcal{H} = \otimes_{k=0}^{N-1} \mathcal{H}_{2k}$ and σ is a state on $\mathcal{H} = \otimes_{k=0}^{N-1} \mathcal{H}_{2k+1}$. The fact that for $nN1$ our distance can be strictly larger than the cb-norm distance is due to the fact that the cb-norm distance is related to discrimination in parallel schemes where the unknown channel is applied to a large entangled state on $\mathcal{H}^{\otimes 2}$ and a collective measurement is finally performed on the resulting state on $\mathcal{H} \otimes \mathcal{H}$, while the distinguishability of two memory channels can be enhanced by using sequential schemes as in Fig. 13.

B. Informationally complete testers

In the present section we introduce *informationally complete* testers, namely, testers $\{P_i\}$ such that the probabilities $p(i|R) := \mathrm{Tr}[P_i^T R]$ is sufficient to completely characterize the (generally probabilistic) quantum comb R on $\otimes_{k=1}^{2n} \mathcal{H}_k$. These testers are particularly important for network tomography, in the very same way as informationally complete POVMs describe possible tomographic experiments for quantum states [24]. Exploiting such testers in Ref. [25] tomography of quantum channels and operations has been optimized. More precisely, the probabilities $p(i|R)$ is sufficient for the reconstruction of R, if $p(i|R)$ allows to evaluate $\mathrm{Tr}[TR]$ for all $T \in \mathcal{L}(\otimes_{k=1}^{2n} \mathcal{H}_k)$ as follows:

$$\mathrm{Tr}[TR] = \sum_i t_i \mathrm{Tr}[P_i^T R] = \sum_i t_i p(i|R). \quad (87)$$

From this condition the following definition comes straightforwardly:

Definition 13 (Informationally complete tester). The tester $\{P_i\}$ is informationally complete if and only if for all $T \in \mathcal{L}(\otimes_{k=1}^{2N} \mathcal{H}_k)$ there exist coefficients t_i such that $T = \sum_i t_i P_i^T$.

It is clear that this definition is an equivalent restatement of the condition in Eq. (87). With the following theorem we prove that informationally complete testers actually exist.

Theorem 13. For $\{\tilde{P}_i\}$ informationally complete POVM, the tester with elements $P_i = \frac{1}{d_1 \ldots d_{2n-1}} \tilde{P}_i$ is informationally complete.

Proof. If the POVM $\{\tilde{P}_i\}$ is informationally complete, then for all operators $T \in \mathcal{L}(\otimes_{k=1}^{2n} \mathcal{H}_k)$ one has

$$T = \sum_i t_i \tilde{P}_i^T. \quad (88)$$

It is straightforward to verify that the coefficients $\tilde{t}_i := d_1 \ldots d_{2n-1} t_i$ expand T on P_i. Moreover, $\{P_i\}$ is a tester, since

$$\sum_i P_i = \frac{I}{d_1 \ldots d_{2n-1}}, \quad (89)$$

which clearly satisfies the conditions in Eq. (67). ■

In the following we will prove some theorems that will help characterizing informationally complete testers.

Theorem 14. The operator $\Theta^{(N)}$ in Eq. (67) providing the normalization of an informationally complete tester is invertible.

Proof. Suppose that $\Theta^{(N)}$ is not invertible. Then the support of P_i is contained in the support of $I \otimes \Theta^{(N)}$. It is then impossible that $\{P_i^T\}$ spans operators on the kernel of $I \otimes \Theta^{(N)}$. ■

Theorem 15. A tester $\{P_i\}$ is informationally complete iff it can be written as $P_i=(I\otimes\sqrt{\Theta^{(N)}})\tilde{P}_i(I\otimes\sqrt{\Theta^{(N)}})$, with $\{\tilde{P}_i\}$ informationally complete POVM and $\Theta^{(N)}$ invertible and satisfying identities (67).

Proof. Let us first suppose that $\{P_i\}$ is informationally complete. Then $\Sigma_i P_i=I\otimes\Theta^{(N)}$ is invertible, and since for all T one has $T=\Sigma_i t_i P_i^T$ one also has

$$(I\otimes\sqrt{\Theta^{(N)T}})T(I\otimes\sqrt{\Theta^{(N)T}})=\sum_i \tilde{t}_i P_i^T. \tag{90}$$

If we now consider the POVM $\tilde{P}_i:=(I\otimes\sqrt{\Theta^{(N)-1}})P_i(I\otimes\sqrt{\Theta^{(N)-1}})$, we have clearly

$$T=\sum_i \tilde{t}_i\tilde{P}_i^T, \tag{91}$$

where the coefficients $\tilde{t}_i$ are the ones in Eq. (90). The set $\{P_i\}$ is then an informationally complete POVM. On the other hand, if $\Theta^{(N)}$ is invertible and satisfies Eq. (67) and $\{\tilde{P}_i\}$ is an informationally complete POVM, clearly $\tilde{P}_i:=(I\otimes\sqrt{\Theta^{(N)}})P_i(I\otimes\sqrt{\Theta^{(N)}})$ is a tester. We can easily prove that it is informationally complete by considering that since $(I\otimes\sqrt{\Theta^{(N)-1T}})T(I\otimes\sqrt{\Theta^{(N)-1T}})=\Sigma_i t_i P_i^T$ for all T, one has also $T=\Sigma_i t_i\tilde{P}_i^T$. ■

In a completely analogous way, we can define informationally complete testers for deterministic combs, which instead of separating the whole $\mathcal{L}(\otimes_{k=1}^{2n}\mathcal{H}_k)$ separate only the subspace $\mathcal{D}$ spanned by deterministic combs. Notice that the set $\mathcal{D}$ is given by $\mathcal{D}=\{X|\mathrm{Tr}_{2N-1}[X]=I\otimes Y\}$. The definition is then the following:

Definition 14. The tester $\{P_i\}$ is informationally complete for deterministic testers if and only if for all $T\in\mathcal{D}$ there exist coefficients t_i such that $T=\Sigma_i t_i P_i^T$.

Notice that this definition requires that the linear span of $\{P_i\}$ contains $\mathcal{D}$ as a subspace. The existence theorem—analogous of Theorem 13—is trivial since any informationally complete tester is also informationally complete for deterministic combs. On the other hand, characterization theorems can be stated, but they are beyond the scope of the present paper.

VI. MULTIPLE-TIME STATES AND MEASUREMENTS

In this section we want to show that quantum combs and generalized instruments allow to treat in a unified and simple framework the objects introduced in Ref. [26] under the definitions of *multiple-time states* and *multiple-time measurements*. Multiple-time states correspond to preparation of a state $|\Psi_0\rangle$ at time t_0 and subsequent postselection by measurements containing the Kraus operators $|\Psi_i\rangle\langle\Phi_i|$ at times t_i, with $i=1,\ldots,N-1$, and finally postselection by a *bra* $\langle\Phi_N|$ at time t_N. The corresponding probabilistic quantum comb is the following:

$$S=\overset{N}{\underset{j=0}{\otimes}}S_j,$$

$$S_N=|\Phi_N^*\rangle\langle\Phi_N^*|,\quad S_0=|\Psi_0\rangle\langle\Psi_0|,$$

$$S_i=|\Psi_i\rangle\langle\Psi_i|\otimes|\Phi_i^*\rangle\langle\Phi_i^*|,\quad 1\leq i\leq N-1. \tag{92}$$

A multiple-time measurement is just a quantum operation with multipartite Kraus operators $K_j^{(i)}$ for outcome i, such that the probability of occurrence of the outcome i for a multiple-time state is provided by the expression

$$p(i|S)=\frac{\sum_j|\langle\Phi_1|\ldots\langle\Phi_N|K_j^{(i)}|\Psi_0\rangle\ldots|\Psi_{N-1}\rangle|^2}{\sum_{lj}|\langle\Phi_1|\ldots\langle\Phi_N|K_j^{(l)}|\Psi_0\rangle\ldots|\Psi_{N-1}\rangle|^2}. \tag{93}$$

In our formalism, a multiple-time measurement is described by a generalized instrument $\{R_i\}$, with $R_i=\Sigma_j|K_j\rangle\rangle\langle\langle K_j|$, providing probabilities for different outcomes on multiple-time states by the generalized Born rule

$$p(i|S)=\frac{\mathrm{Tr}[SR_i^T]}{\sum_j\mathrm{Tr}[SR_j^T]}. \tag{94}$$

What the authors call an *history* is the outcome i of the generalized instrument. We want to stress that the approach to multiple time states and measurements based on quantum combs provides a simple answer to the following three fundamental questions left open by Ref. [26]:

(1) *Given a Kraus operator, can we always find some multiple-time measurement such that this operator represents a particular outcome of the measurement?* The answer is clearly yes, since by Corollary IVA, any positive operator, and in particular a rank one $|K\rangle\rangle\langle\langle K|$, suitably rescaled by a positive factor, provides a probabilistic comb, which in turn by Theorem 9 can be included in a generalized instrument.

(2) *What are the conditions that a set of "histories" must satisfy in order to describe a measurement?* The answer is directly provided by the condition in Definition 10, representing the normalization of an admissible generalized instrument. More precisely, the Choi-Jamiołkowski operators R_i of the histories must add to a deterministic comb, with the normalization conditions given by Eq. (25).

(3) *Is it possible that there are cases of sets of Kraus operators that do not lead to causality violations but still there is no actual way to implement them in quantum mechanics?* In this case the answer is negative. We provided indeed the causal interpretation of conditions (25). Any multiple-time measurement that does not violate causality satisfies the latter condition, and by Theorem 10 this implies that the measurement is feasible as in Fig 12.

As an example we consider the same multiple-time measurement as the authors provide, implementing the measurement of the difference of the values of the operator σ_x at times t_1 and t_2 on a qubit, σ_x denoting the Pauli matrix. The measurement can be summarized by the following generalized instrument:

$$P_{+2}=|-\rangle\langle-|_{t_2}\otimes|-\rangle\langle-|\otimes|+\rangle\langle+|_{t_1}\otimes|+\rangle\langle+|,$$

$$P_{-2}=|+\rangle\langle+|_{t_2}\otimes|+\rangle\langle+|\otimes|-\rangle\langle-|_{t_1}\otimes|-\rangle\langle-|,$$

$$P_0 = |+\rangle\langle+|_{t_2} \otimes |+\rangle\langle+| \otimes |+\rangle\langle+|_{t_1} \otimes |+\rangle\langle+| + |-\rangle\langle-|_{t_2} \otimes |-\rangle\langle-| \otimes |-\rangle\langle-|_{t_1} \otimes |-\rangle\langle-| + |-\rangle\langle+|_{t_2} \otimes |+\rangle\langle-| \otimes |-\rangle \times \langle+|_{t_1} \otimes |+\rangle\langle-| + |+\rangle\langle-|_{t_2} \otimes |-\rangle\langle+| \otimes |+\rangle\langle-|_{t_1} \otimes |-\rangle\langle+|, \tag{95}$$

where $\sigma_x|\pm\rangle = \pm|\pm\rangle$. Clearly, the measurement outcomes ± 2 correspond to $\sigma_x(t_1) - \sigma_x(t_2) = \pm 2$, while P_0 corresponds to $\sigma_x(t_1) - \sigma_x(t_2) = 0$.

VII. OTHER APPLICATIONS

The general theory of quantum combs is useful for many applications, ranging in different branches of quantum mechanics, like Quantum Information theory, quantum game theory and cryptography, quantum metrology, and finally foundations of physics.

In Quantum Information combs provide an efficient and immediate description of networks, which is the most suitable for optimization purposes. For example, quantum algorithms can be thought of as testers on chains of unitaries, representing successive calls of quantum oracles. The optimization of the tester for discrimination of oracle classes would provide the scaling of the performances of the optimal algorithm with respect to the number of oracle calls, allowing for a definite classification of the quantum complexity class for a wide class of problems. An example of application in quantum information is optimal cloning of unitary gates, that was studied in Ref. [27], where combs were used to find the optimal physical device allowing to emulate two uses of the same unknown unitary gate by actually running it only once.

In quantum game theory or quantum cryptography, quantum combs describe all conceivable strategies and protocols of players and users. This has been already noticed in Ref. [7] for protocols in which only quantum systems are exchanged, without classical (i.e., openly known) communication. The use of quantum combs provides a great simplification in the analysis of cryptographic protocols, where one can use the operational definition of distance between strategies of Eq. (82) for search of equilibria and analysis of cheating strategies. Moreover, quantum combs provide the tool for the analysis of all those protocols that involve quantum and classical communication in more than one direction, e.g., for the evaluation of two-way channel capacity. In order to include classical communication parallel to the quantum one needs to consider sets of nonsuperimposable orthogonal states, which can be easily taken into account using a C*-algebraic version of quantum combs, as it is done for channels [28].

In quantum estimation theory and quantum metrology, quantum combs provide the appropriate framework for parameter estimation since in the actual situation it is a unitary transformation that carries the parameter to be estimated. In this case the old approach of Helstrom [20] and Holevo [3] optimizes the POVM for a given class of input states, then optimizes the state within the class, and finally optimizes the class itself. Instead, the quantum tester provides the optimization with a unified procedure, including the case of multiple uses, and even optimizing over all possible dispositions of the uses. Moreover, as proved in Ref. [19], memory effects turn out to be crucial in the discrimination of memory channels.

Regarding the feasibility of quantum combs, all the possible implementations of qubits and their quantum gates already largely explored for quantum computation are eligible also for the implementation of quantum combs. A very promising scalable implementation of quantum combs is provided by optical qubits in silicon waveguides [29].

Finally, we would like to mention one possible development of combs for foundations of physics, in particular for the formulation of a Quantum Theory of causally undetermined spacetime structures. This suggestion comes from a striking analogy between quantum combs and a quantum realization of the *causaloid* of Hardy [30,31] a promising tool for the formulation of quantum gravity.

Admissible maps in general operational settings

In Sec. IV we proved that the whole hierarchy of linear transformations of any order in Quantum Mechanics reduces to one level, corresponding to memory channels. Any admissible transformation is physically achievable by a memory channel, namely, a channel exploiting ancillary systems as quantum memories that correlate successive uses. We proved this feature for the classical and quantum combs. However, our proof exploits the detailed features of the theory, and it may not hold more generally for any probabilistic theory [32]. More precisely, we proved that: (1) all admissible N-maps are realized by memory channels; (2) any admissible map [i.e., (K,L)-map] is indeed an N-map. One may wonder whether such features are generic for any probabilistic theory, or if they are true only for the quantum-classical case.

VIII. CONCLUSION

In conclusion, we introduced a mathematical description of quantum networks in terms of Choi-Jamiołkowski operators, from two complementary points of view. The constructive approach is based on the composition of Choi-Jamiołkowski operators. Within this approach, it is possible to characterize the properties of composite networks by the unified necessary and sufficient condition in Eq. (22).

The axiomatic approach starts from a completely different perspective, and defines admissible maps on quantum objects in a recursive manner, starting from states and quantum operations and rising the level to transformations of transformations, describing them through their Choi-Jamiołkowski operator. We proved that under minimal requirements such transformations correspond to memory channels, and their admissibility implies feasibility.

All details of the theory of quantum networks are explored and thoroughly proved, including properties of generalized instruments, testers and informationally complete testers, along with discriminability criteria and operational distances between networks.

A comprehensive outline of applications and possible implementations of the theoretical objects introduced in the paper is provided, including the description of multiple-time states and multiple-time measurements. In particular, application of quantum combs to the description of multiple-time states and measurements shows the power of this approach, enabling us to answer three important questions left open in Ref. [26].

Finally, we introduce the problem of classification of operational probabilistic theories in terms of the structure of the hierarchy of admissible transformations, which could in principle elucidate the peculiarity of Quantum Mechanics with respect to other theories.

ACKNOWLEDGMENTS

This work has been supported by EU FP7 program through the STREP project COQUIT.

[1] K. Kraus, *States, Effects, and Operations* (Springer, Berlin, 1983).
[2] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1973).
[3] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[4] W. F. Stinespring, Proc. Am. Math. Soc. **6**, 211 (1955).
[5] M. Ozawa, J. Math. Phys. **25**, 79 (1984).
[6] M. A. Naimark, Izv. Akad. Nauk SSSR, Ser. Mat. **4**, 277 (1940).
[7] G. Gutoski and J. Watrous, in *Proceedings of the 39th Annual ACM Symposium on Theory of Computation (STOC)*, 2007 pp. 565–574.
[8] Labeling Hilbert spaces in this way make us able to define the operator analog of the saturated-index pairs for multiplying tensors in general relativity. When multiplying only two operators only on some pairs of isomorphic tensor factors we will use the same name for the two Hilbert spaces, whereas we use different names for all other Hilbert spaces
[9] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. Lett. **101**, 060401 (2008).
[10] Both restrictions are due to the convention that different uses of the same circuit are considered here as different circuits. Notice that the situation in which a quantum system undergoes the evolution $\mathcal{C}^2$ by passing twice through the gate $\mathcal{C}$ does not correspond to a loop, since the two successive uses of channel $\mathcal{C}$ are represented as the cascade of two different channels.
[11] D. Kretschmann and R. F. Werner, Phys. Rev. A **72**, 062323 (2005).
[12] T. Eggeling, D. Schlingemann, and R. F. Werner, Europhys. Lett. **57**, 782 (2002).
[13] D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill, Phys. Rev. A **64**, 052309 (2001).
[14] M. Piani, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **74**, 012305 (2006).
[15] G. Chiribella, G. M. D'Ariano, and P. Perinotti, J. Math. Phys. **50**, 042101 (2009).
[16] I. L. Chuang and M. A. Nielsen, *Quantum Information and Quantum Computation* (Cambridge University Press, Cambridge, 2000).
[17] G. Chiribella, G. M. D'Ariano, and P. Perinotti, EPL **83**, 30004 (2008).
[18] M. Ziman, Phys. Rev. A **77**, 062112 (2008).
[19] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. Lett. **101**, 180501 (2008).
[20] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[21] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Pitman Research Notes in Math. No. 146. (Longman Scientific & Technical, Harlow, 1996).
[22] D. Aharonov, A. Kitaev, and N. Nsan, in *Proceedings of the 30th Annual ACM Symposium on Theory of Computation (STOC)*, 1997 (unpublished) pp. 20–30.
[23] M. F. Sacchi, Phys. Rev. A **71**, 062340 (2005).
[24] E. Prugovečki, Int. J. Theor. Phys. **16**, 321 (1977).
[25] A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, and P. Perinotti, Phys. Rev. Lett. **102**, 010404 (2009).
[26] Y. Aharonov, S. Popescu, J. Tollaksen, and L. Vaidman, e-print arXiv:0712.0320.
[27] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Phys. Rev. Lett. **101**, 180504 (2008).
[28] G. M. D'Ariano, D. Kretschmann, D. M. Schlingemann, and R. F. Werner, Phys. Rev. A **76**, 032328 (2007).
[29] A. Politi, M. J. Cryan, J. G. Rarity, S. Yu, and J. L. O'Brien, Science **320**, 646 (2008).
[30] L. Hardy, J. Phys. A **40**, 3081 (2007).
[31] L. Hardy, e-print arXiv:0804.0054.
[32] H. Barnum, J. Barrett, M. Leifer, and A. Wilce, Phys. Rev. Lett. **99**, 240501 (2007).