Generating qudits with $d=3,4$ encoded on two-photon states

Giacomo Mauro D'Ariano,[1,3] Paolo Mataloni,[2,3] and Massimiliano F. Sacchi[1,3]
[1]*QUIT, Unità INFM and Dipartimento di Fisica "A. Volta," Università di Pavia, I-27100 Pavia, Italy*
[2]*Dipartimento di Fisica, Università di Roma "La Sapienza," I-00185 Roma, Italy*
[3]*CNISM, Consorzio Nazionale Interuniversitario per le Scienze Fisiche della Materia, Italy*
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We present an experimental method to engineer arbitrary pure states of qudits with $d=3,4$ using linear optics and a single nonlinear crystal.

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Many issues in quantum-information theory and processing deal with qudits, namely, d-level quantum systems, instead of qubits. The interest in such more complex systems is both theoretical—the general structure of quantum protocols can be simplified for arbitrary dimension—and practical—some relevant applications perform better using qudits. For example, new quantum cryptographic protocols were recently proposed that deal specifically with qutrits [1–3] and the eavesdropping analysis showed that these systems are more robust against specific classes of eavesdropping attacks [3–6]. A further advantage of using multilevel systems deals with novel fundamental tests of quantum mechanics [7,8].

Recent experimental realizations of qutrits rely on different physical implementations. In interferometric schemes, qutrits are generated by sending an entangled photon pair through a multiarmed interferometer [9], and the number of arms defines the dimensionality of the system. Other techniques exploit the properties of orbital angular momentum of single photons [3,10,11], or perform postselection from four-photon states [12]. All the above techniques, however, provide only partial control over the qutrit state. In the method of Refs. [3,10,11] one needs a specific hologram for a given qutrit state. Also the interferometric scheme [9] is not very flexible in switching between different states. More recently, an experimental realization of arbitrary qutrit states has been reported [13], where the polarization state of a two-photon field has been exploited. Such a realization requires the use of *three* nonlinear crystals pumped by a common coherent source.

In this paper we show an experimental method to engineer arbitrary pure states of qutrits and ququads, using a single nonlinear crystal and linear optical devices as phase wave plates. The qudit is encoded on the polarization of a two-photon state, and is obtained from local (e.g., single-photon) unitary transformations on a pure nonmaximally entangled state which plays the role of a *seed* state. It can be generated from a parametric source of entangled photon states [14,15]. In the present paper we refer to a high-brilliance source [15], with high flexibility in terms of state generation. It has been recently demonstrated that by this it is possible to produce two-photon hyperentangled states, entangled in polarization and momentum [16]. Indeed, the adoption of hyperentangled states can be crucial whenever one is interested in quantum-information applications of qudits, since hyperentanglement in polarization and momentum allows one to perform nontrivial measurements—such as Bell measurements [17]—which are needed for quantum key distribution. In fact, as we will show, it is possible to implement a quantum cryptographic scheme with ququads that exploits two mutually unbiased bases made by two-photon Bell states, and here hyperentanglement allows one to perform Bell measurements.

In the following we first show how to obtain an arbitrary qudit with $d=3,4$, from local unitary transformations on a bipartite pure state of two qubits. Hence, we want to show how to generate a state of the form

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle + \gamma|01\rangle + \delta|10\rangle$$

from the seed state

$$|\chi\rangle = x|00\rangle + \sqrt{1-x^2}|11\rangle$$

by means of two local unitary transformations. In the state $|\psi\rangle$ we can fix α positive, and take β, γ, and δ complex without loss of generality. The state $|\chi\rangle$ is chosen with x positive. Hence, given α, β, γ, and δ we want to find x and two unitaries U and W such that

$$|\psi\rangle = U \otimes W|\chi\rangle. \tag{1}$$

Of course, x, U, and W will depend on the desired parameters α, β, γ, and δ.

We can solve this problem by means of singular value decomposition (SVD), which states that for any matrix A one can find two unitaries U and W such that [18]

$$A = UDW^{\tau}, \tag{2}$$

where τ denotes transposition on the fixed basis, and D is diagonal and positive.

Consider now the matrix Ψ corresponding to the state $|\psi\rangle$,

$$\Psi = \alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1| + \gamma|0\rangle\langle 1| + \delta|1\rangle\langle 0|, \tag{3}$$

through the identity [19]

$$|\psi\rangle = (\Psi \otimes I)(|00\rangle + |11\rangle). \tag{4}$$

From the SVD $\Psi = UDW^{\tau}$ it follows that

$$\begin{aligned}|\psi\rangle &= (UDW^{\tau} \otimes I)(|00\rangle + |11\rangle)\\ &= (UD \otimes W)(|00\rangle + |11\rangle)\\ &= (U \otimes W)(D \otimes I)(|00\rangle + |11\rangle)\\ &= (U \otimes W)(d_1|00\rangle + d_2|11\rangle),\end{aligned} \tag{5}$$

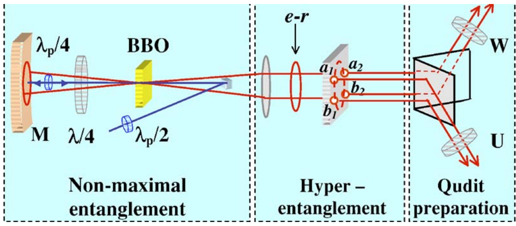

FIG. 1. Layout of the universal source of nonmaximally polarization entangled and polarization-momentum hyperentangled two-photon states. In the left part, nonmaximally entangled states in polarization are generated. In the central part, after division of the entanglement ring (e-r) along a vertical axis by a prismlike two-mirror system, momentum entanglement is realized by a four-hole screen which selects the correlated pairs of modes a_1, b_2 and a_2, b_1. In the right part, qudits are encoded by means of the local unitary transformations U and W on modes a_1, b_1 and a_2, b_2, respectively.

which is equivalent to Eq. (1). The values d_1 and d_2 are the elements of the diagonal matrix D ("the singular values of Ψ"). Notice that

$$1 = \langle\psi|\psi\rangle = \mathrm{Tr}[\Psi^\dagger\Psi] = \mathrm{Tr}[D^2] = d_1^2 + d_2^2, \tag{6}$$

namely, one obtains the correct normalization for the state $|\chi\rangle$. Our result generally holds in arbitrary Hilbert spaces, and hence provides a way to encode a qudit on a bipartite quantum system of $\mathcal{H}\otimes\mathcal{H}$, by means of local unitary transformations, where $d=[\dim(\mathcal{H})]^2$. Notice also that the decomposition in Eq. (2) is not unique, and hence the unitaries U and W in Eq. (1) are not unique either. For example, one has the invariance property $U'=UZ$ and $W'=WZ^\dagger$, where Z is an arbitrary diagonal unitary matrix.

Let us apply the above derivation to the case where the qubits are represented by the polarization state of two photons. The seed state is written

$$|\chi\rangle = x|HH\rangle + \sqrt{1-x^2}|VV\rangle. \tag{7}$$

The state in Eq. (7) represents a nonmaximally entangled polarization state. It is easily obtained from the source sketched in Fig. 1. It is based on a high-stability single-arm interferometer which accomplishes the generation of the polarization entangled state $|\Phi\rangle=(1/\sqrt{2})(|H\rangle|H\rangle+e^{i\theta}|V\rangle|V\rangle)$ by the superposition of degenerate parametric emission cones at wavelength λ (see Fig. 1) of a type-I β-BaB_2O_4 (BBO) crystal, excited in two opposite directions, by a V-polarized laser beam at wavelength $\lambda_p=\lambda/2$. Other basic elements of the source are the following:

(i) a spherical mirror M, reflecting both the parametric radiation and the pump beam, whose micrometric displacement allows one to control the state phase θ $(0\leqslant\theta\leqslant\pi)$;

(ii) a zero-order $\lambda/4$ wave plate (WP), placed within the M-BBO path, which performs the $|HH\rangle\to|VV\rangle$ transformation on the two-photon state belonging to the left cone;

(iii) a positive lens which transforms the conical parametric emission of the crystal into a cylindrical one, whose transverse circular section identifies the so-called entanglement ring.

A zero-order $\lambda_p/4$ WP inserted between M and the BBO crystal, intercepting only the laser beam, allows the engineering of tunable nonmaximally entangled states in the following way. The polarization of the back-reflected pump beam is rotated by an angle $2\theta_p$ with respect to the optical axis of the crystal when the pump WP is rotated by an angle θ_p. As a consequence the emission efficiency of the $|HH\rangle$ contribution is lowered by a coefficient proportional to $\cos^2 2\theta_p$, with θ_p adjusted in the range $0-\pi/4$. Alternatively, we can obtain a lower value of the $|VV\rangle$ contribution with respect to $|HH\rangle$ by inserting a $\lambda_p/2$ WP in the laser beam path before the crystal. By simultaneous rotation of the two WP's, the complete tunability of the entanglement degree can be achieved [20].

The local unitary transformations that are needed to generate the desired state of the qudit can be easily realized by linear optics. In fact, a unitary 2×2 matrix can generally be written as

$$U = \begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta \\ -e^{i\gamma}\sin\theta & e^{i(\beta+\gamma-\alpha)}\cos\theta \end{pmatrix}. \tag{8}$$

Such a unitary can be factorized as follows:

$$U = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{i(\beta+\gamma-\alpha)} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{i(\alpha-\beta)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

Hence, any unitary transformation on the polarization state of a photon can be obtained as a sequence of a phase shift, a rotation of the polarization, and a final phase shift.

The general scheme can be used to engineer mutually unbiased bases [21] of qutrits for cryptographic purposes. For example, the basis

$$|u_I\rangle = |HH\rangle,$$

$$|u_{II}\rangle = |VV\rangle,$$

$$|u_{III}\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) \equiv |\psi^+\rangle \tag{10}$$

is mutually unbiased with the basis

$$|v_I\rangle = \frac{1}{\sqrt{3}}(|HH\rangle + |VV\rangle + |\psi^+\rangle),$$

$$|v_{II}\rangle = \frac{1}{\sqrt{3}}(|HH\rangle + e^{2\pi i/3}|VV\rangle + e^{-2\pi i/3}|\psi^+\rangle),$$

$$|v_{III}\rangle = \frac{1}{\sqrt{3}}(|HH\rangle + e^{-2\pi i/3}|VV\rangle + e^{2\pi i/3}|\psi^+\rangle). \tag{11}$$

It is quite easy to generate the states of the first basis. On the other hand, the states of the second basis can be generated according to the above derivation.

Explicitly one has

$$|v_i\rangle = (U_i \otimes W_i)|\chi\rangle, \tag{12}$$

where the seed state $|\chi\rangle$—the same for all $i=\mathrm{I,II,III}$—is written

$$|\chi\rangle = \frac{\sqrt{2}+1}{\sqrt{6}}|HH\rangle + \frac{\sqrt{2}-1}{\sqrt{6}}|VV\rangle \simeq 0.986|HH\rangle + 0.169|VV\rangle, \tag{13}$$

and the set of unitaries is given by

$$U_I = W_I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$U_{II} = W_{II} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{-2i\pi/3} & e^{i\pi/3} \end{pmatrix},$$

$$U_{III} = W_{III} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{2i\pi/3} & e^{-i\pi/3} \end{pmatrix}. \tag{14}$$

Notice that for the particular chosen basis, the unitaries U_i and W_i are identical. Moreover, using the factorization formula (9), the phase shift on the right reduces to the identity matrix, and hence the U_i's can be implemented by a $\lambda/2$ WP rotated by $\theta=\pi/8$, followed by a further phase delay between H and V, corresponding to $\varphi_I=0$, $\varphi_{II}=\frac{2}{3}\pi$, and $\varphi_{III}=-\frac{2}{3}\pi$, respectively.

Other qutrits of the form

$$|\xi\rangle = \frac{1}{\sqrt{3}}(|HH\rangle + e^{i\psi}|VV\rangle + e^{i\phi}|\psi^{+}\rangle) \tag{15}$$

can be generated by using the general formula $UDW^{\tau}=\xi$, where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\,\arg[\sqrt{2}+e^{i(\phi-\psi/2)}]} & e^{i\,\arg[\sqrt{2}-e^{i(\phi-\psi/2)}]} \\ e^{i\,\arg[e^{i\phi}+\sqrt{2}e^{i\psi/2}]} & e^{i\,\arg[e^{i\phi}-\sqrt{2}e^{i\psi/2}]} \end{pmatrix},$$

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ e^{i\psi/2} & -e^{i\psi/2} \end{pmatrix},$$

$$D = \begin{pmatrix} \sqrt{\frac{1}{2}+\frac{\sqrt{2}}{3}\cos\left(\frac{\psi}{2}-\phi\right)} & 0 \\ 0 & \sqrt{\frac{1}{2}-\frac{\sqrt{2}}{3}\cos\left(\frac{\psi}{2}-\phi\right)} \end{pmatrix},$$

$$\xi = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \frac{e^{i\phi}}{\sqrt{2}} \\ \frac{e^{i\phi}}{\sqrt{2}} & e^{i\psi} \end{pmatrix}. \tag{16}$$

Hence, one can write a relation as in Eq. (12) for two other bases which are mutually unbiased with each other and with those of Eqs. (10) and (11), with the seed state

$$|\chi\rangle = \sqrt{\frac{3+\sqrt{2}}{6}}|HH\rangle + \sqrt{\frac{3-\sqrt{2}}{6}}|VV\rangle \simeq 0.858|HH\rangle + 0.514|VV\rangle, \tag{17}$$

and with unitaries that can be evaluated by Eq. (16).

The above described procedure could produce highly pure qudits. In fact, both the technique used to generate nonmaximally entangled states and the wave plates and phase shifters to realize the unitary operations can be very accurate, and in principle do not introduce any amount of mixedness in the state. Indeed, nonmaximally entangled states generated by this source were recently used to prove the Hardy's ladder theorem on nonlocality up to the 20th step of the ladder [20].

Once qudits are available, one can characterize these states by quantum tomography, or use them for more advanced tests of nonlocality [7,8]. As far as more specific quantum-information applications are concerned, e.g., quantum key distribution, a major difficulty is the need to perform quantum measurements on mutually unbiased bases. The use of qutrits requires highly nontrivial setups at the measurement stage. However, the use of ququads is easier. In this case one should use five mutually unbiased bases, hence generating 20 different states. For a system of two qubits [22,23], one can consider three product bases and two Bell bases. We write explicitly the bases from Ref. [23] (in our scheme, we have $|0\rangle\equiv|H\rangle$ and $|1\rangle\equiv|V\rangle$), namely,

$$|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle, \quad \mathrm{I}$$

$$(|0\rangle+|1\rangle)(|0\rangle\pm|1\rangle),$$

$$(|0\rangle-|1\rangle)(|0\rangle\pm|1\rangle), \quad \mathrm{II}$$

$$(|0\rangle+i|1\rangle)(|0\rangle\pm i|1\rangle),$$

$$(|0\rangle-i|1\rangle)(|0\rangle\pm i|1\rangle), \quad \mathrm{III}$$

$$(|0\rangle+i|1\rangle)|0\rangle\pm(|0\rangle-i|1\rangle)|1\rangle,$$

$$(|0\rangle-i|1\rangle)|0\rangle\pm(|0\rangle+i|1\rangle)|1\rangle, \quad \mathrm{IV}$$

$$|0\rangle(|0\rangle+i|1\rangle)\pm|1\rangle(|0\rangle-i|1\rangle),$$

$$|0\rangle(|0\rangle-i|1\rangle)\pm|1\rangle(|0\rangle+i|1\rangle). \quad \mathrm{V} \tag{18}$$

Clearly, the bases I, II, and III correspond to the measurement of $\sigma_z\otimes\sigma_z$, $\sigma_x\otimes\sigma_x$, and $\sigma_y\otimes\sigma_y$, respectively. The bases IV and V are made of Bell projectors. The generation of the 12 product states is trivial. On the other hand, the above source of entangled photon states very efficiently generates the other eight maximally entangled states.

The problem of realizing Bell measurements can be solved by hyperentangled states [17], which have been realized in the two degrees of polarization and momentum by the same source [16]. Besides polarization entanglement, momentum entanglement is realized by a four-hole screen which allows one to select the correlated pairs of modes a_1, b_2 and a_2, b_1 (Fig. 1) occurring with relative phase $\phi=0$. In this way, in either one of the cones the momentum entangled Bell state $|\psi^{+}\rangle=(1/\sqrt{2})(|a_1,b_2\rangle+|b_1,a_2\rangle)$ can be generated. Note that the four modes a_1, b_2, a_2, and b_1 can be easily separated by mirrors and coupled to single-mode optical fibers, allowing in this way fiber-based cryptographic schemes. In a complete Bell state analysis the polarization state acts as the control qubit and the momentum state $|\psi^{+}\rangle$ as the target qubit [17].

We notice also that a cryptographic protocol with ququads that uses just two instead of all five mutually unbiased bases is characterized by a maximum acceptable error rate that is only slightly lower, while the corresponding key rate is much larger [5]. The nontrivial encoding here is represented by the Bell states of the bases IV and V, and a cryptographic scheme based just on such two bases can be implemented by our source.

In conclusion, we have shown how to obtain an arbitrary qudit up to $d=4$, from local unitary transformations on a bipartite pure state of two qubits by SVD encoding. The theoretical scheme generally holds in arbitrary Hilbert space, encoding the qudit on a bipartite quantum system of $\mathcal{H}\otimes\mathcal{H}$ by means of local unitaries, with $d=[\dim(\mathcal{H})]^2$. Upon representing qubits by the polarization state of photons, the method allows one to generate experimentally qudits with a single nonlinear crystal and linear optics, using the source of Ref. [15]. This allows one to create tunable nonmaximally entangled states that play the role of seed states, from which arbitrary qudit states are generated via SVD using simple linear optics. The hyperentanglement of the generated photons allows one to perform nontrivial measurements—such as Bell measurements—that are crucial for quantum-cryptographic applications.

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