

How to Derive the Hilbert-Space Formulation of Quantum Mechanics From Purely Operational Axioms

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Abstract. In the present paper I show how it is possible to derive the Hilbert space formulation of Quantum Mechanics from a comprehensive definition of *physical experiment* and assuming *experimental accessibility and simplicity* as specified by five simple Postulates. This accomplishes the program presented in form of conjectures in the previous paper [1]. Pivotal roles are played by the *local observability principle*, which reconciles the holism of nonlocality with the reductionism of local observation, and by the postulated existence of *informationally complete observables* and of a *symmetric faithful state*. This last notion allows one to introduce an operational definition for the real version of the "adjoint"—i. e. the transposition—from which one can derive a real Hilbert-space structure via either the Mackey-Kakutani or the Gelfand-Naimark-Segal constructions. Here I analyze in detail only the Gelfand-Naimark-Segal construction, which leads to a real Hilbert space structure analogous to that of (classes of generally unbounded) selfadjoint operators in Quantum Mechanics. For finite dimensions, general dimensionality theorems that can be derived from a local observability principle, allow us to represent the elements of the real Hilbert space as operators over an underlying complex Hilbert space (see, however, a still open problem at the end of the paper). The route for the present operational axiomatization was suggested by novel ideas originated from Quantum Tomography.

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INTRODUCTION

Quantum Mechanics is a sort of "syntactic manual" for physical theories: it is a set of rules that hold for any physical field—electroweak, nuclear, gravitational—and apply to the entire physical domain, from micro to macro, independently of the size and energy scale. Should we consider Quantum Mechanics a General Law of Nature, or, instead, a Logical Necessity, a *Miniature Epistemology*? Indeed, for the first time in the history of Physics, Quantum Mechanics in its very essence addresses the crucial problem of the *Physical Measurement*, problem which is at the core of Physics as an experimental science. It is not the physical description of the specific instrumentation that I'm talking about, but the general process of information retrieval in *any* measurement, via interaction of the measured system with the measuring apparatus. I would say that Quantum Mechanics more generally deals with the description of the *Physical Experiment*, which is indeed the epistemic archetype, the prototype *cognitive act* of interaction with reality.

In the above framework it is mandatory to derive Quantum Mechanics from purely

operational axioms. This is not just for the sake of establishing more general and irreducible foundations, but also to understand the intimate relations between general epistemic issues—such as locality, causality, probability interpretations, holism versus reductionism, and growth of experimental complexity with the “size” of the measured system.

In the present work the starting point for axiomatization is a very comprehensive definition of *physical experiment*. As I have shown in Ref. [1], the adoption of such a general definition of experiment constitutes a very seminal point for axiomatization, entailing a thorough series of notions that are usually considered of quantum nature—such as the same probabilistic notion of state, and the notions of conditional state, local state, pure state, faithful state, instrument, propensity (i. e. "effect"), dynamical and informational equivalence, dynamical and informational compatibility, predictability, discriminability, programmability, locality, a-causality, and even many notions of dimensionality, orthogonality of states, rank of a state, etc: for more details the interested reader is addressed to Ref. [1]. Here we will see how, assuming *experimental accessibility and simplicity* in terms of five simple operational axioms, the present conception of experiment brings his own Hilbert-space formulation, which in turns entails the Quantum Mechanical one. The possibility of deriving the Hilbert-space formulation from experimental simplicity/accessibility was first conjectured in the earlier attempt [1]. As we will see, very interesting roles are played by Postulates numbered as 2, 3, and 5 in the following, namely: (2) the assumed existence of informationally complete measurements, (3) the local observability principle, and (5) the existence of symmetric faithful states. Postulate 2 minimizes the number of different apparatuses that are needed to retrieve any different kind of information. Postulate 3 makes it possible to make joint observations using only the same local measuring apparatuses used for measurements on single systems. This also reconciles the holism of nonlocality with the reductionism of local observation. Postulate 5 (in conjunction with the other two) allows one to calibrate any experimental apparatus by just a single input state preparation. It also allows one to introduce an operational definition for the "real adjoint"—i. e. the transposition—from which one can derive a real Hilbert space structure via either the Mackey-Kakutani [2] (see also Ref. [3]), or the Gelfand-Naimark-Segal [4] constructions. Moreover, the Postulates entail general dimensionality theorems, which are in agreement with the quantum mechanical rule of tensor product of Hilbert spaces for composition of independent systems, and show that the derived real Hilbert space is isomorphic to the real Hilbert space of Hermitian complex matrices representing selfadjoint operators over a complex Hilbert space, which is the Hilbert space formulation of Quantum Mechanics. In deriving one of the dimensionality theorems I have made, however, the implicit assumption that the relation between the affine dimension and the informational dimension for a convex set of state is the same for all physical systems—a sort of informational universality (see the discussion at the end of the paper).

The present research has been stimulated by the recent noteworthy works on axiomatization of Quantum Mechanics by L. Hardy [5, 6] and by C. Fuchs [7]. However, apart from a prominent role played by the informationally complete measurements, the relative implications and connections with these works remain rather obscure to me, and will be object of future studies. Some expert readers will recognize strong affinities of the present work with the program of G. Ludwig [8], who sought operational principles

to select the structure of quantum states from all possible convex structures (see also papers collected in the book [9]). These works didn't have a followup mostly because the convex structure by itself is quite poor mathematically. Here we use new crucial concepts that were almost unknown in those years, concepts originated from the field of Quantum Tomography [10]. In particular, recently it has been shown that it is possible to make a complete quantum calibration of a measuring apparatus [11] or of a quantum operation [12] by using a single pure bipartite state, and, more generally, using a faithful state [13]. This gives us a unique opportunity for deriving the Hilbert space structure from the convex structure in terms of calibrability axioms, relying on the special link between the convex set of transformations and the convex set of states which occurs in Quantum Mechanics, and which make the transformations of a single system closely resemble the states of a bipartite system [14, 15].

THE OPERATIONAL AXIOMATIZATION

General Axiom 1 (On experimental science) *In any experimental science we make experiments to get information on the state of a objectified physical system. Knowledge of such a state will allow us to predict the results of forthcoming experiments on the same object system. Since we necessarily work with only partial a priori knowledge of both system and experimental apparatus, the rules for the experiment must be given in a probabilistic setting.*

General Axiom 2 (On what is an experiment) *An experiment on an object system consists in having it interact with an apparatus. The interaction between object and apparatus produces one of a set of possible transformations of the object, each one occurring with some probability. Information on the "state" of the object system at the beginning of the experiment is gained from the knowledge of which transformation occurred, which is the "outcome" of the experiment signaled by the apparatus.*

Postulate 1 (Independent systems) *There exist independent physical systems.*

Postulate 2 (Informationally complete observable) *For each physical system there exists an informationally complete observable.*

Postulate 3 (Local observability principle) *For every composite system there exist informationally complete observables made only of minimal local informationally complete observables.*

Postulate 4 (Informationally complete discriminating observable) *On every composite system made of two identical physical systems there exists a discriminating observable that gives a minimal informationally complete observable for one of the components, for some preparations of the other component.*

Postulate 5 (Symmetric faithful state) *For every composite system made of two identical physical systems there exist a symmetric joint state that is both dynamically and preparationally faithful.*

The General Axioms 1 and 2 entail a very rich series of notions, including those used in the Postulates—e. g. independent systems, observable, informationally complete observable, etc. Starting from the two General Axioms in the first sections of the paper, I will introduce step by step such notions, starting from the pertaining definitions, and then giving the logically related rules. For a discussion on the General Axioms the reader is addressed to the publication [1], where also the generality of the definition of experiment given in the General Axioms 1 is analyzed in some detail.

TRANSFORMATIONS, STATES, INDEPENDENT SYSTEMS

Performing a different experiment on the same object obviously corresponds to the use of a different experimental apparatus or, at least, to a change of some settings of the apparatus. Abstractly, this corresponds to change the set $\{\mathcal{A}_j\}$ of possible transformations, \mathcal{A}_j , that the system can undergo. Such change could actually mean really changing the "dynamics" of the transformations, but it may simply mean changing only their probabilities, or, just their labeling outcomes. Any such change actually corresponds to a change of the experimental setup. Therefore, the set of all possible transformations $\{\mathcal{A}_j\}$ will be identified with the choice of experimental setting, i. e. with the *experiment* itself—or, equivalently, with the *action* of the experimenter: this will be formalized by the following definition

Definition 1 (Actions/experiments and outcomes) *An action or experiment on the object system is given by the set $\mathbb{A} \equiv \{\mathcal{A}_j\}$ of possible transformations \mathcal{A}_j having overall unit probability, with the apparatus signaling the **outcome** j labeling which transformation actually occurred.*

Thus the action/experiment is just a *complete* set of possible transformations that can occur in an experiment. As we can see now, in a general probabilistic framework the *action* \mathbb{A} is the "cause", whereas the *outcome* j labeling the transformation \mathcal{A}_j that actually occurred is the "effect". The *action* has to be regarded as the "cause", since it is the option of the experimenter, and, as such, it should be viewed as deterministic (at least one transformation $\mathcal{A}_j \in \mathbb{A}$ will occur with certainty), whereas the outcome j —i. e. which transformation \mathcal{A}_j occurs—is probabilistic. The special case of a deterministic transformation \mathcal{A} corresponds to a *singleton action/experiment* $\mathbb{A} \equiv \{\mathcal{A}\}$.

In the following, wherever we consider a nondeterministic transformation \mathcal{A} by itself, we always regard it in the context of an experiment, namely for any nondeterministic transformation there always exists at least a complementary one \mathcal{B} such that the overall probability of occurrence of one of them is always unit. According to General Axiom 1 by definition the knowledge of the state of a physical system allows us to predict the results of forthcoming possible experiments on the system, or, more generally, on another system in the same physical situation. Then, according to the General Axiom 2 a precise knowledge of the state of a system would allow us to evaluate the probabilities of any possible transformation for any possible experiment. It follows that the only possible definition of state is the following

Definition 2 (States) A state ω for a physical system is a rule that provides the probability for any possible transformation, namely

$$\omega : \text{state}, \quad \omega(\mathcal{A}) : \text{probability that the transformation } \mathcal{A} \text{ occurs.} \quad (1)$$

We assume that the identical transformation \mathcal{I} occurs with probability one, namely

$$\omega(\mathcal{I}) = 1. \quad (2)$$

This corresponds to a kind of *interaction picture*, in which we do not consider the free evolution of the system (the scheme could be easily generalized to include a free evolution). Mathematically, a state will be a map ω from the set of physical transformations to the interval $[0, 1]$, with the normalization condition (2). Moreover, for every action $\mathbb{A} = \{\mathcal{A}_j\}$ one has the normalization of probabilities

$$\sum_{\mathcal{A}_j \in \mathbb{A}} \omega(\mathcal{A}_j) = 1 \quad (3)$$

for all states ω of the system. As already noticed in Ref. [1], in order to include also non-disturbing experiments, one must conceive situations in which all states are left invariant by each transformation.

The fact that we necessarily work in the presence of partial knowledge about both object and apparatus requires that the specification of the state and of the transformation could be given incompletely/probabilistically, entailing a convex structure on states and an addition rule for coexistent transformations. The convex structure of states is given more precisely by the rule

Rule 1 (Convex structure of states) *The possible states of a physical system comprise a convex set: for any two states ω_1 and ω_2 we can consider the state ω which is the mixture of ω_1 and ω_2 , corresponding to have ω_1 with probability λ and ω_2 with probability $1 - \lambda$. We will write*

$$\omega = \lambda \omega_1 + (1 - \lambda) \omega_2, \quad 0 \leq \lambda \leq 1, \quad (4)$$

and the state ω will correspond to the following probability rule for transformations \mathcal{A}

$$\omega(\mathcal{A}) = \lambda \omega_1(\mathcal{A}) + (1 - \lambda) \omega_2(\mathcal{A}). \quad (5)$$

Generalization to more than two states is obtained by induction. In the following the convex set of states will be denoted by \mathfrak{S} . We will call *pure* the states which are the extremal elements of the convex set, namely which cannot be obtained as mixture of any two states, and we will call *mixed* the non-extremal ones. As regards transformations, the addition of coexistent transformations and the convex structure will be considered in Rules 4 and 6.

Rule 2 (Transformations form a monoid) *The composition $\mathcal{A} \circ \mathcal{B}$ of two transformations \mathcal{A} and \mathcal{B} is itself a transformation. Consistency of composition of transformations requires associativity, namely*

$$\mathcal{C} \circ (\mathcal{B} \circ \mathcal{A}) = (\mathcal{C} \circ \mathcal{B}) \circ \mathcal{A}. \quad (6)$$

There exists the identical transformation \mathcal{I} which leaves the physical system invariant, and which for every transformation \mathcal{A} satisfies the composition rule

$$\mathcal{I} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{I} = \mathcal{A}. \quad (7)$$

Therefore, transformations make a semigroup with identity, i. e. a monoid.

Definition 3 (Independent systems and local experiments) *We say that two physical systems are independent if on each system we can perform local experiments that do not affect the other system for any joint state of the two systems. This can be expressed synthetically with the commutativity of transformations of the local experiments, namely*

$$\mathcal{A}^{(1)} \circ \mathcal{B}^{(2)} = \mathcal{B}^{(2)} \circ \mathcal{A}^{(1)}, \quad (8)$$

where the label $n = 1, 2$ of the transformations denotes the system undergoing the transformation.

In the following, when we have more than one independent system, we will denote local transformations as ordered strings of transformations as follows

$$\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots \doteq \mathcal{A}^{(1)} \circ \mathcal{B}^{(2)} \circ \mathcal{C}^{(3)} \circ \dots \quad (9)$$

where the list of transformation on the left denotes the occurrence of local transformation \mathcal{A} on system 1, \mathcal{B} on system 2, etc.

CONDITIONED STATES AND LOCAL STATES

Rule 3 (Bayes) *When composing two transformations \mathcal{A} and \mathcal{B} , the probability $p(\mathcal{B}|\mathcal{A})$ that \mathcal{B} occurs conditional on the previous occurrence of \mathcal{A} is given by the Bayes rule*

$$p(\mathcal{B}|\mathcal{A}) = \frac{\omega(\mathcal{B} \circ \mathcal{A})}{\omega(\mathcal{A})}. \quad (10)$$

The Bayes rule leads to the concept of *conditional state*:

Definition 4 (Conditional state) *The conditional state $\omega_{\mathcal{A}}$ gives the probability that a transformation \mathcal{B} occurs on the physical system in the state ω after the transformation \mathcal{A} has occurred, namely*

$$\omega_{\mathcal{A}}(\mathcal{B}) \doteq \frac{\omega(\mathcal{B} \circ \mathcal{A})}{\omega(\mathcal{A})}. \quad (11)$$

In the following we will make extensive use of the functional notation

$$\omega_{\mathcal{A}} \doteq \frac{\omega(\cdot \circ \mathcal{A})}{\omega(\mathcal{A})}, \quad (12)$$

where the centered dot stands for the argument of the map. Therefore, the notion of conditional state describes the most general *evolution*.

Definition 5 (Local state) *In the presence of many independent systems in a joint state Ω , we define the **local state** $\Omega|_n$ of the n -th system the state that gives the probability for any local transformation \mathcal{A} on the n -th system, with all other systems untouched, namely*

$$\Omega|_n(\mathcal{A}) \doteq \Omega(\mathcal{I}, \dots, \mathcal{I}, \underbrace{\mathcal{A}}_{nth}, \mathcal{I}, \dots). \quad (13)$$

For example, for two systems only, (which is equivalent to group $n - 1$ systems into a single one), we just write $\Omega|_1 = \Omega(\cdot, \mathcal{I})$. Notice that generally commutativity of local transformations (i. e. Definition 3) does not imply that a transformation on system 2 does not affect the conditioned local state on system 1. We also emphasize that acausality of local actions is not logically entailed by system independence (for a discussion about acausality see Ref. [1]).

Remark 1 (Linearity of evolution) *At this point it is worth noticing that the present definition of “state”, which logically follows from the definition of experiment, leads to a notion of evolution as state-conditioning. In this way, each transformation acts linearly on the state space. In addition, since states are probability functionals on transformations, by dualism (equivalence classes of) transformations are linear functionals over the state space.*

For the following it is convenient to extend the notion of state to that of *weight*, namely nonnegative bounded functionals $\tilde{\omega}$ over the set of transformations with $0 \leq \tilde{\omega}(\mathcal{A}) \leq \tilde{\omega}(\mathcal{I}) < +\infty$ for all transformations \mathcal{A} . To each weight $\tilde{\omega}$ it corresponds the properly normalized state

$$\omega = \frac{\tilde{\omega}}{\tilde{\omega}(\mathcal{I})}. \quad (14)$$

Weights make the convex cone $\tilde{\mathfrak{S}}$ which is generated by the convex set of states \mathfrak{S} .

Definition 6 (Linear real space of generalized weights) *We extend the notion of weight to that of negative weight, by taking differences. Such generalized weights span the affine linear space \mathfrak{W} of the convex cone of weights.*

Remark 2 *The transformations \mathcal{A} act as linear transformations over the space of weights as follows*

$$\mathcal{A} \tilde{\omega} = \tilde{\omega}(\mathcal{B} \circ \mathcal{A}). \quad (15)$$

We are now in position to introduce the concept of *operation*.

Definition 7 (Operation) *To each transformation \mathcal{A} we can associate a linear map $\text{Op}_{\mathcal{A}} : \mathfrak{S} \longrightarrow \tilde{\mathfrak{S}}$, which sends a state ω into the unnormalized state $\tilde{\omega}_{\mathcal{A}} \doteq \text{Op}_{\mathcal{A}} \omega \in \tilde{\mathfrak{S}}$, defined by the relation*

$$\text{Op}_{\mathcal{A}} \omega \doteq \tilde{\omega}_{\mathcal{A}}, \quad \tilde{\omega}_{\mathcal{A}}(\mathcal{B}) = \omega(\mathcal{B} \circ \mathcal{A}). \quad (16)$$

Similarly to a state, the linear form $\tilde{\omega}_{\mathcal{A}} \in \tilde{\mathfrak{S}}$ for fixed \mathcal{A} maps from the set of transformations to the interval $[0, 1]$. It is not strictly a state only due to lack of normalization, since $0 < \tilde{\omega}_{\mathcal{A}}(\mathcal{I}) \leq 1$. The operation Op gives the conditioned state through the state-reduction rule

$$\omega_{\mathcal{A}} = \frac{\tilde{\omega}_{\mathcal{A}}}{\omega(\mathcal{A})} \equiv \frac{\text{Op}_{\mathcal{A}} \omega}{\text{Op}_{\mathcal{A}} \omega(\mathcal{I})}. \quad (17)$$

DYNAMICAL AND INFORMATIONAL STRUCTURE

From the Bayes rule, or, equivalently, from the definition of conditional state, we see that we can have the following complementary situations:

1. There are different transformations which produce the same state change, but generally occur with different probabilities;
2. There are different transformations which always occur with the same probability, but generally affect a different state change.

The above observation leads us to the following definitions of dynamical and informational equivalences of transformations.

Definition 8 (Dynamical equivalence of transformations) *Two transformations \mathcal{A} and \mathcal{B} are dynamically equivalent if $\omega_{\mathcal{A}} = \omega_{\mathcal{B}}$ for all possible states ω of the system. We will denote the equivalence class containing the transformation \mathcal{A} as $[\mathcal{A}]_{\text{dyn}}$.*

Definition 9 (Informational equivalence of transformations) *Two transformations \mathcal{A} and \mathcal{B} are informationally equivalent if $\omega(\mathcal{A}) = \omega(\mathcal{B})$ for all possible states ω of the system. We will denote the equivalence class containing the transformation \mathcal{A} as $[\mathcal{A}]$.*

Definition 10 (Complete equivalence of transformations/experiments) *Two transformations/experiments are completely equivalent iff they are both dynamically and informationally equivalent.*

Notice that even though two transformations are completely equivalent, in principle they can still be different experimentally, in the sense that they are achieved with different apparatus. However, we emphasize that outcomes in different experiments corresponding to equivalent transformations always provide the same information on the

state of the object, and, moreover, the corresponding transformations of the state are the same. The concept of dynamical equivalence of transformations leads one to introduce a convex structure also for transformations. We first need the notion of informational compatibility.

Definition 11 (Informational compatibility or coexistence) *We say that two transformations \mathcal{A} and \mathcal{B} are coexistent or informationally compatible if one has*

$$\omega(\mathcal{A}) + \omega(\mathcal{B}) \leq 1, \quad \forall \omega \in \mathfrak{S}, \quad (18)$$

The fact that two transformations are coexistent means that, in principle, they can occur in the same experiment, namely there exists at least an action containing both of them. We have named the present kind of compatibility "informational" since it is actually defined on the informational equivalence classes of transformations.

We are now in position to define the "addition" of coexistent transformations.

Rule 4 (Addition of coexistent transformations) *For any two coexistent transformations \mathcal{A} and \mathcal{B} we define the transformation $\mathcal{S} = \mathcal{A}_1 + \mathcal{A}_2$ as the transformation corresponding to the event $e = \{1, 2\}$, namely the apparatus signals that either \mathcal{A}_1 or \mathcal{A}_2 occurred, but does not specify which one. By definition, one has the distributivity rule*

$$\forall \omega \in \mathfrak{S} \quad \omega(\mathcal{A}_1 + \mathcal{A}_2) = \omega(\mathcal{A}_1) + \omega(\mathcal{A}_2), \quad (19)$$

whereas the state conditioning is given by

$$\forall \omega \in \mathfrak{S} \quad \omega_{\mathcal{A}_1 + \mathcal{A}_2} = \frac{\omega(\mathcal{A}_1)}{\omega(\mathcal{A}_1 + \mathcal{A}_2)} \omega_{\mathcal{A}_1} + \frac{\omega(\mathcal{A}_2)}{\omega(\mathcal{A}_1 + \mathcal{A}_2)} \omega_{\mathcal{A}_2}. \quad (20)$$

Notice that the two rules in Eqs. (19) and (20) completely specify the transformation $\mathcal{A}_1 + \mathcal{A}_2$, both informationally and dynamically. Eq. (20) can be more easily restated in terms of operations as follows:

$$\forall \omega \in \mathfrak{S} \quad \text{Op}_{\mathcal{A}_1 + \mathcal{A}_2} \omega = \text{Op}_{\mathcal{A}_1} \omega + \text{Op}_{\mathcal{A}_2} \omega. \quad (21)$$

Addition of compatible transformations is the core of the description of partial knowledge on the experimental apparatus. Notice also that the same notion of coexistence can be extended to "propensities" as well (see Definition 12).

Rule 5 (Multiplication of a transformation by a scalar) *For each transformation \mathcal{A} the transformation $\lambda \mathcal{A}$ for $0 \leq \lambda \leq 1$ is defined as the transformation which is dynamically equivalent to \mathcal{A} , but which occurs with probability $\omega(\lambda \mathcal{A}) = \lambda \omega(\mathcal{A})$.*

Notice that according to Definition 10 two transformations are completely characterized operationally by the informational and dynamical equivalence classes to which they belong, whence Rule 5 is well posed.

Remark 3 (Algebra of generalized transformations) *Using Eqs. (19) and (21) one can extend the addition of coexistent transformations to generic linear combinations: the*

generalized transformations. *The generalized transformations constitute a real vector space, which is the affine space of the convex space \mathfrak{T} . Composition of transformations can be extended via linearity to generalized transformations, making their space a real algebra \mathcal{A} , the algebra of generalized transformations. Notice that every generalized transformation belongs to the dynamical equivalence class of a physical transformation, since the conditioned state is always defined.*

It is now natural to introduce a norm over transformations as follows.

Theorem 1 (Norm for transformations) *The following quantity*

$$\|\mathcal{A}\| = \sup_{\omega \in \mathfrak{S}} \omega(\mathcal{A}), \quad (22)$$

is a norm on the set of transformations. In terms of such norm all transformations are contractions.

Proof. We remind the axioms of norm: i) Sub-additivity $\|\mathcal{A} + \mathcal{B}\| \leq \|\mathcal{A}\| + \|\mathcal{B}\|$; ii) Multiplication by scalar $\|\lambda \mathcal{A}\| = \lambda \|\mathcal{A}\|$; iii) $\|\mathcal{A}\| = 0$ implies $\mathcal{A} = 0$. The quantity in Eq. (22) satisfy the sub-additivity relation i), since

$$\|\mathcal{A} + \mathcal{B}\| = \sup_{\omega \in \mathfrak{S}} [\omega(\mathcal{A}) + \omega(\mathcal{B})] \leq \sup_{\omega \in \mathfrak{S}} \omega(\mathcal{A}) + \sup_{\omega' \in \mathfrak{S}} \omega'(\mathcal{B}) = \|\mathcal{A}\| + \|\mathcal{B}\|. \quad (23)$$

Moreover, it obviously satisfies axiom ii). Finally, axiom iii) corresponds to identify all transformations that never occur (occur with zero probability) with the zero transformation $\mathcal{A} = 0$. It is also clear that, by definition, for each transformation \mathcal{A} one has $\|\mathcal{A}\| \leq 1$, namely transformations are contractions. ■

We remind that the multiplication of a transformation \mathcal{A} by a scalar is still a transformation only for scalar $0 \leq \lambda \leq \|\mathcal{A}\|^{-1}$.

Theorem 2 *The norm in Eq. (22) satisfies the following inequality*

$$\text{iv) } \|\mathcal{B} \circ \mathcal{A}\| \leq \|\mathcal{B}\| \|\mathcal{A}\|.$$

Proof. Using the definition of conditional state in Eq. (11) we have

$$\begin{aligned} \|\mathcal{B} \circ \mathcal{A}\| &= \sup_{\omega \in \mathfrak{S}} \omega(\mathcal{B} \circ \mathcal{A}) = \sup_{\omega \in \mathfrak{S}} \omega_{\mathcal{A}}(\mathcal{B}) \omega(\mathcal{A}) \leq \sup_{\omega \in \mathfrak{S}} \omega_{\mathcal{A}}(\mathcal{B}) \sup_{\zeta \in \mathfrak{S}} \zeta(\mathcal{A}) \\ &\leq \sup_{\omega \in \mathfrak{S}} \omega(\mathcal{B}) \sup_{\zeta \in \mathfrak{S}} \zeta(\mathcal{A}) = \|\mathcal{B}\| \|\mathcal{A}\|. \end{aligned} \quad (24)$$

■

The linear space of generalized weights \mathfrak{W} can also be equipped with a norm. For this we need to introduce the following notion of experimentally sufficient set of transformations.

Theorem 3 (Norm over generalized weights) *The following is a norm over generalized weights*

$$\|\tilde{\omega}\| = \sup_{\mathcal{A} \in \mathfrak{T}} |\tilde{\omega}(\mathcal{A})|. \quad (25)$$

Proof. The quantity in Eq. (25) satisfies the sub-additivity relation $\|\tilde{\omega} + \tilde{\zeta}\| \leq \|\tilde{\omega}\| + \|\tilde{\zeta}\|$, since

$$\begin{aligned} \|\tilde{\omega} + \tilde{\zeta}\| &= \sup_{\mathcal{A} \in \mathfrak{S}} [|\tilde{\omega}(\mathcal{A}) + \tilde{\zeta}(\mathcal{A})|] \leq \sup_{\mathcal{A} \in \mathfrak{S}} [|\tilde{\omega}(\mathcal{A})| + |\tilde{\zeta}(\mathcal{A})|] \\ &\leq \sup_{\mathcal{A} \in \mathfrak{S}} |\tilde{\omega}(\mathcal{A})| + \sup_{\mathcal{A} \in \mathfrak{S}} |\tilde{\zeta}(\mathcal{A})| = \|\tilde{\omega}\| + \|\tilde{\zeta}\|. \end{aligned} \quad (26)$$

Moreover, it obviously satisfies the identity

$$\|\lambda \tilde{\omega}\| = |\lambda| \|\tilde{\omega}\|. \quad (27)$$

Finally, $\|\tilde{\omega}\| = 0$ implies that $\tilde{\omega} = 0$, since either $\tilde{\omega}$ is a positive linear form, i. e. it is proportional to a true state, whence at least $\tilde{\omega}(\mathcal{S}) > 0$, or $\tilde{\omega}$ is the difference of two positive linear forms, whence the two corresponding states must be equal by definition, since their probability rules are equal, which means that, again, $\tilde{\omega} = 0$. ■

Remark 4 (Banach space of generalized weights) Closure with respect to the norm (25) makes the real vector space of generalized weights \mathfrak{W} a Banach space, which we will name the Banach space of generalized weights. The norm closure correspond to assume the possibility of preparing states with probabilities close to that of a given one, with the approximability criterion defined by the norm.

Remark 5 (Norms, approximability criteria, and norm closure) Norms defined as in Eq. (22) or Eq. (25) (see also other norms in the following) operationally correspond to approximability criteria. The norm closure is not operationally required, but, as any other kind of extension, it is mathematically convenient. Therefore, in the following we should remind that if norm closure is not operationally assumed in terms of a separate postulate (clearly not of operational nature), then the Banach space element—e. g. the limit of a Cauchy sequence—does not necessarily correspond to a physically achievable quantity.

In terms of the norm (22) for transformations one can equivalently define coexistence (informational compatibility) using the following corollary

Corollary 1 Two transformations \mathcal{A} and \mathcal{B} are coexistent iff $\mathcal{A} + \mathcal{B}$ is a contraction.

Proof. If the two transformations are coexistent, then from Eqs. (18) and (22) one has that $\|\mathcal{A} + \mathcal{B}\| \leq 1$. On the other hand, if $\|\mathcal{A} + \mathcal{B}\| \leq 1$, this means that Eq. (22) is satisfied for all states, namely the transformations are coexistent. ■

Corollary 2 The transformations $\lambda \mathcal{A}$ and $(1 - \lambda) \mathcal{B}$ are compatible for any couple of transformations \mathcal{A} and \mathcal{B} .

Proof. Clearly $\|\lambda \mathcal{A} + (1 - \lambda) \mathcal{B}\| \leq \lambda \|\mathcal{A}\| + (1 - \lambda) \|\mathcal{B}\| \leq 1$. ■

The last corollary implies the rule

Rule 6 (Convex structure of transformations) *Transformations form a convex set, namely for any two transformations \mathcal{A}_1 and \mathcal{A}_2 we can consider the transformation \mathcal{A} which is the mixture of \mathcal{A}_1 and \mathcal{A}_2 with probabilities λ and $1 - \lambda$. Formally, we write*

$$\mathcal{A} = \lambda \mathcal{A}_1 + (1 - \lambda) \mathcal{A}_2, \quad 0 \leq \lambda \leq 1, \quad (28)$$

with the following meaning: the transformation \mathcal{A} is itself a probabilistic transformation, occurring with overall probability

$$\omega(\mathcal{A}) = \lambda \omega(\mathcal{A}_1) + (1 - \lambda) \omega(\mathcal{A}_2), \quad (29)$$

meaning that when the transformation \mathcal{A} occurred we know that the transformation dynamically was either \mathcal{A}_1 with (conditioned) probability λ or \mathcal{A}_2 with probability $(1 - \lambda)$.

We have seen that the transformations form a convex set, more specifically, a spherically truncated convex cone, namely we can always add transformations or multiply a transformation by a positive scalar if the result is a contraction. In the following we will denote the spherically truncated convex cone of transformations as \mathfrak{T} .

Remark 6 *The norm (22) can be extended to the whole algebra \mathcal{A} of generalized transformations as follows*

$$\|\mathcal{A}\| = \sup_{\omega \in \mathfrak{S}} |\omega(\mathcal{A})|. \quad (30)$$

It is then easy to check the axioms i), and ii) of norm. However, axiom iii) does not hold anymore, since one has $\|\mathcal{C}\| = 0$ for $\mathcal{C} = \mathcal{A} - \mathcal{B}$ with \mathcal{A} and \mathcal{B} informationally equivalent transformations. Therefore, the norm extension in Eq. (30) is only a seminorm. Also the bound (2) is not meaningful for the extension, since for the same \mathcal{A} above one would have $\omega(\mathcal{A}) = 0$. We conclude that we cannot introduce the structure of Banach algebra over \mathcal{A} . A Banach space structure can, however, be introduced for the affine space of propensities (see the following).

PROPENSITIES

Informational equivalence allows one to define equivalence classes of transformations, which we may want to call *propensities*, since they give the occurrence probability of a transformation for each state, i. e. its “disposition” to occur.

Definition 12 (Propensities) *We call **propensity** an informational equivalence class of transformations.*

It is easy to see that the present notion of propensity corresponds closely to the notion of “effect” introduced by Ludwig [8]. However, we prefer to keep a separate word, since the “effect” has been identified with a quantum mechanical notion and a precise mathematical object (i. e. a positive contraction). In the following we will denote propensities with underlined symbols as $\underline{\mathcal{A}}$, $\underline{\mathcal{B}}$, etc., and we will use the notation $[\underline{\mathcal{A}}]$

for the propensity containing the transformation \mathcal{A} , and also write $\mathcal{A}_0 \in [\mathcal{A}]$ to say that \mathcal{A}_0 is informationally equivalent to \mathcal{A} . Thus, by definition one has $\omega(\mathcal{A}) \equiv \omega([\mathcal{A}])$, and one can legitimately write $\omega(\underline{\mathcal{A}})$. Similarly, one has $\tilde{\omega}_{\mathcal{A}}(\mathcal{B}) \equiv \tilde{\omega}_{\mathcal{A}}([\mathcal{B}])$ which implies that $\omega(\mathcal{B} \circ \mathcal{A}) = \omega([\mathcal{B}] \circ \mathcal{A})$ which gives the chaining rule

$$[\mathcal{B}] \circ \mathcal{A} \subseteq [\mathcal{B} \circ \mathcal{A}]. \quad (31)$$

One also has the locality rule

$$[(\mathcal{A}, \mathcal{B})] \supseteq ([\mathcal{A}], [\mathcal{B}]), \quad (32)$$

where we used notation (9). It is clear that $\lambda \mathcal{A}$ and $\lambda \mathcal{B}$ belong to the same equivalence class iff \mathcal{A} and \mathcal{B} are informationally equivalent. This means that also for propensities multiplication by a scalar can be defined as $\lambda [\mathcal{A}] = [\lambda \mathcal{A}]$. Moreover, since for $\mathcal{A}_0 \in [\mathcal{A}]$ and $\mathcal{B}_0 \in [\mathcal{B}]$ one has $\mathcal{A}_0 + \mathcal{B}_0 \in [\mathcal{A} + \mathcal{B}]$, we can define addition of propensities as $[\mathcal{A}] + [\mathcal{B}] = [\mathcal{A} + \mathcal{B}]$ for any choice of representatives \mathcal{A} and \mathcal{B} of the two added propensities. Also, since all transformations of the same equivalence class have the same norm, we can extend the definition (22) to propensities as $\|[\mathcal{A}]\| = \|\mathcal{A}\|$ for any representative \mathcal{A} of the class. It is easy to check sub-additivity on classes, which implies that it is indeed a norm. In fact, one has

$$\|[\mathcal{A}] + [\mathcal{B}]\| = \|\mathcal{A} + \mathcal{B}\| \leq \|\mathcal{A}\| + \|\mathcal{B}\| = \|[\mathcal{A}]\| + \|[\mathcal{B}]\|. \quad (33)$$

Therefore, it follows that also propensities form a spherically truncated convex cone, which we will denote by \mathfrak{P} .

Remark 7 (Banach space of generalized propensities) *The norm for propensities can be extended to the embedding affine space of \mathfrak{P} . One can see that in this case all axioms of norm hold, and one can construct a Banach space, with the norm-closure corresponding to an approximation criterion for propensities (see also Remark 5).*

Remark 8 (Duality between the convex sets of states and of propensities) *From the Definition 2 of state it follows that the convex set of states \mathfrak{S} and the convex sets of propensities \mathfrak{P} are dual each other, and the latter can be regarded as the set of positive linear contractions over the set of states, namely the set of positive functionals l on \mathfrak{S} with unit upper bound, and with the functional $l_{[\mathcal{A}]}$ corresponding to the propensity $[\mathcal{A}]$ being defined as*

$$l_{[\mathcal{A}]}(\omega) \doteq \omega(\mathcal{A}). \quad (34)$$

In the following we will often identify propensities with their corresponding functionals, and denote them by lowercase letters a, b, c, \dots , or l_1, l_2, \dots . Finally, notice that the notion of coexistence (informational compatibility) extends naturally to propensities.

Definition 13 (Observable) *We call observable a set of propensities $\mathbb{L} = \{l_i\}$ which is informationally equivalent to an action $\mathbb{L} \in \underline{\mathbb{A}}$, namely such that there exists an action $\mathbb{A} = \{\mathcal{A}_j\}$ for which one has $l_i \in \underline{\mathcal{A}}_j$.*

Clearly, the generalized observable is normalized to the constant unit functional, i. e. $\sum_i l_i = 1$.

Definition 14 (Informationally complete observable) *An observable $\mathbb{L} = \{l_i\}$ is informationally complete if each propensity can be written as a linear combination of the elements of \mathbb{L} , namely for each propensity l there exist coefficients $c_i(l)$ such that*

$$l = \sum_i c_i(l) l_i. \quad (35)$$

We call the informationally complete observable minimal when its propensities are linearly independent.

Clearly, using an informationally complete observable one can reconstruct any state ω from just the probabilities $l_i(\omega)$, since one has

$$\omega(\mathcal{A}) = \sum_i c_i(l_{\mathcal{A}}) l_i(\omega). \quad (36)$$

Based on the notion of informationally complete observable, we can introduce the following one

Definition 15 (Experimentally sufficient set of transformations) *We call a set of transformations \mathfrak{t} experimentally sufficient if it has a subset that is in correspondence with an informationally complete observable.*

Using the above notion we can introduce a norm $\|\cdot\|_{\mathfrak{t}}$ for generalized weights, generalizing the norm given in Eq. (25), by taking the supremum over \mathfrak{t} instead of \mathfrak{T} . The fact that the set of transformations is experimentally sufficient guarantees that $\|\tilde{\omega}\|_{\mathfrak{t}} = 0$ implies that $\tilde{\omega} = 0$. The restriction to a set \mathfrak{t} of transformations may be operationally motivated. An analogous restriction may be considered for the norm of generalized transformations, by restricting the set of states \mathfrak{S} .

Definition 16 (Predictability and resolution) *We will call a transformation \mathcal{A} —and likewise its propensity—predictable if there exists a state for which \mathcal{A} occurs with certainty and some other state for which it never occurs. The transformation (propensity) will be also called resolved if the state for which it occurs with certainty is unique—whence pure. An action will be called predictable when it is made only of predictable transformations, and resolved when all transformations are resolved.*

The present notion of predictability for propensity corresponds to that of "decision effects" of Ludwig [8]. For a predictable transformation \mathcal{A} one has $\|\mathcal{A}\| = 1$. Notice that a predictable transformation is not deterministic, and it can generally occur with nonunit probability on some state ω . Predictable propensities \mathcal{A} correspond to affine functions $f_{\mathcal{A}}$ on the state space \mathfrak{S} with $0 \leq f_{\mathcal{A}} \leq 1$ achieving both bounds. Their set will be denoted by \mathfrak{P}_p .

Definition 17 (Perfectly discriminable set of states) *We call a set of states $\{\omega_n\}_{n=1,N}$ perfectly discriminable if there exists an action $\mathbb{A} = \{\mathcal{A}_j\}_{j=1,N}$ with transformations*

$\mathcal{A}_j \in I_j$ corresponding to a set of predictable propensities $\{l_n\}_{n=1,N}$ satisfying the relation

$$l_n(\omega_m) = \delta_{nm}. \quad (37)$$

Definition 18 (Informational dimensionality) We call informational dimension of the convex set of states \mathfrak{S} , denoted by $\text{idim}(\mathfrak{S})$, the maximal cardinality of perfectly discriminable set of states in \mathfrak{S} .

Definition 19 (Discriminating observable) An observable $\mathbb{L} = \{I_j\}$ is discriminating for \mathfrak{S} when it discriminates a set of states with cardinality equal to the informational dimension $\text{idim}(\mathfrak{S})$ of \mathfrak{S} .

FAITHFUL STATE

Definition 20 (Dynamically faithful state) We say that a state Φ of a composite system is dynamically faithful for the n th component system when acting on it with a transformation \mathcal{A} results in an (unnormalized) conditional state that is in one-to-one correspondence with the dynamical equivalence class $[\mathcal{A}]_{\text{dyn}}$ of \mathcal{A} , namely the following map is one-to-one:

$$\tilde{\Phi}_{\mathcal{I}, \dots, \mathcal{I}, \mathcal{A}, \mathcal{I}, \dots} \leftrightarrow [\mathcal{A}]_{\text{dyn}}, \quad (38)$$

where in the above equation the transformation \mathcal{A} acts locally only on the n th component system.

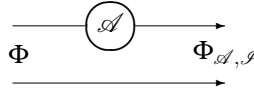


FIGURE 1. Illustration of the notion of dynamically faithful state: the conditioned state $\Phi_{\mathcal{A}, \mathcal{I}}$ is in one-to-one correspondence with the dynamical equivalence class of the transformation \mathcal{A} .

In the following for simplicity we restrict attention to two component systems, and take the first one for the n th. Using the definition 4 of conditional state, we see that the state Φ is dynamically faithful when the map $\Phi(\cdot \circ [\mathcal{A}, \mathcal{I}]_{\text{dyn}})$ is invertible over the set of dynamical equivalence classes of transformations, namely when

$$\forall \mathcal{B}, \Phi(\mathcal{B} \circ (\mathcal{A}_1, \mathcal{I})) = \Phi(\mathcal{B} \circ (\mathcal{A}_2, \mathcal{I})) \iff \mathcal{A}_1 \in [\mathcal{A}_2]_{\text{dyn}}. \quad (39)$$

Definition 21 (Preparationally faithful state) We will call a state Φ of a bipartite system preparationally faithful if all local states of one component can be achieved by a suitable local transformation of the other, namely for every state ω of the first party there exists a local transformation \mathcal{T}_ω of the other party for which the conditioned local state coincides with ω , namely

$$\forall \omega \in \mathfrak{S} \quad \exists \mathcal{T}_\omega : \quad \omega = \Phi_{\mathcal{T}_\omega, \mathcal{I}}|_2 \doteq \frac{\Phi(\mathcal{T}_\omega, \cdot)}{\Phi(\mathcal{T}_\omega, \mathcal{I})}. \quad (40)$$

In Postulate 5 we also use the notion of *symmetric* joint state, defined as follows.

Definition 22 (Symmetric joint state of two identical systems) *We call a joint state of two identical systems symmetric if for a particular choice of local informationally complete measurements one has*

$$\Phi(\underline{\mathcal{X}}_i, \underline{\mathcal{X}}_j) = \Phi(\underline{\mathcal{X}}_j, \underline{\mathcal{X}}_i), \quad \forall i, j. \quad (41)$$

We clearly have

$$\Phi(\underline{\mathcal{A}}, \underline{\mathcal{B}}) = \Phi(\underline{\mathcal{B}}, \underline{\mathcal{A}}), \quad (42)$$

for any couple of propensities $\underline{\mathcal{A}}$ and $\underline{\mathcal{B}}$. Therefore, the choice of the local informationally complete measurement is irrelevant. Moreover, for a symmetric faithful state we have

$$\Phi|_1(\underline{\mathcal{A}}) = \Phi|_1(\underline{\mathcal{A}}') = \Phi|_2(\underline{\mathcal{A}}) = \Phi|_2(\underline{\mathcal{A}}'), \quad (43)$$

and for a symmetric preparationally faithful state we have

$$\omega = \Phi_{\mathcal{T}_\omega, \mathcal{I}}|_2 = \Phi_{\mathcal{I}, \mathcal{T}_\omega}|_1. \quad (44)$$

THE BLOCH REPRESENTATION

In this section we introduce an affine-space representation based on the existence of a minimal informational complete observable. Such representation generalizes the popular Bloch representation used in Quantum Mechanics.

Let's fix a minimal informationally complete observable, denoted by $\{n_j\}$, in terms of which we can expand (in a unique way) any propensity as follows

$$l_{\underline{\mathcal{A}}} = \sum_j m_j(\underline{\mathcal{A}}) n_j. \quad (45)$$

It is convenient to replace one element of the informationally complete observable $\{n_j\}$ with the *normalization functional* n_0 defined as

$$n_0(\tilde{\omega}) = \tilde{\omega}(\underline{\mathcal{I}}), \quad \forall \tilde{\omega} \in \tilde{\mathfrak{S}}, \quad (46)$$

[$n_0(\omega) = 1$ for normalized states ω]. We will then use the Minkowskian notation

$$n \doteq (n_0, n), \quad m \doteq (m_0, m), \quad mn \doteq \sum_j m_j n_j = m \cdot n + m_0 n_0. \quad (47)$$

In the following we will also denote $q \doteq m_0$. Therefore, for any propensity $\underline{\mathcal{A}}$, we will write

$$l_{\underline{\mathcal{A}}}(\omega) = m(\underline{\mathcal{A}})n(\omega) \equiv m(\underline{\mathcal{A}}) \cdot n(\omega) + q(\underline{\mathcal{A}}). \quad (48)$$

Clearly one can extend the convex set of propensities \mathfrak{P} to the complexification $\mathbb{C}\mathfrak{P}$ of the underlying affine space, by keeping the coefficients m_j of the expansion as complex, namely a generic element $l \in \mathbb{C}\mathfrak{P}$ will be given by

$$l = \sum_j m_j n_j, \quad m_j \in \mathbb{C}. \quad (49)$$

Notice that $n(\omega)$ gives a complete description of the state ω , since for any transformation \mathcal{A} one can write

$$\omega(\mathcal{A}) = m(\mathcal{A}) \cdot n(\omega) + q(\mathcal{A}). \quad (50)$$

On the other hand, by denoting with \mathcal{X}_j and l_j the propensity such that $[m(\mathcal{X}_j)]_l = \delta_{jl}$ we have

$$n_j(\omega) = l_{\mathcal{X}_j}(\omega) \doteq l_j(\omega). \quad (51)$$

Notice that $\mathcal{X}_0 \equiv \mathcal{I}$. We will call $n(\omega)$ the Bloch vector representing the state ω . The Bloch representation is *faithful* (i. e. one-to-one), since the informationally complete observable $\{l_j\}$ is minimal, namely the functionals l_j are linearly independent. We also emphasize that the representation is trivially extended to generalized weights, transformations and propensities.

We now recover the linear transformation describing conditioning, given in terms of the *operation*, which we remind is given in terms of the unnormalized state $\text{Op}_{\mathcal{A}} \omega \equiv \tilde{\omega}_{\mathcal{A}}$ defined as follows

$$\text{Op}_{\mathcal{A}} \omega(\mathcal{B}) \equiv \tilde{\omega}_{\mathcal{A}}(\mathcal{B}) = \omega(\mathcal{B} \circ \mathcal{A}) = \omega(\underline{\mathcal{B}} \circ \mathcal{A}) \equiv l_{\underline{\mathcal{B}}}(\tilde{\omega}_{\mathcal{A}}). \quad (52)$$

From linearity of transformations (see Eq. (21) and Remark 3), upon introducing a matrix $\{M_{jl}(\mathcal{A})\}$, one can write

$$\omega(\mathcal{X}_j \circ \mathcal{A}) = \sum_l M_{jl}(\mathcal{A}) l_l(\omega) + M_{j0}(\mathcal{A}), \quad (53)$$

and, in particular,

$$\omega(\mathcal{X}_0 \circ \mathcal{A}) \equiv \omega(\mathcal{A}) = \sum_l M_{0l}(\mathcal{A}) n_l(\omega) \equiv m(\mathcal{A}) \cdot n(\omega) + q(\mathcal{A}), \quad (54)$$

from which we derive the identities

$$M_{0l}(\mathcal{A}) \equiv [m(\mathcal{A})]_l, \quad M_{00}(\mathcal{A}) \equiv q(\mathcal{A}). \quad (55)$$

The real matrices $M_{jl}(\mathcal{A})$ are a *representation* of the real algebra of transformations \mathcal{A} . The first row of the matrix is a representation of the propensity \mathcal{A} (see Fig. 2).

In the Bloch-vector notation, one has

$$n_j(\tilde{\omega}_{\mathcal{A}}) = l_{\mathcal{X}_j}(\tilde{\omega}_{\mathcal{A}}) = \omega(\mathcal{X}_j \circ \mathcal{A}), \quad n_0(\tilde{\omega}_{\mathcal{A}}) = l_{\mathcal{X}_0}(\tilde{\omega}_{\mathcal{A}}) = \omega(\mathcal{A}). \quad (56)$$

$$\begin{aligned} n(\tilde{\omega}_{\mathcal{A}}) &= M(\mathcal{A})n(\omega) + k(\mathcal{A}), \\ k_j(\mathcal{A}) &\doteq q(\mathcal{X}_j \circ \mathcal{A}), \quad n_0(\tilde{\omega}_{\mathcal{A}}) = m(\mathcal{A}) \cdot n(\omega) + q(\mathcal{A}), \end{aligned} \quad (57)$$

$$\tilde{\omega}_{\mathcal{A}}(\mathcal{B}) = m(\mathcal{B}) \cdot n(\tilde{\omega}_{\mathcal{A}}) + q(\mathcal{B})n_0(\tilde{\omega}_{\mathcal{A}}) \quad (58)$$

The matrix representation of the transformation is synthesized in Fig. 2. Since the

$$M_{ij}(\mathcal{A}) = \begin{pmatrix} q(\underline{\mathcal{A}}) & m(\underline{\mathcal{A}}) \\ k(\underline{\mathcal{A}}) & M(\underline{\mathcal{A}}) \end{pmatrix}$$

FIGURE 2. Matrix representation of the real algebra of transformations \mathcal{A} . The first row represents the propensity $\underline{\mathcal{A}}$ of the transformation \mathcal{A} . It gives the transformation of the zero-component of the Bloch vector $n_0(\underline{\omega}_{\mathcal{A}}) \equiv \omega(\underline{\mathcal{A}}) = m(\underline{\mathcal{A}}) \cdot n(\omega) + q(\underline{\mathcal{A}})$, namely the probability of the transformation. The following rows represent the affine transformation of the Bloch vector $n(\omega)$ corresponding to the quantum operation $\text{Op}_{\mathcal{A}}$, the first column giving the translation $k(\underline{\mathcal{A}})$, and the remaining square matrix $M(\underline{\mathcal{A}})$ the linear part. Overall, the Bloch vector of the state ω is transformed as $n(\text{Op}_{\mathcal{A}} \omega) = Mn(\omega) + k(\underline{\mathcal{A}})$.

Bloch representation is faithful, then the dimension of the affine space of the Bloch vector $n(\omega)$ is just the affine dimension $\text{adm}(\mathfrak{S})$ of the convex set of states \mathfrak{S} .

Therefore, summarizing we have the following conditioning transformation

$$n(\omega) \longrightarrow n(\omega_{\mathcal{A}}) = \frac{M(\underline{\mathcal{A}})n(\omega) + k(\underline{\mathcal{A}})}{m(\underline{\mathcal{A}}) \cdot n(\omega) + q(\underline{\mathcal{A}})}, \quad (59)$$

with the transformation occurring with probability given by

$$p(\underline{\mathcal{A}}; \omega) = m(\underline{\mathcal{A}}) \cdot n(\omega) + q(\underline{\mathcal{A}}). \quad (60)$$

Using a joint local informationally complete observable, we can build a Bloch representation of joint states and of transformations of the composed system. We introduce the dual tensor notation $n \odot n$ with the following meaning

$$(n \odot n)_{ij}(\Phi) \equiv n_i \odot n_j(\Phi) \doteq l_{\underline{\mathcal{X}}_i, \underline{\mathcal{X}}_j}(\Phi), \quad i, j = 0, 1, \dots \quad (61)$$

and with the matrix composition rule

$$(M(\underline{\mathcal{A}}) \odot M(\underline{\mathcal{B}}))(n \odot n)(\Phi) = (M(\underline{\mathcal{A}})n \odot M(\underline{\mathcal{B}})n)(\Phi), \quad (62)$$

corresponding to the probability rule

$$\Phi(\underline{\mathcal{X}}_i \circ \underline{\mathcal{A}}, \underline{\mathcal{X}}_j \circ \underline{\mathcal{B}}) = (M(\underline{\mathcal{A}})n \odot M(\underline{\mathcal{B}})n)_{ij}(\Phi) \quad (63)$$

which follows from Eq. (53) along with the conditioning rule and the notion of local state. For example, more explicitly for $i, j = 1, 2, \dots$, one has

$$\begin{aligned} \Phi(\underline{\mathcal{X}}_i \circ \underline{\mathcal{A}}, \underline{\mathcal{X}}_j \circ \underline{\mathcal{B}}) &= (M(\underline{\mathcal{A}})n \odot M(\underline{\mathcal{B}})n)_{ij}(\Phi) + (k(\underline{\mathcal{A}}))n_0 \odot M(\underline{\mathcal{B}})n)_{ij}(\Phi) \\ &\quad + (M(\underline{\mathcal{A}})n \odot k(\underline{\mathcal{B}})n_0)_{ij}(\Phi) + k_i(\underline{\mathcal{A}})k_j(\underline{\mathcal{B}}) \end{aligned} \quad (64)$$

where we used the identity $(n_0 \odot n_0)(\Phi) = 1$. It is easy to see that the representation of the local states $\Omega|_1 = \Omega(\cdot, \mathcal{I})$ and $\Omega|_2 = \Omega(\mathcal{I}, \cdot)$ are simply given by

$$n(\Omega|_1) = (n \odot n_0)(\Omega), \quad n(\Omega|_2) = (n_0 \odot n)(\Omega). \quad (65)$$

OPERATIONAL ADJOINT AND REAL HILBERT SPACE STRUCTURE

In this section we will see how it is possible to define operationally a real adjoint map (i. e. a transposition) using a symmetric faithful state, and how using such adjoint one can introduce a Hilbert space structure via two different constructions: the Mackey-Kakutani and the Gelfand-Naimark-Segal constructions.

Twin involution

We now define the *twin* involution over transformations.

Definition 23 *For a faithful bipartite state Φ , the twin \mathcal{A}' of the transformation \mathcal{A} is that transformation which when applied to the second component system gives the same conditioned state and with the same probability than the transformation \mathcal{A} operating on the first system. In equations, one has*

$$\tilde{\Phi}_{\mathcal{A}, \mathcal{I}} = \tilde{\Phi}_{\mathcal{I}, \mathcal{A}'} \quad (66)$$

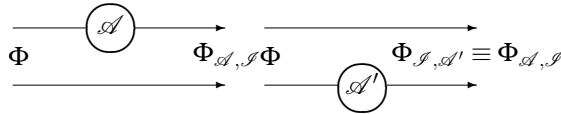


FIGURE 3. Illustration of the concept of *twin* involution.

Notice that, by definition, independently on the faithful state Φ we always have trivially

$$\mathcal{I}' = \mathcal{I}. \quad (67)$$

We now derive the Bloch matrix representation of the twin involution. The bipartite state in the Bloch form is represented by the matrix

$$F_{ij} \doteq n_i \odot n_j(\Phi). \quad (68)$$

The matrix F is real and invertible, as a consequence of faithfulness of state Φ (by definition the correspondence $\tilde{\Phi}_{\mathcal{A}, \mathcal{I}} \leftrightarrow \mathcal{A}$ is one-to-one). Indeed, a transformation \mathcal{A} on the first system is described by the matrix multiplication

$$n_i \odot n_j(\tilde{\Phi}_{\mathcal{A}, \mathcal{I}}) = \sum_k A_{ik} F_{kj} = (AF)_{ij}, \quad (69)$$

where $A \doteq M(\mathcal{A})$. On the other hand, a transformation \mathcal{A} on the second system is represented as

$$n_i \odot n_j(\tilde{\Phi}_{\mathcal{I}, \mathcal{A}}) = \sum_k A_{jk} F_{ik} = (FA^\tau)_{ij}. \quad (70)$$

One can also check the composition rules

$$n_i \odot n_j(\tilde{\Phi}_{\mathcal{B} \circ \mathcal{A}, \mathcal{I}}) = (BAF)_{ij}, \quad (71)$$

$$n_i \odot n_j(\tilde{\Phi}_{\mathcal{I}, \mathcal{B} \circ \mathcal{A}}) = (FA^\tau B^\tau)_{ij} \equiv (F(BA)^\tau)_{ij}. \quad (72)$$

Also, if one considers another faithful state Ψ which is obtained by applying an invertible deterministic transformation \mathcal{M} to the first system in the joint state Φ , namely

$$\Psi = \Phi_{\mathcal{M}, \mathcal{I}}, \quad (73)$$

then the matrix F in Eqs. (71) and (72) is substituted by the matrix MF . The defining identity (66) now corresponds to the matrix identity

$$AF = F(A')^\tau, \quad (74)$$

namely the twin involution is given by

$$A' = F^\tau A^\tau F^{\tau-1}. \quad (75)$$

If the faithful state Φ is also *symmetric*, the twin involution satisfies all four axioms of *generalized adjoint*:

Definition 24 (Generalized adjoint) 1. $(A+B)' = A' + B'$, 2. $(A')' = A$, 3. $(AB)' = B'A'$, 4. $A'A = 0 \implies A = 0$.

Indeed, a faithful symmetric state has a Bloch representation in terms of a symmetric matrix F in Eq. (68). Therefore, the first three axioms are obvious. We just need to check the last axiom. For this purpose we need the following simple lemma

Lemma 1 *The following implication holds*

$$A^\tau A = 0 \implies A = 0. \quad (76)$$

Proof. Using the real polar decomposition $A = PR$, with $P \geq 0$ positive symmetric and $RR^\tau = R^\tau R = I$ (rotation matrix), one has that $A^\tau A = R^\tau P^2 R$ has all positive eigenvalues, each one is the square of the corresponding eigenvalue of P , whence $A^\tau A = 0$ if all eigenvalues of P are zero, namely $P = 0$, or, equivalently, $A = PR = 0$, since R is invertible. ■

We can now check that axiom 4. for the real adjoint holds for symmetric F , namely Postulate 5 implies the existence of a *transposition* (the real equivalent of the *adjoint*), which can be operationally defined via the twin involution on a faithful symmetric state.

Theorem 4 (Operational adjoint) *The existence of a symmetric faithful bipartite states guarantees the existence of a transposition on the real algebra \mathcal{A} of transformations.*

Proof. Suppose that there exists a symmetric faithful state Φ . Its matrix F is symmetric invertible. Then also F^{-1} is symmetric. By real polar decomposition of A , we write

$$A'A = F^\tau R^\tau P F^{-1/2} F^{-1/2} P R, \quad (77)$$

and invertibility of F implies that $A'A = 0$ is equivalent to

$$R^\tau P F^{-1/2} F^{-1/2} P R = 0, \quad (78)$$

and using Lemma 1 one has

$$F^{-1/2} P R = 0, \quad (79)$$

namely $A = P R = 0$. This proves identity 4. of Definition 24, completing the list of requirements that the twin involution must satisfy in order to be a generalized adjoint. ■

Lemma 2 *For a faithful symmetric state Φ the following identities hold*

$$\check{\Phi}_{\mathcal{A}, \mathcal{B}} = \check{\Phi}_{\mathcal{I}, \mathcal{B} \circ \mathcal{A}'} = \check{\Phi}_{\mathcal{A} \circ \mathcal{B}', \mathcal{I}}. \quad (80)$$

Proof.

$$\check{\Phi}_{\mathcal{A}, \mathcal{B}} = (\check{\Phi}_{\mathcal{A}, \mathcal{I}})_{\mathcal{I}, \mathcal{B}} = (\check{\Phi}_{\mathcal{I}, \mathcal{A}'})_{\mathcal{I}, \mathcal{B}} = \check{\Phi}_{\mathcal{I}, \mathcal{B} \circ \mathcal{A}'} = \check{\Phi}_{\mathcal{A} \circ \mathcal{B}', \mathcal{I}}. \quad (81)$$

■

Definition 25 (Real positive form) *A linear form φ over the algebra of transformations \mathcal{A} is called real positive (with respect to the real adjoint $\mathcal{A} \rightarrow \mathcal{A}'$) if $\forall \mathcal{A} \in \mathcal{A}$ it satisfies the following identities*

- a) $\varphi(\mathcal{A}') = \varphi(\mathcal{A})$,
- b) $\varphi(\mathcal{A}' \circ \mathcal{A}) \geq 0$.

Theorem 5 *The local state $\Phi|_1 = \Phi|_2$ of a symmetric faithful state Φ is a real positive form over \mathcal{A} .*

Proof. From identity (43) we have that $\Phi|_1 = \Phi|_2$. Condition a) also follows from the same identity. On the other hand, the condition b) holds also for generalized transformations, since a generalized transformation is always a multiple of a physical one by a real scalar. ■

Mackey-Kakutani (MK) construction of real Hilbert space structure

In the following we will show how the existence of a generalized adjoint over transformations allows us to derive a structure of real Hilbert space over generalized weights. For this purpose we need the following two theorems by Mackey and Kakutani[2].

Theorem 6 (Mackey-Kakutani I) *[Ref. [2]]. Let \mathfrak{B} be a real Banach space, and \mathfrak{K} the ring of continuous linear transformations of \mathfrak{B} into itself. Then \mathfrak{B} is isomorphic to a (generally non separable) real Hilbert space \mathbb{H} if and only if there is an operation $\mathcal{T} \rightarrow \mathcal{T}'$ from \mathfrak{K} to \mathfrak{K} which has the properties of definition (24).*

Theorem 7 (Mackey-Kakutani II) [Ref. [2]]. *The isomorphism in Theorem 6 may be set up in such a manner that the correspondence $\mathcal{T} \rightarrow \mathcal{T}'$ goes over into the correspondence between its operator and its adjoint. In other words, \mathfrak{B} may be provided with a positive definite symmetric bilinear inner product (x, y) such that the new norm $\|x\|$ in \mathfrak{B} defined by the equation $\|x\| = \sqrt{(x, x)}$ is equivalent to the given norm $\|x\|$ and such that for all x and y in \mathfrak{B} $(T(x), y) = (x, T'(y))$.*

Theorems 6 and 7 entail the following Hilbert space formulation:

Remark 9 (Hilbert space structure for the Banach space of generalized weights)
Take for \mathfrak{B} the Banach space of generalized weights \mathfrak{W} and for \mathfrak{R} the ring of linear transformations of \mathfrak{W} according to Eq. (15). Then Theorems 6 and 7 assert that the space of generalized weights \mathfrak{W} is isomorphic to a real Hilbert space H , and that it is possible to choose the scalar product in such a way that the twin transform corresponds to the real-adjoint—i. e. the transposition—and the norm is equivalent to the one induced by the scalar product. The Riesz theorem implies that the affine space of generalized propensities (linear real forms over states or, equivalently, over generalized weights) is itself a real Hilbert space isomorphic to H .

Gelfand-Naimark-Segal (GNS) construction of real Hilbert space structure

With the introduction of a generalized adjoint given in Definition in 24 corresponding to the operational concept of twin involution, the real algebra \mathcal{A} of generalized transformations becomes a real $*$ -algebra. Then each real positive form φ over the $*$ -algebra \mathcal{A} —e. g. the local state $\varphi \doteq \Phi|_1$ of a faithful symmetric state Φ —defines a Hilbert space H_φ and a representation π_φ of \mathcal{A} by linear operators acting on H_φ . Indeed, \mathcal{A} is a linear space over \mathbb{R} and φ defines a symmetric (positive semi-definite) scalar product on \mathcal{A} as follows

$$\varphi\langle \mathcal{A} | \mathcal{B} \rangle_\varphi \doteq \varphi(\mathcal{A}' \circ \mathcal{B}) \equiv \Phi(\mathcal{A}', \mathcal{B}'), \quad \mathcal{A}, \mathcal{B} \in \mathcal{A}, \quad (82)$$

where we remind the use of notation defined in Eq. (9). Indeed, condition a) of Definition 25 implies the symmetry $\varphi\langle \mathcal{B} | \mathcal{A} \rangle_\varphi = \varphi\langle \mathcal{A} | \mathcal{B} \rangle_\varphi$, whereas condition b) implies the positivity $\varphi\langle \mathcal{A} | \mathcal{A} \rangle_\varphi \geq 0$. Also, it is easy to check that

$$\varphi\langle \mathcal{C}' \circ \mathcal{A} | \mathcal{B} \rangle_\varphi = \varphi\langle \mathcal{A} | \mathcal{C} \circ \mathcal{B} \rangle_\varphi, \quad (83)$$

as it can be derived from the definition (82) as follows

$$\begin{aligned} \varphi\langle \mathcal{C}' \circ \mathcal{A} | \mathcal{B} \rangle_\varphi &= \Phi(\mathcal{A}' \circ \mathcal{C}, \mathcal{B}') = \tilde{\Phi}_{\mathcal{C}, \mathcal{A}}(\mathcal{A}', \mathcal{B}') = \tilde{\Phi}_{\mathcal{A}, \mathcal{C}'}(\mathcal{A}', \mathcal{B}') \\ &= \Phi(\mathcal{A}', \mathcal{B}' \circ \mathcal{C}') = \varphi\langle \mathcal{A} | \mathcal{C} \circ \mathcal{B} \rangle_\varphi \end{aligned} \quad (84)$$

Symmetry and positivity imply the bounding

$$\varphi\langle \mathcal{A} | \mathcal{B} \rangle_\varphi \leq \sqrt{\varphi\langle \mathcal{A} | \mathcal{A} \rangle_\varphi \varphi\langle \mathcal{B} | \mathcal{B} \rangle_\varphi}. \quad (85)$$

Using the bounding (85) for the scalar product $\varphi\langle \mathcal{A}' \circ \mathcal{A} \circ \mathcal{X} | \mathcal{X} \rangle_\varphi$ we can easily see that the set $\mathcal{J} \subseteq \mathcal{A}$ consisting of all elements $\mathcal{X} \in \mathcal{A}$ with $\varphi(\mathcal{X}' \circ \mathcal{X}) = 0$ is a left ideal, i. e. a linear subspace of \mathcal{A} which is stable under multiplication by any element of \mathcal{A} on the left (i. e. $\mathcal{X} \in \mathcal{J}$, $\mathcal{A} \in \mathcal{A}$ implies $\mathcal{A} \circ \mathcal{X} \in \mathcal{J}$). The set of equivalence classes \mathcal{A}/\mathcal{J} thus becomes a real pre-Hilbert space equipped with a symmetric scalar product, an element of the space being an equivalence class. Notice that the scalar product does not depend on the algebraic representatives chosen for classes, namely

$$\varphi\langle \{\mathcal{A}\} | \{\mathcal{B}\} \rangle_\varphi = \varphi\langle \mathcal{A} | \mathcal{B} \rangle_\varphi, \quad \forall \mathcal{A} \in \{\mathcal{A}\}, \forall \mathcal{B} \in \{\mathcal{B}\}, \quad (86)$$

$\{\mathcal{A}\}$ denoting the equivalence class containing \mathcal{A} . For the equivalence classes we can define the norm

$$\|\mathcal{X}\|_\varphi^2 \doteq \varphi\langle \mathcal{X} | \mathcal{X} \rangle_\varphi, \quad \mathcal{X} \in \mathcal{A}/\mathcal{J}. \quad (87)$$

We keep the subindex φ for the norm in order to distinguish it from the previously defined norm (22). The Hilbert space is then obtained by completion of \mathcal{A}/\mathcal{J} in the norm topology (the Hilbert space closure is not operationally relevant: see Remark 5). The product in \mathcal{A} defines the action of \mathcal{A} on the vectors in \mathcal{A}/\mathcal{J} , by associating to each element $\mathcal{A} \in \mathcal{A}$ the linear operator $\pi_\varphi(\mathcal{A})$ defined on the dense domain $\mathcal{A}/\mathcal{J} \subseteq \mathsf{H}_\varphi$ as follows

$$\pi_\varphi(\mathcal{A})|\mathcal{X}\rangle_\varphi \doteq |\{\mathcal{A} \circ \mathcal{B}\}\rangle_\varphi, \quad \mathcal{X} = \{\mathcal{B}\}. \quad (88)$$

The norm (87) can be extended to a seminorm on the whole \mathcal{A} as follows

$$\|\mathcal{A}\|_\varphi \doteq \|\{\mathcal{A}\}\|_\varphi = \sqrt{\varphi\langle \{\mathcal{A}\} | \{\mathcal{A}\} \rangle_\varphi}. \quad (89)$$

On the other hand, on \mathcal{A}/\mathcal{J} one can easily verify that $\|\cdot\|_\varphi$ indeed satisfies all axioms of norm, since clearly $\|\mathcal{A}\|_\varphi = 0$ implies that $\mathcal{A} \in \mathcal{J}$, corresponding to the null vector, and

$$\begin{aligned} \|\{\lambda\mathcal{A}\}\|_\varphi &= \|\lambda\{\mathcal{A}\}\|_\varphi = \lambda\|\{\mathcal{A}\}\|_\varphi, \\ \|\{\mathcal{A} + \mathcal{B}\}\|_\varphi &= \|\{\mathcal{A}\} + \{\mathcal{B}\}\|_\varphi \leq \|\{\mathcal{A}\}\|_\varphi + \|\{\mathcal{B}\}\|_\varphi. \end{aligned} \quad (90)$$

If \mathcal{A} were a Banach *-algebra the domain of definition of $\pi_\varphi(\mathcal{A})$ could be easily extended to the whole H_φ by continuity, since to a Cauchy sequence $\mathcal{X}_n \in \mathcal{A}/\mathcal{J}$ there correspond Cauchy sequences $\mathcal{A}_n, \mathcal{B}_n \in \mathcal{X}_n$ as a consequence of the norm bounding

$$\begin{aligned} \|\pi_\varphi(\mathcal{A})\mathcal{X}_n - \pi_\varphi(\mathcal{A})\mathcal{X}_m\|_\varphi &= \|\{\mathcal{A}(\mathcal{B}_n - \mathcal{B}_m)\}\|_\varphi = \|\mathcal{A}(\mathcal{B}_n - \mathcal{B}_m)\|_\varphi \\ &\leq \|\mathcal{A}\|_\varphi \|\mathcal{B}_n - \mathcal{B}_m\|_\varphi. \end{aligned} \quad (91)$$

However, the last step is not necessarily true, since conditions $\|\mathcal{B} \circ \mathcal{A}\|_\varphi \leq \|\mathcal{B}\|_\varphi \|\mathcal{A}\|_\varphi$, and $\|\mathcal{A}'\|_\varphi = \|\mathcal{A}\|_\varphi$ do not necessarily hold, whence the possibility of representing generalized transformations as operators over H_φ remains an open problem for the infinite dimensional case. Also, the use of the seminorm (30) closure is not of much help, since one can just prove that

$$\|\mathcal{A}\|_\varphi \leq \|\mathcal{A}'\|, \quad \|\mathcal{A}\|_\varphi^2 \leq \|\mathcal{A}'\| \|\mathcal{A}\|, \quad (92)$$

but we cannot prove a bounding $\|\mathcal{B}\| \leq \|\mathcal{X}\|_\varphi$, $\mathcal{B} \in \mathcal{X}$. The first bound in Eq. (92) can be derived as follows

$$\|\mathcal{A}\|_\varphi = \Phi(\mathcal{A}', \mathcal{A}') = \Phi_{\mathcal{I}, \mathcal{A}'}(\mathcal{A}', \mathcal{I})\Phi(\mathcal{I}, \mathcal{A}') = \Phi_{\mathcal{I}, \mathcal{A}'}|_1(\mathcal{A}')\Phi|_2(\mathcal{A}') \leq \|\mathcal{A}'\|^2, \quad (93)$$

where Φ is any faithful state corresponding to φ . The second bound in Eq. (92) is implied by the inequality

$$\|\mathcal{A}\|_\varphi^2 = \varphi(\mathcal{A}' \circ \mathcal{A}) \leq \|\mathcal{A}'\mathcal{A}\| \leq \|\mathcal{A}'\|\|\mathcal{A}\|. \quad (94)$$

Also we do not have that $\|\mathcal{A}'\| = \|\mathcal{A}\|$, not even $\|\mathcal{A}'\| \leq \|\mathcal{A}\|$.

In terms of the faithful state Φ and of its Bloch representation the scalar product (82) rewrites as

$$\varphi\langle \mathcal{A} | \mathcal{B} \rangle_\varphi = \Phi(\mathcal{A}', \mathcal{B}') = (A'FB'^\tau)_{00} = (F^\tau A^\tau F^{\tau-1}BF)_{00}. \quad (95)$$

Remark 10 (Pairing between states and propensities) *From the definition (82) of the scalar product we have*

$$\varphi\langle \mathcal{A}' | \mathcal{B} \rangle_\varphi = \varphi_{\mathcal{B}}(\mathcal{A})\varphi(\mathcal{B}) = \Phi_{\mathcal{I}, \mathcal{B}'}|_1(\mathcal{A}), \quad (96)$$

and if we assume that the state Φ is preparationally faithful, then for every state ω there exists a transformation \mathcal{T}_ω such that $\omega = \Phi_{\mathcal{I}_\omega, \mathcal{I}}|_2 = \Phi_{\mathcal{I}, \mathcal{I}_\omega}|_1 = \varphi_{\mathcal{I}'_\omega}$ with $\varphi(\mathcal{I}_\omega) \neq 0$. Then one has

$$\omega(\mathcal{A}) = \varphi\langle \mathcal{A}' | \tilde{\mathcal{T}}_\omega \rangle_\varphi = \varphi\langle \underline{\mathcal{A}}' | \tilde{\mathcal{T}}_\omega \rangle_\varphi, \quad \tilde{\mathcal{T}}_\omega = \frac{\mathcal{I}'_\omega}{\varphi(\mathcal{I}_\omega)}, \quad (97)$$

and we recover the pairing between states and propensities in terms of the scalar product.

Notice that state φ is cyclic. Eq. (97) along with the bounds in Eq. (92) imply the following theorem

Theorem 8 *Two (bounded) generalized transformations belong to the same equivalence class in \mathcal{A}/\mathcal{I} if and only if they are informationally equivalent, namely $\mathcal{A} \in \{\mathcal{B}\} \Leftrightarrow \mathcal{A} \in \underline{\mathcal{B}}$.*

Proof. If \mathcal{A} is informational equivalent to \mathcal{B} , then $\omega(\mathcal{A} - \mathcal{B}) = 0 \forall \omega \in \mathfrak{S}$, which implies that $\|\mathcal{A} - \mathcal{B}\| = 0$, whence, according to the second bound in Eq. (92), $\|\mathcal{A} - \mathcal{B}\|_\varphi = 0$ if both \mathcal{A} and \mathcal{B} are bounded (for any generalized transformation with bounded norm $\|\mathcal{A}\|$ one has $\|\mathcal{A}'\| < \infty$, since one can write $\mathcal{A} = \lambda \mathcal{T}$, with \mathcal{T} a true transformation and $|\lambda| < \infty$, and \mathcal{T}' bounded, being \mathcal{T}' a true transformation by definition of the real adjoint). This means that $\mathcal{A} = \mathcal{B} + \mathcal{X}$, with $\mathcal{X} \in \mathcal{I}$, namely $\mathcal{A} \in \{\mathcal{B}\}$. Reversely, if $\mathcal{A} \in \{\mathcal{B}\}$, then one has $\mathcal{A} = \mathcal{B} + \mathcal{X}$, with $\varphi\langle \mathcal{X} | \mathcal{X} \rangle_\varphi = 0$. Using Eq. (97) we have the bounding

$$\omega(\mathcal{X}) = \varphi\langle \mathcal{X}' | \tilde{\mathcal{T}}_\omega \rangle_\varphi \leq \|\mathcal{X}\|_\varphi \|\tilde{\mathcal{T}}_\omega\|_\varphi, \quad (98)$$

whence if $\|\mathcal{X}\|_\varphi = 0$, then $\omega(\mathcal{A} - \mathcal{B}) = \omega(\mathcal{X}) = 0$ for all states ω , namely \mathcal{A} is informationally equivalent to \mathcal{B} . ■

Therefore, the vectors of the Hilbert space H_φ are in one-to-one correspondence with generalized propensities. From the bounding (98) we can also see that if the state φ satisfies $\|\mathcal{F}_\omega\|_\varphi \leq C_\varphi$ for some constant $C_\varphi \geq 0$ depending only on φ , then one can also reversely bound the two inequivalent norms $\|\cdot\|$ and $\|\cdot\|_\varphi$ as follows

$$\|\mathcal{A}\| \leq C_\varphi \|\mathcal{A}\|_\varphi. \quad (99)$$

In such case one the domain of definition of $\pi_\varphi(\mathcal{A})$ can be extended to the whole Hilbert space H_φ .

DIMENSIONALITY THEOREMS

We will now consider the consequences of Postulates 3 and 4. We will see that they entail dimensionality theorems that agree with the tensor product rule for Hilbert spaces for composition of independent systems in Quantum Mechanics. Moreover, Postulate 4, in particular, shows that the real Hilbert space H_φ is isomorphic to the real Hilbert space of Hermitian complex matrices representing selfadjoint operators over a complex Hilbert space H of dimensions equal to $\text{idim}(\mathfrak{S})$, finally leading to the Hilbert space formulation of Quantum Mechanics.

The *local observability principle* 3 is operationally crucial, since it reduces enormously the complexity of informationally complete observations on composite systems, by guaranteeing that only local (although jointly executed!) experiments are sufficient for retrieving a complete information, also any correlations between the component systems. This principle directly implies the following upper bound for the affine dimension of a composed system

$$\text{adm}(\mathfrak{S}_{12}) \leq \text{adm}(\mathfrak{S}_1) \text{adm}(\mathfrak{S}_2) + \text{adm}(\mathfrak{S}_1) + \text{adm}(\mathfrak{S}_2). \quad (100)$$

In fact, if the number of outcomes of a minimal informationally complete observable on \mathfrak{S} is N , the affine dimension is given by $\text{adm}(\mathfrak{S}) = N - 1$ (since the number of outcomes must equal the dimension of the affine space embedding the convex set of states \mathfrak{S} plus another dimension for the normalization functional n_0). Now, consider a global informationally complete measurement made of two local minimal informationally complete observables measured jointly. It has number of outcomes $[\text{adm}(\mathfrak{S}_1) + 1][\text{adm}(\mathfrak{S}_2) + 1]$. However, we are not guaranteed that the joint observable is itself minimal, whence the bound (100) follows.

We now translate the concept of dynamically faithful state in the Bloch representation. If the state Φ is (dynamically) faithful, then the output state $\Phi_{\mathcal{A}, \mathcal{I}}$ (conditioned that the transformation \mathcal{A} occurred locally on the first system) is in one-to-one correspondence with the transformation \mathcal{A} . Therefore, one can completely determine the transformation by determining the output state. We need to determine the matrix $M(\mathcal{A})$ plus the vectors $k(\mathcal{A})$ and $m(\mathcal{A})$, plus the parameter $q(\mathcal{A})$, namely $\text{adm}(\mathfrak{S})^2 + 2\text{adm}(\mathfrak{S}) + 1$ parameters. However, one parameter, say $q(\mathcal{A})$ is determined by the overall probability of occurrence of \mathcal{A} on the state Φ , from which the conditioned state is independent. Therefore, in order to have a joint faithful state we need to have at least $\text{adm}(\mathfrak{S})[\text{adm}(\mathfrak{S}) + 2]$

independent parameters for the joint state, namely we have the lower bound for the affine dimension of the joint system

$$\text{adm}(\mathfrak{S}^{\times 2}) \geq \text{adm}(\mathfrak{S})[\text{adm}(\mathfrak{S}) + 2]. \quad (101)$$

If we put the two bounds (100) and (101) together, for a bipartite system made of two identical systems we obtain

$$\text{adm}(\mathfrak{S}^{\times 2}) = \text{adm}(\mathfrak{S})[\text{adm}(\mathfrak{S}) + 2], \quad (102)$$

which agrees with the dimensionality of composite systems in Quantum Mechanics coming from the tensor product. The Bloch representation can be obtained experimentally by performing a joint informationally complete measurement on both systems at the output, and then:

1. determining the probability of occurrence of the transformation \mathcal{A} on the state Φ , which is given by

$$\Phi(\mathcal{A}, \mathcal{I}) = \Phi(\underline{\mathcal{X}}_0 \circ \mathcal{A}, \underline{\mathcal{X}}_0) = (m(\underline{\mathcal{A}}) \cdot n \odot n_0)(\Phi) + q(\underline{\mathcal{A}}); \quad (103)$$

2. determining the following probabilities

$$\begin{aligned} \Phi(\underline{\mathcal{X}}_j \circ \mathcal{A}, \underline{\mathcal{X}}_k) &= \frac{[(M(\mathcal{A})n)_j \odot n_k](\Phi) + k_j(\mathcal{A})(n_0 \odot n_k)(\Phi)}{\Phi(\mathcal{A}, \mathcal{I})}, \\ \Phi(\underline{\mathcal{X}}_0 \circ \mathcal{A}, \underline{\mathcal{X}}_j) &= (m(\underline{\mathcal{A}}) \cdot n \odot n_j)(\Phi) + q(\underline{\mathcal{A}}), \quad \begin{array}{l} j = 1, \dots, \text{adm}(\mathfrak{S}), \\ k = 0, 1, \dots, \text{adm}(\mathfrak{S}); \end{array} \end{aligned} \quad (104)$$

3. invert the above equations in terms of $M(\mathcal{A})$, $k(\mathcal{A})$, $m(\underline{\mathcal{A}})$, and $q(\underline{\mathcal{A}})$.

Assuming now Postulate 4 gives a bound for the informational dimension of the informational dimension of convex sets of states. In fact, if for any bipartite system made of two identical components and for some preparations of one component there exists a discriminating observable that is informationally complete for the other component, this means that $\text{adm}(\mathfrak{S}) \geq \text{idim}(\mathfrak{S}^{\times 2}) - 1$, with the equal sign if the informationally complete observable is also minimal, namely

$$\text{adm}(\mathfrak{S}) = \text{idim}(\mathfrak{S}^{\times 2}) - 1. \quad (105)$$

By comparing this with the affine dimension of the bipartite system, we get

$$\begin{aligned} \text{adm}(\mathfrak{S}^{\times 2}) &= \text{adm}(\mathfrak{S})[\text{adm}(\mathfrak{S}) + 2] = [\text{idim}(\mathfrak{S}^{\times 2}) - 1][\text{idim}(\mathfrak{S}^{\times 2}) + 1] \\ &= \text{idim}(\mathfrak{S}^{\times 2})^2 - 1, \end{aligned} \quad (106)$$

which, generalizing to any convex set gives the identification

$$\text{adm}(\mathfrak{S}) = \text{idim}(\mathfrak{S})^2 - 1, \quad (107)$$

corresponding to the dimension of the quantum convex sets \mathfrak{S} originated from Hilbert spaces. Moreover, upon substituting Eq. (105) into Eq. (107) one obtain

$$\text{idim}(\mathfrak{S}^{\times 2}) = \text{idim}(\mathfrak{S})^2, \quad (108)$$

which is the *tensor product rule* for informational dimensionalities.

According to Theorem 8 we have the identity

$$\dim(H_\varphi) = \text{adm}(\mathfrak{S}) + 1, \quad (109)$$

since H_φ is identified with the vector space of the generalized propensities, namely the space of the linear functionals over states which has one more dimension than the convex set of states corresponding to normalization. From Eqs. (107) and (109) we now have

$$\dim(H_\varphi) = \text{idim}(\mathfrak{S})^2. \quad (110)$$

Then, for finite dimensions the real Hilbert space H_φ is isomorphic to the real Hilbert space of Hermitian complex matrices representing selfadjoint operators over a complex Hilbert space H of dimensions $\dim(H) = \text{idim}(\mathfrak{S})$, with scalar product corresponding to the trace pairing used in the Born rule, and with the convex cones of propensities and states corresponding to the convex cone of positive matrices. This is the Hilbert space formulation of Quantum Mechanics. In infinite dimensions the selfadjoint operators are generally unbounded, since norm $\|\cdot\|$ is not necessarily bounded, and boundedness of probabilities is provided by the faithful state Φ .

In deriving Eq. (107) I have implicitly assumed that the relation between the affine dimension and the informational dimension which holds for bipartite systems must hold for any system. Indeed, one can prove independently that

$$\text{idim}(\mathfrak{S}^{\times 2}) \geq \text{idim}(\mathfrak{S})^2, \quad (111)$$

since locally perfectly discriminable states are also jointly discriminable, and the existence of a preparationally faithful state guarantees the existence of $\text{idim}(\mathfrak{S})^2$ jointly discriminable states, the bound in place of the identity coming from the fact that we are not guaranteed that the set of jointly discriminable states made of local ones is maximal. At the present stage of this research in progress it is still not clear if the mentioned implicit assumption is avoidable, and, if not, how relevant it is. One may need to add another postulate requiring a kind of universality of informational laws— such as $\text{adm}(\mathfrak{S}) = \text{idim}(\mathfrak{S})^2 - 1$ —independently on the physical system, i. e. on the convex set of states \mathfrak{S} . It is also possible that in this way Postulate 4 can be avoided. These issues will be analyzed in detail in a forthcoming publication.

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