Quantum Tomography of Optical Devices

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Abstract

We propose an experimental procedure for measuring the Hamiltonian operator (or the Liouvillian super–operator, in the case of open systems) of an optical device. The setup is based on the tomographic reconstruction of quantum states, and is composed of a non degenerate optical amplifier, a homodyne detector and a linear high efficiency photodetector. Numerical Monte Carlo simulation of experiments are given for the measurement of the Liouvillian of a two photon phase insensitive amplifier.

Introduction

"Quantum homodyne tomography" [1, 2, 3] proved to be a very powerful technique for experimentally reconstructing the quantum state of radiation [4, 5]. Here we propose to apply this technique to the measurement of the Hamiltonian operator of a quantum device. In fact, by impinging a complete set of input states and accurately measuring the respective output states, it is possible to recover the full dynamical characterization of a device, namely to reconstruct the quantum operator governing its dynamics. Since in the general case the dynamical evolution is non–unitary, one has to resort to the notion of Liouvillian $\mathcal{L}$ super–operator. The Liouvillian is responsible for the evolution of the density matrix ρ of a generally open quantum system, through the master equation

$$\frac{d}{dt}\rho(t) = \mathcal{L}[\rho(t)] \ , \tag{1}$$

which is formally solved as

$$\rho(t) = \mathcal{G}_t[\rho(0)], \text{ with } \mathcal{G}_t \doteq \exp(t\mathcal{L}) \ , \tag{2}$$

where $\mathcal{G}_t$ is the so-called Green super–operator. Hence $\mathcal{G}_t$ is a map (in the Liouville space of density matrices $\mathcal{F}$) between the input and output states for the device: $\rho_{out} = \mathcal{G}_t[\rho_{in}]$. By using a complete set of states ρ_{in} in $\mathcal{F}$, and by measuring the resulting ρ_{out}, it is possible to reconstruct the mapping $\mathcal{G}_t$, and hence the Liouvillian $\mathcal{L} = \frac{1}{t}\log \mathcal{G}_t$.

The outline of the present paper is as follows. In Sect. 1 we describe the experimental setup and give the procedure for the measurement of the Liouvillian matrix. A short discussion is given for the case in which the optical detectors involved in the experiment have non–unit quantum

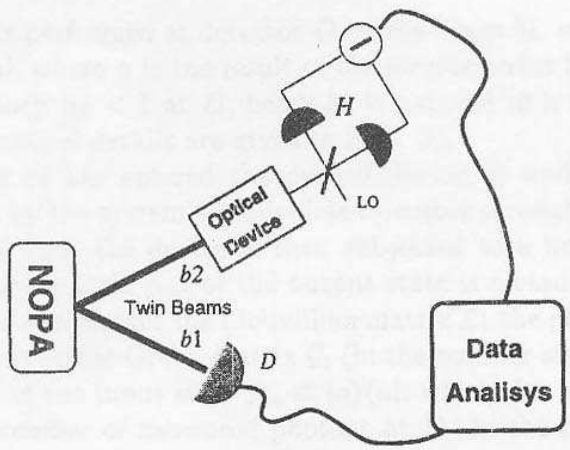

Figure 1: Sketch of the proposed experimental setup. Two correlated twin beams ($b1$ and $b2$) exit from a non degenerate optical amplifier (NOPA). A direct measurement is performed on $b1$ at detector D, thus reducing the beam $b2$ to a number state (or a mixture of number states if the quantum efficiency of the detector D is $\eta_D < 1$). In this way, a number state (mixture) actually impinges into the optical device. After being evolved by the device, the beam $b2$ emerges, and is subject to a homodyne measurement at detector H. The Liouvillian matrix can now be reconstructed by data analysis.

efficiency. In Sect. 2 the underlying mathematical backbone is briefly presented. In Sect. 3 a Monte Carlo numerical simulation of the experiment is given by analyzing the measurement of the Liouvillian of a two photon phase insensitive amplifier. Finally in Sect. 4 we give the conclusions and analyze the actual feasibility of the experiment.

1 Experimental Apparatus

The sketch of the experimental setup is given in Fig. 1. In this case the set of states impinging into the device are number states $\{|n\rangle\langle n|\}$. Since this set is not complete in the Liouville space $\mathcal{F}$, the setup presented here is capable of reconstructing only the diagonal part (in the number state representation) of the Liouvillian $\mathcal{L}$, *i.e.* the part where the 4 indexes of $\mathcal{L}$ super–operator matrix are two by two equal. The extension of the method to the reconstruction of the whole four-dimensional matrix $\mathcal{L}$ is under development, and is subject to the realization of a device capable of preparing the appropriate input states. However, there is a vast class of devices for which the Liouvillian $\mathcal{L}$ is diagonal. These are the so–called phase insensitive devices, which transforms dephased states (states that commute with the number operator $a^\dagger a$ of the electromagnetic mode) into dephased states. Typical examples of such devices are the laser and the phase insensitive linear amplifier (PIA).

The preparation of number states is rather critical, and resorts to the state–reduction mechanism. By means of a non degenerate optical amplifier (NOPA) with vacuum input and strong classical pump, two distinct quantum correlated twin–beams ($b1$ and $b2$) are created. A very accurate direct measurement is performed at detector D on the beam $b1$, so that beam $b2$ is reduced to a number state $\rho = |n\rangle\langle n|$, where n is the result of the measurement in detector D. In the case of non–unit quantum efficiency $\eta_D < 1$ at D, beam $b2$ is reduced to a mixture of number states $\rho = \sum_i c_i|i\rangle\langle i|$ (the mathematical details are given in Sect. 2).

Once the reduced beam $b2$ has entered the optical device, it undergoes a (generally non–unitary) evolution governed by the system's Liouvillian operator through the master equation (1). The output beam emerging from the device is then subjected to a homodyne measurement at detector H, where the density matrix ρ_{out} of the output state is measured. Now only some data analysis is needed in order to reconstruct the Liouvillian matrix $\mathcal{L}$: the photon–number probability measured in H is the n-th row of the Green matrix $\mathcal{G}_t$ (in the number state representation), where n is the number of photons of the input state $\rho_{in} \equiv |n\rangle\langle n|$, which, for unit quantum efficiency at detector D, is exactly the number of measured photons at D. In the case of non–unit quantum efficiency $\eta_D \neq 1$, one considers the most probable input state $\rho_{in} \equiv |\bar{n}\rangle\langle\bar{n}|$, given the outcome n. The Green matrix $\mathcal{G}_t$, constructed in this way, is the input–output transformation matrix for the phase–insensitive device. The Liouvillian matrix $\mathcal{L}$ is obtained by taking the natural logarithm of $\mathcal{G}_t$, and dividing it by the evolution time t. The non–unit quantum efficiencies η_D and η_H at detectors D and H yield rather different effects: while the non–unit quantum efficiency $\eta_D < 1$ at detector D tends to shuffle the rows of the matrix $\mathcal{G}_t$, the non–unit quantum efficiency $\eta_H < 1$ at detector H tends to smear out the details of each row of the reconstructed matrix $\mathcal{G}_t$.

2 Mathematical Backbone

We briefly give the mathematical details of the physical processes outlined in Sect. 1. We first describe the input number state preparation procedure, and then the evolution process in the number state representation inside the optical device. For a complete description of quantum homodyne tomography we address the reader to the review paper [6].

2.1 Number State Preparation

At the output of the NOPA, the state of the correlated twin beams is [7] $|\psi\rangle = \sum_{n=0}^{\infty} \frac{\kappa^n}{\sqrt{1-|\kappa|^2}}|n, n\rangle$, where κ is the pump-parameter for the NOPA. This state is reduced by the photodetector D with quantum efficiency η_D. Such a photodetector can be modeled as a perfect ($\eta = 1$) photodetector preceded by a beam splitter of transmissivity η_D that entangles the input mode with a vacuum mode [6]. The unitary evolution operator U for the beam splitter is ,$U = \exp\left[\left(\mathrm{arctg}\sqrt{\eta_D^{-1} - 1}\right)\left(ad^\dagger - a^\dagger d\right)\right]$, where η is the transmissivity of the beam splitter, a is the annihilation operator of the relevant $b1$ mode, and d is the annihilation operator of the vacuum mode. To obtain the state of the reduced beam $b2$, we must project the state $U|\psi\rangle$ that exits the beam splitter on the eigenstate $|n\rangle$ of the number operator, thus describing a perfect ($\eta = 1$) direct measurement with outcome n. Moreover, it is necessary to trace out the spurious mode corresponding to the evolution of the vacuum mode. The final state, which is actually the one entering into the optical device, is thus, the mixture

$$\rho_{in} = \frac{1}{\mathcal{N}} \sum_{m=0}^{\infty} |\kappa|^{2m} (1-\eta_D)^m \frac{(n+m)!}{m!} \; |n+m\rangle\langle n+m| \; , \tag{3}$$

where $\mathcal{N}$ is a normalization and n is the result of the measurement in the photodetector D. Notice that in the case of perfect detector D (i.e. $\eta_D = 1$), we obtain the number state $\rho_{in} = |n\rangle\langle n|$; otherwise a mixture of number states results, and the most probable state is $\rho_{in} = |\bar{n}\rangle\langle\bar{n}|$, where $\bar{n}$ maximizes the mixture–state coefficients. Notice that η_D must be not too small in order to reduce the spread of the mixture state distribution, and thus to reduce the probability of shuffling the rows of the reconstructed matrix $\mathcal{G}_t$.

2.2 Number state evolution in a phase insensitive device

The most general evolution of radiation through an optical device is a non–unitary evolution governed by a Green super–operator of the form given in Eq. (2). Given a complete set of vectors in the Liouville space $\mathcal{F}$, the general expansion for the super–operator $\mathcal{G}_t$ is the four–index tensor G^{ik}_{jl} defined as

$$\mathcal{G}\Big[|i\rangle\langle j|\Big] = \sum_{kl=0}^{\infty} G^{ik}_{jl} |k\rangle\langle l| \; . \tag{4}$$

In the presently proposed experimental setup we are only interested in the diagonal part of the Green super–operator in the number representation, since the subspace in $\mathcal{F}$ of diagonal operators is invariant for phase insensitive devices. Thus we only need the two–index matrix $G^0_{nm} \doteq G^{nm}_{nm}$ obtained by expanding $\mathcal{G}_t$ on the number states $\{|n\rangle\langle n|\}$. It is immediate to see that the m-th row of G^0_{nm} is the output photon number probability, when the input is the number state $\rho_{in} = |m\rangle\langle m|$:

$$G^0_{nm} = \langle n|\mathcal{G}_t\Big[|m\rangle\langle m|\Big]|n\rangle = p_{out}(n) \; , \tag{5}$$

where $\rho_{out} = \mathcal{G}_t\rho_{in} = \sum_n \; p_{out}(n) \; |n\rangle\langle n|$. In the practical case, one always needs to truncate the set of states $\{|n\rangle\langle n|\}$ at sufficiently large n. Once G^0_{nm} has been reconstructed through the measured probabilities, we calculate the diagonal Liouvillian $L^0_{nm} = \langle n|\mathcal{L}\Big[|m\rangle\langle m|\Big]|n\rangle$ by taking the natural logarithm of the matrix G^0_{nm}, and dividing it by the evolution time t as shown in Eq. (2). For the theoretical matrix this operation is straightforward, since all eigenvalues of the Markoff matrix G^0_{nm} are within the interval $(0,1]$. However, for the evaluation of the logarithm of the experimental matrix G^0_{nm} one needs to consider that for sizeable times t the quantum homodyne noise can yield a measured Green matrix with negative eigenvalues, so that the principal logarithm of G^0_{nm} may be undefined. For this reason, it is preferable to evaluate $\mathcal{G}_t$ for small t, and to calculate the logarithm by series expansion in order to keep to the principal branch. Other methods [8] for the calculus of the logarithm have also been used.

3 Numerical Simulation of the experiment

We now present a numerical simulation of the experiment we proposed. In this example we will analyze a two photon phase insensitive amplifier, with Liouvillian

$$\mathcal{L} = A\mathcal{D}[a^\dagger] + B\mathcal{D}[a] + C\mathcal{D}[a^2] + D\mathcal{D}[(a^\dagger)^2] \; , \tag{6}$$

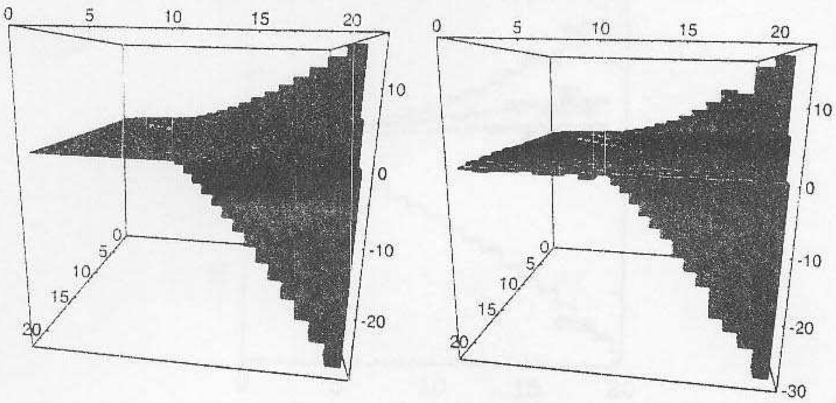

Figure 2: On the left the theoretical matrix L^0_{nm} is plotted versus m and n from Eq. (7). The parameters of the two photon phase insensitive amplifier are $A = 0.3$; $B = 0.15$; $C = 5 \cdot 10^{-4}$; $D = 4 \cdot 10^{-2}$. On the right the result of a Monte Carlo simulation of the measurement of the same Liouvillian is given. The reconstruction of L^0_{nm} is simulated by using only 15000 homodyne measurements divided into 3 statistical blocks of data for each input state.

where a is the destruction operator for the field mode, A and D are the one–photon and two–photon absorption coefficients respectively, B and C are the one– and two–photon creation coefficients, and $\mathcal{D}[\theta]\rho \doteq \theta\rho\theta^\dagger - 1/2(\theta^\dagger\theta\rho + \rho\theta^\dagger\theta)$ denotes the Lindblad superoperator for the (complex) operator θ. The Liouvillian matrix L^0_{nm} is

$$
\begin{aligned}
L^0_{nm} = {} & A(m+1)[\delta_{nm+1} - \delta_{nm}] + Bm[\delta_{nm-1} - \delta_{nm}] + \\
& Cm(m+1)[\delta_{nm-2} - \delta_{nm}] + D(m+2)(m+1)[\delta_{nm+2} - \delta_{nm}] \quad . \qquad (7)
\end{aligned}
$$

In Fig. 2 the theoretical Liouvillian (7) is plotted versus the reconstructed Liouvillian obtained by a Monte Carlo simulation of the proposed experiment. In this case both detectors have unit quantum efficiency ($\eta_D = \eta_H = 1$). In Fig. 3 the non–zero diagonals of matrix L^0_{nm} are plotted with the statistical error bars versus the theoretical value. One can see that the details of the Liouvillian are very well recovered.

4 Conclusions

In conclusion we have proposed an experiment to reconstruct the Liouvillian super–operator of a phase insensitive quantum optical device. The actual feasibility of such an experiment is mostly related to the availability of a good photodetector D that is able to resolve single photons with high quantum efficiency (possibly better than $\eta_D = .5$: see Ref.[3]), a device that may be available in a few years within the superconducting–detectors technology. Currently we are working on the extension of the method to measuring non–diagonal Liouvillians and on error correction techniques to reduce experimental errors coming from non–unit quantum efficiency $\eta_D < 1$ at detector D. Notice that a correction scheme for the case of non–unit quantum efficiency $\eta_H < 1$

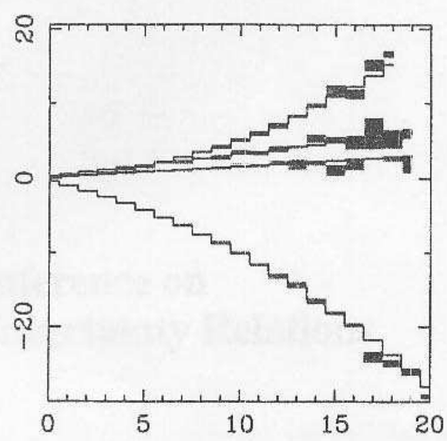

Figure 3: The four non–zero diagonals of the reconstructed Liouvillian matrix L^0_{nm} of Fig. 2 are plotted with the statistical error bars. The dark line is the theoretical value from Eq. (7).

at the homodyne detector H is already available by using special η–dependent pattern functions for the homodyne tomography [3], and by increasing the number of homodyne data. We think that a prototype experiment will be feasible soon, and that it will represent a useful experimental test for theoretical models of quantum optical devices like lasers, parametric amplifiers, and the quantum gates of the forthcoming quantum computer.

References

[1] D. T. Smithey, M. Beck, M. G. Raymer and A. Faridani, Phys. Rev. Lett. **70**,1244 (1993)

[2] G. M. D'Ariano, C. Macchiavello, M. G. A. Paris, Phys. Rev. A **50**, 4298 (1994)

[3] G. M. D'Ariano, U. Leonhardt, M. Paul, Phys. Rev. A **52** R1881 (1995)

[4] U. Leonhardt, M. Munroe, T. Kiss, Th. Richter, and M. G. Raymer, Opt. Comm. **127**, 144 (1996).

[5] S. Schiller, G. Breitenbach, S. F. Pereira, T. Müller, S. Mlynek, Phys. Rev. Lett. **77**, 2933 (1996)

[6] G. M. D'Ariano, *Measuring Quantum States*, in *Quantum Optics and Spectroscopy of Solids*, ed. by T. Hakioğlu and A. S. Shumovsky, (Kluwer Academic Publisher, Amsterdam 1997), p. 175-202

[7] G. M. D'Ariano and M. F. Sacchi, Phys. Rev. A **52** R4309 (1995)

[8] L. Dieci, B. Morini, A. Papini, Siam J. Matrix Anal. Appl. Vol. 17 **3**, 570 (1996); L. Dieci, B. Morini, A. Papini, A. Pasquali, submitted to Applied Numerical Mathematics.