Amplification in small phase-shifts measurements

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Abstract. We study the performance of a quantum amplifier in order to improve the sensitivity of small phase shifts measurements. This amplifier is based on ideal photon number attenuation. Phase measurement is analyzed in the context of quantum estimation theory. We consider quantum communication channels and, for binary hypothesis testing, we observe sizeable bit-error-rate reduction and increase of information retrieved from the measurement.

Phase amplification has never been considered in the past. Indeed, phase-modulation based communications rely upon on-off 0-π phase shifts keying (Barry and Lee 1990) and a phase amplifier is useless, because any kind of loss that decreases the radiation field amplitude does not decrease the average phase, but just enhances noise. The situation, however, is very different when the channel is intrinsically signal-limited, namely when the information is unavoidably coded on small phase shifts, as, for example, in interferometric high sensitive measurements. As we will see in the following, in this case phase amplification represents a convenient strategy to improve the overall sensitivity.

The words "phase amplification" can be given a precise meaning in the context of the quantum estimation theory (Helstrom 1976). In our case the problem is the estimation of the phase shift φ of a pure state $|\psi\rangle$ that undergoes the unitary transformation

$$|\psi\rangle \rightarrow \exp(ia^\dagger a\varphi)|\psi\rangle \ , \tag{1}$$

with $a^\dagger a$ denoting the number operator of the mode a of the electromagnetic field. The state $|\psi\rangle$ itself is supposed to have a well defined phase (say φ') namely in the number representation it is written as

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle \,, \qquad \psi_n \equiv |\psi_n| \exp(in\varphi') \,. \tag{2}$$

Without loss of generality in the following we will consider $\varphi' = 0$, i.e. all ψ_n are real nonnegative. A phase amplifier multiplies the shift φ by a fixed gain g: in general, this can be achieved at the expense of introducing some additional noise and of partially destroying the coherence of the input state. We will see that these undesired effects can be avoided by using phase coherent input states.

The quantum description of an apparatus for detecting a phase shift is given by a Born's rule in the following form

$$p(\phi|\varphi)\mathrm{d}\phi = \mathrm{Tr}\left[e^{ia^\dagger a\varphi}|\psi\rangle\langle\psi|e^{-ia^\dagger a\varphi}\,\mathrm{d}\hat{\mu}(\phi)\right] \,, \tag{3}$$

where $p(\phi|\varphi)$ is the probability density of detecting a phase shift ϕ "conditioned" by the actual value φ, whereas $\mathrm{d}\hat{\mu}(\phi)$ denotes the POM (probability operator-valued measure) that pertains the apparatus. When there is no *a priori* preferred phase the conditional probability density should have the form $p(\phi|\varphi) \equiv p(\phi - \varphi)$. In this case the POM has the *covariant* form

$$\mathrm{d}\hat{\mu}(\phi) = e^{ia^\dagger a\phi}\hat{\zeta}e^{-ia^\dagger a\phi}\frac{\mathrm{d}\phi}{2\pi} \,, \tag{4}$$

with $\hat{\zeta}$ being a fixed positive operator. For covariant measurements, the operator $\hat{\zeta}$ corresponding to the ideal POM is (Holevo 1982)

$$\hat{\zeta} = \sum_{n,m=0}^{\infty} |n\rangle\langle m| \,. \tag{5}$$

The phase amplifier considered here is based on ideal photon number attenuation. In Ref. (D'Ariano 1992a) the Hamiltonian of the ideal photon number amplifier (PNA) (Yuen 1986a, 1987) is derived. When the PNA is used in the inverse way as ideal number attenuator, it works also as a phase amplifier. Ideal number attenuation and simultaneous phase amplification are described by the unitary operator (D'Ariano 1992a)

$$\hat{U}_g = \sum_{\nu=0}^{g-1} \sum_{n,m=0}^{\infty} |n\rangle\langle gn+\nu| \otimes |gm+\nu\rangle\langle m| \tag{6}$$

that acts on the enlarged Hilbert space $\mathcal{H} \otimes \mathcal{H}_i$ including the signal Hilbert space $\mathcal{H}$ and the space $\mathcal{H}_i$ of an additional idler mode, needed to preserve unitarity and responsible for "mixing" the state. We introduce the amplifying maps $\mathcal{A}_g$ and $\mathcal{A}_g^\vee$ pertaining the signal Hilbert space $\mathcal{H}$ only, and obtained by partially tracing over the idler mode. These are given by

$$\begin{aligned}\mathcal{A}_g(|\psi\rangle\langle\psi|) &= \mathrm{Tr}_i\left\{\hat{U}_g|\psi\rangle\langle\psi|\otimes\hat{\rho}_i\hat{U}_g^\dagger\right\}\,,\\ \mathcal{A}_g^\vee(\mathrm{d}\hat{\mu}(\phi)) &= \mathrm{Tr}_i\left\{\hat{U}_g^\dagger\mathrm{d}\hat{\mu}(\phi)\otimes\hat{1}\hat{U}_g\hat{1}\otimes\hat{\rho}_i\right\}\,,\end{aligned}\tag{7}$$

with $\hat{\rho}_i$ denoting the density matrix of the idler mode. From Eqs.(6) and (7) one can see that the ideal number attenuator achieves the phase amplification

$$\begin{aligned}&\mathcal{A}_g^\vee(\mathrm{d}\hat{\mu}(\phi)) =\\ &\tfrac{\mathrm{d}\phi}{2\pi g}\textstyle\sum_{\lambda=0}^{g-1} e^{ia^\dagger a g^{-1}(\phi+2\pi\lambda)}\sum_{n,m=0}^{\infty}|n\rangle\langle[n/g]|\hat{\varsigma}|[m/g]\rangle\langle m|e^{-ia^\dagger a g^{-1}(\phi+2\pi\lambda)}\,,\end{aligned}\tag{8}$$

where $[x]$ represents the integer part of x. Notice that for the ideal POM (5) Eq. (8) is just a 2π-periodic rescaling of the POM: in this sense the present phase amplification can be considered ideal.

Now we examine the performance of the phase amplifier for phase coherent input states. Phase coherent states (Shapiro and Shepard 1991) are defined as follows

$$|\xi\rangle = (1-|\xi|^2)^{1/2}\sum_{n=0}^{\infty}\xi^n|n\rangle\,,\quad \xi = e^{i\varphi}|\xi|,\quad |\xi|<1\,,\tag{9}$$

where the complex number ξ also carries the phase-shift information φ. Then, from Eqs.(7) and (9) one has

$$\mathcal{A}_g^\vee(|\xi\rangle\langle\xi|) = |\xi^g\rangle\langle\xi^g|\,.\tag{10}$$

Thus, phase coherence is preserved under amplification. The phase-coherent state $|\xi\rangle$ has average number of photons $\langle\hat{n}\rangle = |\xi|^2/(1-|\xi|^2)$. For ideal phase detection the output phase probability after amplification is given by

$$p_{out}(\phi|\varphi) = \frac{1}{2\pi}|\langle\xi^g|e^{i\phi}\rangle|^2 = \frac{1}{2\pi}\frac{1-|\xi|^{2g}}{1+|\xi|^{2g}-2|\xi|^g\cos(\phi-g\varphi)}\,.\tag{11}$$

In the limit $|\xi|\to 1^-$ one has $p_{out}(\phi)\to\delta_{2\pi}(\phi-g\varphi)$, $\delta_{2\pi}$ denoting the periodicized delta. All quantities of interest can be evaluated analytically for $|\xi| = 1-\epsilon$ with $g\epsilon \ll 1$ and $g\varphi\in[-\pi,\pi]$. The average phase is amplified as

$$\langle\phi\rangle_{out} = g\varphi + \mathcal{O}(g\epsilon)\,,\tag{12}$$

whereas the r.m.s. fluctuations

$$\langle\Delta\phi^2\rangle_{out} = 2g\epsilon + \mathcal{O}(g^2\epsilon^2)\tag{13}$$

are amplified by only a factor g.

As a preliminary information about the goodness of the amplifier, we evaluate the noise figure

$$\mathcal{R} = \frac{(\mathcal{S}^2/\mathcal{N})_{in}}{(\mathcal{S}^2/\mathcal{N})_{out}}, \tag{14}$$

where $\mathcal{S}$ and $\mathcal{N}$ denote respectively the signal $\langle\phi\rangle$ and the noise $\langle\Delta\phi^2\rangle$. From Eqs. (12) and (13) we have

$$\mathcal{R} = \frac{1}{g}. \tag{15}$$

This good performance of phase-coherent states is simply due to the fact that such states exhibit shot noise $\langle\Delta\phi^2\rangle \propto \langle\hat{n}\rangle^{-1}$ and, at the same time, they are preserved under amplification. Hence, when attenuating $\langle\hat{n}\rangle$ one gets a phase noise which is amplified by only a factor g.

The typical situation in which one takes advantage of amplification occurs when the signal $\langle\phi\rangle$ is very low, below the detection threshold, and the amplifier is used to enhance the signal above the threshold. However, as amplification also increases noise, the net benefit must be evaluated carefully, by comparing the values of BER (bit error rate) and mutual information (Blahut 1987) obtained with and without amplification. We consider now a binary channel that pertains the phase detection of a low signal. The measurement consists of testing the hypothesis that a phase-shifting event has occurred, assigning the "true" value to every outcome above a fixed threshold φ_s. We denote by $|`0'\rangle \equiv |\psi\rangle$ the reference zero-phase state and by $|`1'\rangle \equiv \exp(ia^\dagger a\varphi)|\psi\rangle$ the shifted state. The input signal is very weak ($\varphi \ll 1$): the threshold φ_s is taken above φ due to limitations of the detector sensitivity and in order to achieve a low value of the "false alarm probability" $Q_{1|0}$ (Helstrom 1976; Blahut 1987)

$$Q_{1|0} = \int_{\varphi_s}^{\pi} \mathrm{d}\phi\, p(\phi|0), \tag{16}$$

namely the probability of detecting '1' given state $|`0'\rangle$. It is clear that amplification will increase $Q_{1|0}$ as a consequence of the spread of the right tail of the '0' distribution; however, it will simultaneously enhance the "detection probability" $Q_{1|1}$ (Helstrom 1976; Blahut1987)

$$Q_{1|1} = \int_{\varphi_s}^{\pi} \mathrm{d}\phi\, p(\phi|\varphi), \tag{17}$$

namely the probability that '1' is correctly detected given state $|`1'\rangle$. An improvement of the binary test measurement is determined by a decrease of the bit-error-rate

$$B = 1 + Q_{1|0} - Q_{1|1}, \tag{18}$$

or, equivalently, by an enhancement of the mutual information (Blahut 1987)

$$I = \sum_{j,k=0}^{1} p_j Q_{k|j} \ln \frac{Q_{k|j}}{\sum_{i=0}^{1} p_i Q_{k|i}} , \tag{19}$$

after specifying the *a priori* probabilities $\{p_j\}_{j=0,1}$ of input states $|`j'\rangle$, and considering all possible conditional probabilities $Q_{k|j}$ of detecting '$k$' given $|`j'\rangle$.

Now we evaluate the conditional probabilities $Q_{1|0}$ and $Q_{1|1}$ with $(g > 1)$ and without $(g = 1)$ amplification for input phase coherent states. For $g\epsilon \ll 1$ we have

$$\begin{aligned} Q_{1|0} &= \tfrac{g\epsilon}{2\pi} \cot\left(\tfrac{\varphi_s}{2}\right) , \\ Q_{1|1} &= \tfrac{1}{\pi}\left\{\tfrac{\pi}{2} - \arctan\left[\tfrac{2}{g\epsilon}\tan\left(\tfrac{\varphi_s - g\varphi}{2}\right)\right]\right\} . \end{aligned} \tag{20}$$

These probabilities give the BER and the mutual information plotted in Fig. 1 as functions of the gain for different values of input number of photons $\langle\hat{n}\rangle_{in}$. The case of very weak input signal $\varphi \ll \varphi_s$ has been considered. One can see that the BER exhibits a steep decrease and that, at the same time, the mutual information shows a rapid increase near the gain $g_s = \varphi_s/\varphi$. These features are further enhanced when the mean input photon number is increased. For the mutual information we refer to the situation of rare events, i.e. $p_1 = 1 - p_0 \ll 1$, which is of interest for example in interferometric detection of gravitational waves: the behavior of I, however, does not qualitatively depend on p_1, apart from the range of variation. For large input signal $\varphi > \varphi_s$, on the contrary, one could see that the mutual information would monotonically decrease versus g, whereas there would be essentially no reduction of the BER.

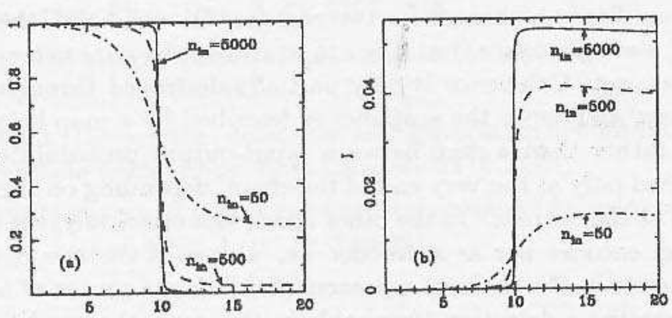

Figure 1: Bit error rate (a) and mutual information (b) versus gain g for phase-coherent states with input number of photons $\langle\hat{n}\rangle_{in} = 50, 500, 5000$. The phase shift is $\varphi = 0.05$, whereas the threshold phase is $\varphi_s = 0.5$. For the mutual information a probability $p_1 = 0.01$ has been used (rare events).

For generic input states phase amplification changes the kind of state and partially destroys coherence: for example, phase amplification does not preserve coherent or squeezed states. However, especially for nonideal phase detection, one can gain much benefit from phase amplification as well. This occurs because the amplifier partially recovers the effective loss due to nonideal measurement. The qualitative behavior of I and B for coherent and squeezed states is the same as for phase coherent states (D'Ariano, Macchiavello, Sterpi, and Yuen 1996). The only differences are that the variations of B and I are less steep, and the amplifier efficiency is much reduced for low numbers of input photons: these phenomena are distinctive of a partial loss of coherence of the amplified state.

We emphasize that phase amplification is advantageous only for measurements of small phase shifts φ, and not too large gains g, such that $g\varphi \ll 1$. In fact, the transformation (8) folds the probability distribution at the boundaries of the 2π window in order to maintain the distribution as 2π-periodic after the stretching along the direction of abscissa. In this way, in the limit of large gains any probability distribution would converge to the uniform probability on the 2π window.

Some comments are now in order on the apparent violation of the data processing theorem (Blahut 1987) inherent an improvement of mutual information. Indeed, the theorem states the impossibility of improving the mutual information by performing any kind of data processing. More precisely, for a channel described by a map $X \to Y$ between input-output random variables X and Y, the mutual information $I(X;Y)$ between X and Y can be improved neither by any kind of "encoding" $U \to X$, nor by any "decoding" $Y \to V$, where U and V are new random variables. In other words: the end-to-end mutual information $I(U;V)$ in the Markov chain of random variables $U \to X \to Y \to V$ is never greater than $I(X;Y)$. However, the data processing theorem does not pertain the case of insertion of a quantum amplifier in a channel for two reasons. On one hand, the amplifier is not a "classical" data processor, i.e. it is not equivalent to a measurement followed by a data processing. Coherence is only partially destroyed throughout the amplification process, and hence the amplifier is described by a map between probability amplitudes rather than a map between input-output probabilities. Probabilities are determined only at the very end of the chain, depending on the observable that is measured at the output. On the other hand, also classically, the amplifier is used neither as an encoder nor as a decoder–i.e. at one of the two ends of the chain–but it is *inserted* in the chain as a *preamplifier* before a source of additive noise, in order to overcome a detection threshold: in this case, the conditions for the data processing theorem are not fulfilled. In practice, the amplifier is used to reshape the information channel to approach conditions for optimal information transfer. This is the case when the channel is suboptimal, as when it is intrinsically signal limited by limitations of the detector sensitivity, because the detection threshold is above the maximum incoming signal.

In conclusion, we have proposed a scheme for amplifying small phase shifts below the detection threshold. In such scheme the BER is reduced and the information retrieved from the measurement is increased. The best performance is achieved by phase-coherent states. Finally, we point out that, as suggested in Ref. (D'Ariano 1992b), phase-coherent states can be ideally achieved using a phase insensitive amplifier (PIA) and a photon number duplicator (PND) (Yuen 1986b) in cascade, whereas the PND can be approached by a sum-frequency up-converter.

References

Barry, J.R. and Lee, E.A. (1990) Performance of coherent optical receivers. Proc. IEEE 78: 1369-1394.

Blahut, R.E. (1987) Principles and Practice of Information Theory. Addison-Wesley.

D'Ariano, G.M. (1992a) Hamiltonians for the photon-number-phase amplifiers. Phys. Rev. A 45:3224-3227.

D'Ariano, G.M. (1992b) Number-phase squeezed states and photon fractioning. Int. J. Mod. Phys. B 6: 1291-1354.

D'Ariano, G.M., Macchiavello, C., Sterpi, N., and Yuen, H.P. (1996) unpublished.

Helstrom, C.W. (1976) Quantum Detection and Estimation Theory. Academic Press.

Holevo, A.S. (1982) Probabilistic and Statistical Aspects of Quantum Theory. North Holland, Amsterdam.

Shapiro, J.H. and Shepard, S.R. (1991) Quantum phase measurement: A system-theory perspective. Phys. Rev. A 43: 3795-3818.

Yuen, H.P. (1986a) Amplification of quantum states and noiseless photon amplifiers. Phys. Lett. A 113: 405-407.

Yuen, H.P. (1986b) Generation, detection, and application of high-intensity photon-number-eigenstate fields. Phys. Rev. Lett. 56: 2176-2179.

Yuen, H.P. (1987) Design of transparent optical networks by using novel quantum amplifiers and sources. Opt. Lett. 12: 789-791.