HOMODYNING AND HETERODYNING
THE QUANTUM PHASE

G. M. D'Ariano, C. Macchiavello and M. G. A. Paris
Dipartimento di Fisica 'Alessandro Volta'
Via Bassi 6, I-27100 Pavia, Italy

ABSTRACT: The double-homodyne and the heterodyne detection schemes for phase shifts between two synchronous modes of the electromagnetic field are analyzed in the framework of quantum estimation theory. The probability operator-valued measures (POM's) of the detectors are evaluated and compared with the ideal one in the limit of strong local reference oscillator. The present operational approach leads to a reasonable definition of phase measurement, whose sensitivity is actually related to the output r.m.s. noise of the photodetector. We emphasize that the simple-homodyne scheme does not correspond to a proper phase-shift measurements as it is just a zero-point detector. The sensitivity of all detection schemes are optimized at fixed energy with respect to the input state of radiation. It is shown that the optimal sensitivity can be actually achieved using suited squeezed states.

1 Introduction

Weak forces on macroscopic bodies in interferometric arrangements or, more generally, minute variations of environmental parameters in optical fibers are detected through the induced changes in the optical paths of the light beams. The detection of the induced phase-shift represents one of the most sensitive measurement on radiation in order to monitor such small perturbations. The back-action effect on the measured parameter due to the radiation pressure imposes limitations on the radiation intensity, and improvements of the sensitivity can only be achieved by suited preparation of the input quantum state.

In this paper a narrowband analysis of some relevant phase detection schemes is presented (a multimode wideband analysis can be found in [1]). Classically a phase-shift measurement in an interference experiment can be directly related to the polar angle between two quadratures of one field-mode, which in turn are given by two output photocurrents. Quantum mechanically the quadratures of the field are noncommuting observables and their relative polar angle cannot be interpreted as a selfadjoint operator, as in the early Dirac's heuristic approach [2]. The oscillator phase is not an observable in usual sense, and the problem of identifying its quantum dynamical counterpart has provoked many discussions in the literature (see, for example, [3, 4, 5] and references therein). Among the numerous attempts the limiting procedure of Pegg and Barnett [3] has become the most popular technique, because it allows the evaluation of expected values with very simple and reliable rules. However, despite its simplicity and effectiveness as a mathematical tool, this approach has no obvious physical interpretation, and leaves most of the conceptual problems on phase detection still open. The actual problem of a phase measurement description does not concerns with a definition of a selfadjoint operator, but with an operative recipe to evaluate the phase statistics in a real measurement, starting from the knowledge of the density matrix of the input radiation. On these lines the most appropriate approach to the phase detection of the field is the quantum estimation theory of [4]. Even though it easy to show that this method is equivalent in the end to the Pegg and Barnett procedure (see [8] for more details), nevertheless it provides a physically meaningful scheme for the phase measurement where all conceptual problems disappear. Despite it has long been recognised as the most natural framework for analyzing any kind of quantum detection, the quantum estimation theory has not

gained the necessary popularity yet, perhaps due to the fact that its main ingredient—the probability operator measure (POM)—is generally a nonorthogonal spectral decomposition, and thus appears to be in conflict with the conventional dictum of quantum mechanics that only "observables"—i.e. orthogonal POM's—can be measured. This point has been well clarified in some papers (see for example [8]), where it is shown that nonorthogonal POM's correspond to actual observables on a larger Hilbert space which includes also modes pertaining the measuring apparatus (all together referred to as "probe"). It is clear that this assertion provides the proper quantum setting for the operational point of view of [11], where the dependence of the measured operator on the detection scheme just corresponds to the involvement of the probe variables in the measuring process, involvement which becomes unavoidable when the phase of the field is detected.

Any quantum measurement needs a classical final stage and for measurement on radiation this is essentially a photodetection. Moreover a proper measurement of the quantum phase has to be related to the detection of a quantity which itself is a phase, i.e. is defined on the unit circle. In this sense we distinguish between two main different classes: the genuine phase detection schemes and the measurements of a single phase-dependent observable. In the former class, the phase-shift of the field is related to the polar angle between two measured photocurrents which, in turn, correspond to a couple of two conjugated quadratures of the field. Such scheme is the only viable one for phase detection, and corresponds equivalently to either heterodyning or double-homodyning the field. This also clarifies the subtle nature of the phase itself which, despite being a single real parameter, nonetheless requires a joint measures of two conjugated operators. In contrast, in the second class of measurements, only a single observable is detected—typically, when homodyning a single-quadrature of the field. Here we want emphasize that a single-quadrature measurement cannot be used to infer the value of the phase, because the knowledge of a quadrature would require an additional measurement on the field—essentially its intensity—which unavoidably would destroy the information on phase. Thus, the single-homodyning scheme can be used only as a zero-phase monitoring technique, which, however, is the essential of a typical interferometric measurement. In order to stress the operational nature of POM approach here we also present, as an example, a numerical simulation of a real experiment, reproducing the classical output photocurrents due to a particular quantum state, and then evaluating the phase statistics as the polar angle distribution.

After selecting a measurement procedure—either ideal or feasible—the statistics of the detected phase can be further improved at fixed total energy by looking for optimal states of the field. We show that the r.m.s. phase sensitivity versus the average photon number $\bar{n}$ is bounded by the ideal limit $\Delta\phi \sim \bar{n}^{-1}$, whereas for the feasible schemes the bound is $\Delta\phi \sim \bar{n}^{-2/3}$, in between the shot noise level $\Delta\phi \sim \bar{n}^{-1/2}$ and the ideal bound. The latter can actually be achieved by single-homodyning suitable squeezed states, but only in the neighborhood of a fixed zero-phase working point. The requirement of detecting the whole phase probability distribution makes the proper phase measurement less sensitive than in the case of a zero-point detection and the state achieving the two bounds are dramatically different: they are weakly squeezed (about 2% of squeezing photons) for the double-quadrature measurement, whereas they become strongly squeezed (50%) for one-quadrature detection.

Sect.2 is devoted to the theory of phase measurements, with a detailed analysis of the various schemes. Subsect. 2.1 is a brief review of the quantum estimation approach. Subsect. 2.2 presents some remarks and criticisms about the different definitions of sensitivity. Detection of the phase through simultaneous measurement of two quadratures of the field are discussed in Subsects. 2.3 and 2.4, which are dedicated to double-homodyne and heterodyne detection (it is shown that the two schemes are fully equivalent). Subsect. 2.5 examines the measurement of the quantum sine and cosine of the phase, with a comparison between the present quantum estimation approach and earlier treatments [7].In Subsect. 2.6 we analyze the homodyne detection scheme. In Sect.3 the optimal states of the field are given, which maximize sensitivity for all the schemes of Sect.2, also indicating how to actually achieve such states.

2 Quantum measurement of the phase

At presently it is fairly universally accepted that the quantum description of a phase measurement on a single mode of the e. m. field cannot be approached by means of the usual concept of observable. As a matter of fact, even though selfadjoint operators can be defined on the Fock space, none of them can appropriately describe the actual statistics of phase measurements. The quantum estimation theory of Helstrom [4] provides the most general—nonetheless operational—framework to analyze any kind of measurement in the quantum context, and, in particular, the detection of a phase differences between synchronous oscillators. The main ingredient of such theory is represented by the mathematical concept of probability-operator-valued measure (POM) on the Hilbert space $\mathcal{H}_S$ of the system, which extends the conventional description by selfadjoint operators. Using a notation which is familiar to physicists—even though not strictly legitimate from the mathematical point of view [8]— given a set of (generally complex) parameters z to be measured, a POM $d\hat{\lambda}(z)$ is a self-adjoint measure with the following properties

$$d\hat{\lambda}(z) \geq 0 , \qquad \int_Z d\hat{\lambda}(z) = \hat{1} , \tag{2.1}$$

where Z denotes the space of the parameters z. Eqs.(2.1) assure that using a POM one has an operational recipe which univocally relates the density-matrix state $\hat{\rho}$ of the system to a probability distributions of the parameters z corresponding to a particular experimental setup. In formulas one has

$$dP(z) = \text{tr}\{\hat{\rho}d\hat{\lambda}(z)\} . \tag{2.2}$$

Also a set of selfadjoint operators $\hat{\Lambda}$ can be defined

$$\hat{\Lambda} = \int_Z z d\hat{\lambda}(z) , \tag{2.3}$$

and, more generally, operator functions $\widehat{f(\Lambda)}$ as

$$\widehat{f(\Lambda)} = \int_Z f(z) d\hat{\lambda}(z) . \tag{2.4}$$

When the POM $d\hat{\lambda}(z) = |z\rangle\langle z|dz$ is given in terms of orthogonal $|z\rangle$'s, then it corresponds to the customary measure of the commuting observables $\hat{\Lambda}$, whose corresponding selfadjoint operators obey the function calculus

$$\widehat{f(\Lambda)} = f(\hat{\Lambda}) . \tag{2.5}$$

On the contrary, the relation (2.5) no longer holds true for a generic nonorthogonal POM. As a consequence, the selfadjoint operators $\hat{\Lambda}$ only provide the expected values of the parameters z, whereas the higher moments of the probability distributions differ from the corresponding moments of the operators $\hat{\Lambda}$ themselves, and can be evaluated only through Eq.(2.2). One should notice that, despite the POM's generally describe measurements that do not correspond to observables in the usual sense, nonetheless there is no conflict with the basic assertion of quantum mechanics that only observables can be measured. In fact, the Naimark theorem assures that every POM can be obtained as a partial trace of a customary projection-valued measure on a larger Hilbert space [8] which itself represents the original system interacting with an appropriate measuring apparatus. Upon denoting by $|\psi(z)\rangle \in \mathcal{H}_S \otimes \mathcal{H}_P$ a complete orthonormal set of eigenvectors of commuting selfadjoint operators acting on the enlarged space—including also the apparatus (*probe*) space $\mathcal{H}_P$—the probability distribution

$$dP(z) = \text{tr}\left(\hat{\rho}_S \otimes \hat{\rho}_P |\psi(z)\rangle\langle\psi(z)|\right) = \text{tr}_S\left[\hat{\rho}_S \text{tr}_P\left(\hat{\rho}_P |\psi(z)\rangle\langle\psi(z)|\right)\right] , \tag{2.6}$$

corresponds to the POM on the Hilbert space $\mathcal{H}_S$ of the system only

$$d\hat{\lambda}(z) = \text{tr}_P\left(\hat{\rho}_P|\psi(z)\rangle\langle\psi(z)|\right) \equiv \langle\psi(z)|\hat{\rho}_P|\psi(z)\rangle .$$ (2.7)

Notice that the above extensions of the system Hilbert space—hence, physically, the experimental realization of the POM—is not necessarily unique.

In Subsect. 2.1 we review the quantum estimation theory of the phase which leads to the optimal POM representing the most accurate measurement. This, however, is only an ideal limit as no viable schemes implementing such POM has been envisaged yet. Therefore, in Subsects.2.3 and 2.4 experimentally achievable detection schemes which correspond to a sub-optimal POM (double-homodyne and heterodyne) are analyzed in detail, whereas in Subsect.2.6 the customary homodyne detection is considered. The latter exits from the present quantum estimation approach, however it is in order, due to the relevance of this scheme in any interferometric setup.

2.1 Canonical Measurement

The quantum estimation theory analyzes the possible strategies for estimating a parameter on the basis of an error-cost-function: the optimum POM is the one which minimizes the total average cost. For a maximum-likelihood criterion, the optimum POM for the phase is

$$d\hat{\mu}(\phi) = \frac{d\phi}{2\pi}|e^{i\phi}\rangle\langle e^{i\phi}| ,$$ (2.8)

$|e^{i\phi}\rangle$ being the Susskind-Glogower phase states [7]

$$|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle .$$ (2.9)

It is worth noticing that in the present case the maximum likelihood criterion is equivalent to the Gaussian-cost-function one for the bounded case ($\sin^2$-cost)[4]. Such generality of the optimal POM $d\hat{\mu}(\phi)$ justifies the term *Canonical Measurement* here adopted for this POM approach. Also notice that the Pegg and Barnett [3] approach is totally equivalent to the present one as regards evaluations of statistics. However, there is no physical interpretation for the mathematical tricks on which their method relies. Some examples of commuting pairs of self-adjoint operators achieving the optimum POM (2.8) on a system-probe Hilbert space have been proposed in [9] and in [5]: however, no viable method for experimentally implementing a corresponding setup has been devised yet, and hence the POM (2.8) only represents an ideal limit. Corresponding to the optimal POM (2.8) one defines the selfadjoint phase operator

$$\hat{\phi} = \int_{-\pi}^{\pi} \phi \, d\hat{\mu}(\phi) = -i\sum_{n\neq m}(-)^{n-m}\frac{1}{n-m}|n\rangle\langle m|$$ (2.10)

and the squared operator

$$\widehat{\phi^2} = \int_{-\pi}^{\pi} \phi^2 d\hat{\mu}(\phi) = \frac{\pi^2}{3} + 2\sum_{n\neq m}(-)^{n-m}\frac{1}{(n-m)^2}|n\rangle\langle m| .$$ (2.11)

Notice that, as announced,

$$\widehat{\phi^2} \neq \hat{\phi}^2 ,$$ (2.12)

and more generally $\widehat{f(\phi)} \neq f(\hat{\phi})$. Such fail of the operator function calculus also holds true for the selfadjoint operators defined through the experimentally feasible non-optimal POM's. The fact that there is no orthogonal optimum POM for the phase, physically corresponds to the impossibility of defining the measurement of the phase independently on the apparatus. This assertion clarifies and formalizes the operational nature of the phase detection which has been pointed out by Mandel et al.in [11].

2.2 Phase Sensitivity

Usually the sensitivity of a measurement of a parameter say $x \in \mathbb{R}$ is assumed equivalent to the r.m.s. of the experimental probability distribution $dP(x)$, namely

$$\overline{\Delta x^2} = \int_{\mathbb{R}} dP(x)x^2 - \left(\int_{\mathbb{R}} dP(x)x\right)^2 . \tag{2.13}$$

On the other hand, the phase variable ϕ is defined in the bounded domain $[-\pi, \pi)$ with 2π-periodicity: this peculiar property of the phase has lead many authors to the conclusion that the r.m.s. of the phase is not the appropriate quantity to be considered as an evaluation of the phase sensitivity, claiming that it is not invariant under phase variable translation $\phi \longrightarrow \phi + \chi$. Thus, different definitions for the sensitivity have been adopted, which would be equivalent for an unbounded Gaussian-distributed variable. Here, after critical reviewing such quantities, we show that an operational definition of a phase measurement procedure leads unequivocally to adopt the r.m.s. itself as the correct sensitivity parameter.

a. Phase dispersion D[12, 13].
Dispersion D is defined as follows

$$D \equiv (1 - \langle\cos\phi\rangle^2 - \langle\sin\phi\rangle^2)^{\frac{1}{2}} = 1 - \left|\sum_{n=0}^{\infty} c_{n+1}^* c_n\right|^2 , \tag{2.14}$$

where c_n are the coefficient of the number representation of the state and the sine and cosine operators are defined according to Eq.(2.4) as follows

$$\widehat{\cos\phi} = \int_{-\pi}^{\pi} \cos\phi \, d\hat{\mu}(\phi) , \tag{2.15}$$

$$\widehat{\sin\phi} = \int_{-\pi}^{\pi} \sin\phi \, d\hat{\mu}(\phi) , \tag{2.16}$$

and coincide with the sine and cosine operators of Susskind and Glogower [7]. The definition (2.14) follows from elementary error-propagation calculus, the phase ϕ being regarded as a function of the two "independent variables" $\sin\phi$ and $\cos\phi$ as follows

$$\phi = -i\ln(\cos\phi + i\sin\phi) . \tag{2.17}$$

In Eq.(2.17) the correct logarithm branch is selected in order to obtain the desired domain for ϕ. A part the minor point that Eq.(2.14) would lead to dispersion $D = 1$ for constant distributions—instead of $\overline{\Delta\phi^2} = \pi^2/3$—the main criticism is that $\sin\phi$ and $\cos\phi$ cannot be considered as independent variables, because they correspond to a noncommuting pair of operators which are jointly measured when detecting ϕ.

b. Reciprocal peak likelihood $\delta\phi$[5].
The peak likelihood $P(\phi|\phi)$ is the maximum height of the probability distribution. Its inverse, namely

$$\delta\phi = \frac{1}{P(\phi|\phi)} = 2\pi \left(\sum_{n=0}^{\infty} |c_n|\right)^{-2} , \tag{2.18}$$

has been introduced in [5] as a measure of the width of the distribution, coherently with the maximum-likelihood strategy used in the quantum estimation theory. Here, the following criticisms are in order: i) $\delta\phi$ is a *local* criterion, namely it checks only one point of the distribution, whereas there is no control on the global behaviour as, for example, on the eventual occurrence of high tails. The most degenerate situation occurs when the tails are so high that the distribution itself converges to the $P(\phi) = 1/2\pi$ apart from

one point with infinite probability density and zero integral [14], thus leading to vanishing $\delta\phi$ instead of $\delta\phi = \pi^2/3$; ii) the coherence of this sensitivity definition with the maximum-likelihood strategy [5] cannot be considered as a valid argument, in view of the aforementioned equivalence between the likelihood strategy and the (quasi)-Gaussian one; iii) recent numerical results [6] have shown that the simulated sensitivity does actually not correspond to $\delta\phi$.

c. POM r.m.s. $\langle \widehat{\Delta\phi^2} \rangle$

Given a physical apparatus (or an ideal detector) one has a corresponding POM and, in turn, a probability distribution $dP(\phi)$ according to Eq.(2.2). Such probability has a r.m.s. error (2.13) given by

$$\langle \widehat{\Delta\phi^2} \rangle = \langle \widehat{\phi^2} \rangle - \langle \widehat{\phi} \rangle^2 . \tag{2.19}$$

Here $\langle \ldots \rangle$ denotes the ensemble quantum average on the system space $\mathcal{H}_S$, and the operators $\widehat{\phi}$ and $\widehat{\phi^2}$ depend on the considered POM (for the optimum POM they are given in Eqs.(2.10,2.11)). Notice that there is no ambiguity in choosing between the two operators $\widehat{\phi^2} \neq \widehat{\phi}^2$, because Eq.(2.19) directly follows from the probability (2.2). For a random-phase state—namely a constant probability distribution—one correctly has $\langle \widehat{\Delta\phi^2} \rangle = \pi^2/3$.

As regards the problem of invariance under phase shifts, here we stress that this actually is not a problem. In fact, the only concern is the correspondence between experimental and theoretical quantities, and the circular topology of the phase arises at both experimental and theoretical levels in the same way. Whatever procedure is considered for measuring the phase, the information on it has always to be inferred from a joint sine-cosine measurement, and hence the experimental equipment itself has to be tuned on a selected 2π-window. Once the domain is fixed, the experimental noise is, by definition, the r.m.s. noise on such domain. Therefore, different choices of the 2π-window actually lead to different experimented amounts of noise, and also theoretically the r.m.s. noise has to be evaluated on the chosen domain (hereafter we will always use the $[-\pi, \pi)$ window).

2.3 Double-homodyne Detection

The double-balanced-homodyne [11] (DBH) detection provides a way for simultaneously measuring a couple of field-quadratures for one mode of e. m. field. The schematic diagram of the experimental set-up is reported in Fig. 1. There are four 50-50 beam splitters and four identical photocounters, and a $\pi/2$ phase shifter is inserted in one arm. The mode supporting the phase is a, whereas a stable reference for the phase is provided by a local oscillator (LO) which is synchronous with a and is prepared in a highly excited coherent state $|z\rangle$.

The DBH scheme can also perform a phase measurement, however with a probability distribution which does not correspond to the ideal case due to unavoidable addition of "instrumental" noise. The DBH phase distribution is obtained through the following procedure. Each experimental event consists of a simultaneous detection of the two difference photocurrents $\hat{I}_1 = \hat{n}_6 - \hat{n}_5$ and $\hat{I}_2 = \hat{n}_4 - \hat{n}_3$ which "trace" two field-quadratures. Each event thus corresponds to a point plotted in the complex plane of the field amplitude. The phase relative to the event is nothing but the polar angle of the point itself. An experimental histogram of the phase distributions is thus obtained upon dividing the plane into small ("infinitesimal") angular bins of equal width $\delta\phi$, from $-\pi$ to π, then counting the number of points which fall into each bin. In formulas, one has the statistical frequency P_n for the n-th bin $\theta_n \equiv [-\pi + n\delta\phi, -\pi + (n+1)\delta\phi)$

$$P_n = \frac{1}{N} \{ \# \text{ of events with } I_1 = \rho\cos\phi , I_2 = \rho\sin\phi , \phi \in \theta_n \} , \tag{2.20}$$

where $\rho = \sqrt{I_1^2 + I_2^2}$ and N is the total number of experimental points.

In Fig. 2, as an example, a computer simulation of the above experimental procedure is illustrated for a squeezed state with equal number $\langle n \rangle = 10$ of signal and squeezing photons. The experimental histogram
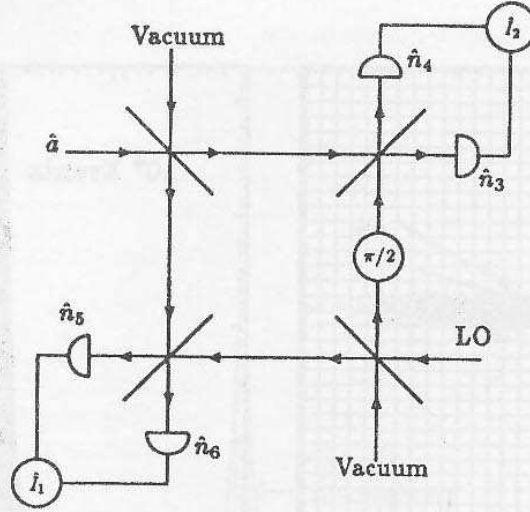
Figure 1: Outline of scheme of a double-homodyne detectors

(10^4 events) is compared with the theoretical results from the POM for the DBH detection. This can be obtained as follows. The difference photocurrents $\hat{I}_1$ and $\hat{I}_2$ are commuting operators with factorized probability $P(I_1, I_2) = P(I_1)P(I_2)$. Introducing the *reduced current* $\hat{\mathcal{I}} = \hat{I}/|z|$ for each homodyne detector, one has the probability distribution in terms of the Fourier-transform of the generating function for the moments $\langle e^{i\lambda\hat{\mathcal{I}}} \rangle$

$$P(\mathcal{I}) = \int_{-\pi|z|}^{\pi|z|} \frac{d\lambda}{2\pi} \, \text{tr}\{\hat{\rho} e^{i\lambda(\hat{\mathcal{I}}-\mathcal{I})}\}. \tag{2.21}$$

The phase distribution is the marginal probability integrated over the modulus ρ

$$P(\phi) = \int_0^\infty \rho d\rho \, P_1(\rho\cos\phi) P_2(\rho\sin\phi) \,. \tag{2.22}$$

Using Eq.(2.21) one has

$$P(\phi) = \int_0^\infty \rho d\rho \int_{-\pi|z|}^{\pi|z|} \frac{d\mu}{2\pi} \int_{-\pi|z|}^{\pi|z|} \frac{d\nu}{2\pi} \, \text{tr}\{\hat{\rho}_S \otimes \hat{\rho}_P \, e^{i\mu(\hat{\mathcal{I}}_1 - \rho\cos\phi) + i\nu(\hat{\mathcal{I}}_2 - \rho\sin\phi)}\} \tag{2.23}$$

$\hat{\rho}_S$ being the density matrix of the mode a (the system) and

$$\hat{\rho}_P = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |z\rangle\langle z| \tag{2.24}$$

the density matrix of the probe. From Eqs.(2.6,2.7) one can see that the "experimental" POM is obtained upon tracing over the probe Hilbert space $\mathcal{H}_P$, thus obtaining the operator which acts solely on the system space $\mathcal{H}_S$

$$d\hat{\mu}_D(\phi) = d\phi \int_0^\infty \rho d\rho \int_{-\pi|z|}^{\pi|z|} \frac{d\mu}{2\pi} \int_{-\pi|z|}^{\pi|z|} \frac{d\nu}{2\pi} \, \text{tr}_P\{\hat{1}_S \otimes \hat{\rho}_P \, e^{i\mu(\hat{\mathcal{I}}_1 - \rho\cos\phi) + i\nu(\hat{\mathcal{I}}_2 - \rho\sin\phi)}\}. \tag{2.25}$$
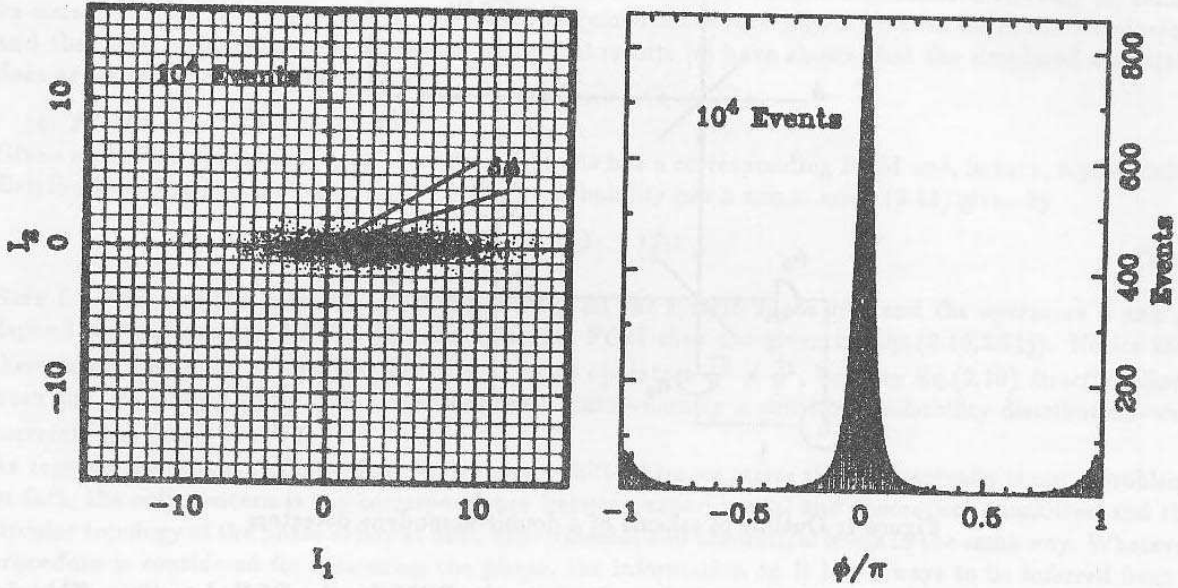
135

Figure 2: Computer simulation of the DBH experimental procedure for a squeezed state with equal number $\langle n \rangle = 10$ of signal and squeezing photons. The experimental histogram (10^4 events) is compared with the theoretical results from the POM in Eq.(2.26)

Using the coherent state resolution of the identity, the following closed formula is obtained (the detailed derivation is reported in the appendix)

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{2\pi} \sum_{n \neq m} e^{i(n-m)\phi} \frac{\Gamma(\frac{n+m}{2}+1)}{\sqrt{n!m!}} |n\rangle\langle m| , \qquad (2.26)$$

where $\Gamma(z)$ is the Eulero's gamma function.

The POM in Eq.(2.26) for the DBH detector corresponds to an *effective* measured phase operator which is given by

$$\hat{\phi}_D = \int \phi \, d\hat{\mu}_D(\phi) = -i \sum_{n \neq m} (-)^{n-m} \frac{1}{n-m} \frac{\Gamma(\frac{n+m}{2}+1)}{\sqrt{n!m!}} |n\rangle\langle m| , \qquad (2.27)$$

and the squared one

$$\widehat{\phi_D^2} = \int \phi^2 \, d\hat{\mu}_D(\phi) = \frac{\pi^2}{3} + 2 \sum_{n \neq m} (-)^{n-m} \frac{1}{(n-m)^2} \frac{\Gamma(\frac{n+m}{2}+1)}{\sqrt{n!m!}} |n\rangle\langle m| , \qquad (2.28)$$

needed for evaluation of the instrumental sensitivity $\langle \widehat{\Delta\phi_D^2} \rangle$. For any state of the mode a one can simply verify that

$$\langle \widehat{\Delta\phi_D^2} \rangle \geq \langle \widehat{\Delta\phi^2} \rangle , \qquad (2.29)$$

namely the DBH scheme adds extrinsic instrumental noise, as it does not implement the optimal canonical measurement of the phase. However, we stress again that the DBH detection is the best available method for detecting the phase. In Fig.3 a comparison between the canonical (ideal) and DBH (feasible) phase probability distributions is given for the same state of the computer simulation in Fig.2, showing that the former is sharper and higher than the latter.
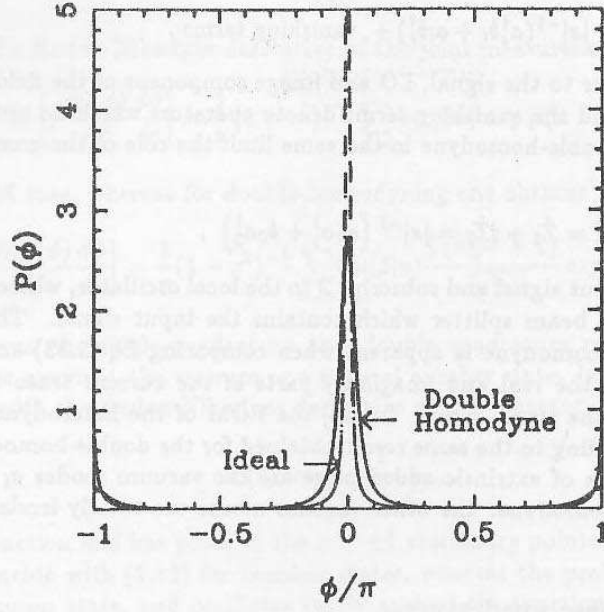
Figure 3: Comparison between the ideal and double-homodyne phase probability distributions for the same squeezed state of Fig.2.

2.4 Heterodyne Detection

The first proposed method to perform simultaneous measurements of two field-quadratures was the heterodyne detection. Here we synthetically analyze this scheme, only in order to make a connection with the double-homodyne detector and show that that the two apparatus are completely equivalent from the point of view of the measured physical quantities. The input field E_{IN} impinges into a beam splitter and has nonzero photon number only at the frequency $\omega_0 + \omega_{IF}$. The local oscillator works at the different frequency ω_0, and the output the photocurrent $\hat{I}_{OUT}$ is measured at the *intermediate frequency* ω_{IF}. The measured photocurrent is given by

$$\hat{I}_{OUT}(t) = \hat{E}_{OUT}^-(t)\hat{E}_{OUT}^+(t), \tag{2.30}$$

where $E^\pm$ denote the usual positive and negative frequency components of the field. The component at frequency ω_{IF} is given by

$$
\begin{aligned}
\hat{I}_{OUT}(\omega_{IF}) &= \int dt\, \hat{I}_{OUT}(t) e^{i\omega_{IF}t} \\
&= \int d\omega\, \hat{E}_{OUT}^-(\omega + \omega_{IF})\hat{E}_{OUT}^+(\omega).
\end{aligned} \tag{2.31}
$$

For a nearly transparent beam splitter, and in the limit of strong LO in the coherent state $|z\rangle$ one can define the reduced complex current $\hat{y}$

$$\hat{y} = \lim_{\substack{\eta \to 1, |z| \to \infty \\ \gamma = \text{cost.}}} \gamma^{-1}\hat{I}_{OUT}(\omega_{IF}), \tag{2.32}$$

where $\gamma = |z|\sqrt{\eta(1-\eta)}$. In this limit the expression of $\hat{y}$ is given by

$$\hat{y} = |z|^{-1}(a_s^\dagger b_l + a_i b_l^\dagger) + \text{ vanishing terms} ,\qquad (2.33)$$

where the subscript s, l and i refer to the signal, LO and image component of the field respectively, a are signal modes, b the LO modes, and the vanishing terms denote operators which do not give contributions in the strong LO limit. In the double-homodyne in the same limit the role of the complex current (2.33) is played by

$$\hat{\mathcal{I}} = \hat{\mathcal{I}}_1 + i\hat{\mathcal{I}}_2 = |z|^{-1}\left(a_2 a_1^\dagger + b_0 a_2^\dagger\right) ,\qquad (2.34)$$

where subscript 1 refers to the input signal and subscript 2 to the local oscillator, whereas b_0 is the vacuum mode at the unused port of the beam splitter which contains the input signal. The fully equivalence between heterodyne and double-homodyne is apparent when comparing Eq.(2.33) and Eq.(2.34). As in the double-homodyne case, now the real and imaginary parts of the current trace the two conjugated quadratures $a_{s\phi}$ and $a_{s\phi+\pi/2}$ of the signal mode. In [16] the POM of the heterodyne detector has been derived in a different context, leading to the same result obtained for the double-homodyne in Subsect.2.3. We notice that the actual sources of extrinsic added noise are the vacuum modes a_i for the heterodyne detector and b_0 for the double-homodyne: the other vacuum modes are totally irrelevant in the limit of strong LO.

2.5 Measurement of the phase quadratures

The POM approach naturally leads to well defined operator functions of the phase which obey the trigonometric calculus at the operator level, and, hence, also at the level of expectation values. In particular, the sine and cosine operators are defined as in (2.16). Such definitions coincide, in the case of optimum POM, with the sine and cosine operators $\hat{s}$ and $\hat{c}$ introduced by Susskind and Glogower [7]

$$\hat{s} = \frac{1}{2i}(\hat{e}^- - \hat{e}^+) , \qquad \hat{c} = \frac{1}{2}(\hat{e}^- + \hat{e}^+) ,\qquad (2.35)$$

where $\hat{e}_\pm$ denote the raising and lowering operators $\hat{e}_+|n\rangle = |n+1\rangle$, $\hat{e}_- \equiv (\hat{e}_+)^\dagger$. Notice, however, that this equivalence between operators fails for higher powers, namely

$$\hat{c}^n \neq \widehat{\cos^n}\phi , \qquad \hat{s}^n \neq \widehat{\sin^n}\phi , \qquad \text{for } n > 1 \qquad (2.36)$$

Here some remarks are in order, regarding relevant differences between a conventional measurement of a single phase-quadrature—say the cosine $\hat{c}$—and a joint measurement of both sine-cosine quadratures which have been analyzed in previous Subsections. A single phase-quadrature measurement leads to violation of the trigonometric calculus for expectation values. In fact, for a general density matrix state $\hat{\rho}$ one has that

$$\text{Tr}[\hat{\rho}(\hat{c}^2 + \hat{s}^2)] = 1 - \frac{1}{2}\langle 0|\hat{\rho}|0\rangle ,\qquad (2.37)$$

whereas for a joint measurement one obtains

$$\text{Tr}[\hat{\rho}(\widehat{\sin^2}\phi + \widehat{\cos^2}\phi)] = 1 .\qquad (2.38)$$

We stress again that, however, the linear operators coincide in the two cases, and thus one gets the same average values. However, the probability distribution of the outcomes from single phase-quadrature measurement exhibits unphysical features for nonclassical states, whereas the probability distribution from the joint measurement does not. In the single-quadrature measurement one has

$$P(c) = \text{tr}\{\hat{\rho}|c\rangle\langle c|\} ,\qquad (2.39)$$

where the eigenstates of $\hat{c}$ are given by [7, 15]

$$|c\rangle = \sqrt{\frac{2}{\pi}}(1 - c^2)^{-1/4} \sum_{n=0}^{\infty} \sin[(n+1)\arccos c]|n\rangle \tag{2.40}$$

On the other hand, the Radon-Nikodym derivative of the joint measurement POM's leads to

$$P(c) = \text{Tr}\left[\hat{\rho}\frac{d\hat{\mu}(\phi)}{d\phi}\frac{d\phi}{dc}\right] = \frac{1}{\pi}(1 - c^2)^{-\frac{1}{2}}\sum_{n,m}\langle m|\hat{\rho}|n\rangle \exp\left[i(n-m)\arccos c\right], \tag{2.41}$$

for the optimum POM case, whereas for double-homodyning one obtains

$$P(c) = \text{Tr}\left[\hat{\rho}\frac{d\hat{\mu}_D(\phi)}{d\phi}\frac{d\phi}{dc}\right] = \frac{1}{\pi}(1 - c^2)^{-\frac{1}{2}}\sum_{n,m}\langle m|\hat{\rho}|n\rangle \frac{\Gamma(\frac{m+n}{2}+1)}{\sqrt{n!m!}}\exp\left[i(n-m)\arccos c\right]. \tag{2.42}$$

The differences between the single-quadrature and double-quadrature probabilities become striking for isotropic states, as, for example, the vacuum or a general number state. In this case the above distribution should be compared with the Radon-Nikodym derivative of the constant distribution

$$P(c) = \frac{1}{\pi}\frac{1}{\sqrt{1 - c^2}}, \tag{2.43}$$

which is a concave function and has poles at the $c = \pm 1$ stationary points of the cosine. The probabilities (2.41) and (2.42) coincide with (2.43) for number states, whereas the probability (2.39) has the opposite curvature for the vacuum state, and oscillates fastly around the function (2.43) for nonvacuum number states. These undesired physical features disappear for highly excited coherent states, where, however, the main quantum features are lost.

2.6 Homodyne Detection

This Subsection is devoted to the customary homodyne detector, which, despite it exits from the present phase estimation treatment, however it is the most relevant device in any interferometric setup. Actually, the homodyne detector belongs to the class of the *zero-point* measurement schemes, and thus *is not a measurement of phase*. The balanced homodyne scheme measures one quadrature of a field mode, which in turn is related to its phase difference with respect to the synchronous LO. Generally one is interested in the measure of the phase shift χ of the signal state

$$|\psi\rangle_\chi = \exp(-i\chi\hat{n})|\psi\rangle_0, \tag{2.44}$$

where, without loss of generality, the input state is assumed of the form

$$|\psi\rangle_0 = \sum_{n=0}^{\infty} c_n|n\rangle, \quad c_n \geq 0. \tag{2.45}$$

The expectation value of the quadrature is given by

$$\langle \hat{a}_\phi \rangle_\chi = \sum_{n=0}^{\infty} \sqrt{n+1}c_n c_{n+1}\cos(\phi - \chi) = \langle \hat{a}_0 \rangle_0 \cos(\phi - \chi). \tag{2.46}$$

The quadrature $\hat{a}_\phi$ is proportional to the cosine of the phase with a proportionality "constant" $\langle \hat{a}_0 \rangle_0$ which can be evaluated from the knowledge of the fixed input state. Notice that, however, when the present scheme is regarded as a measure of the phase of the state $|\psi\rangle_\chi$ itself the state-depending "constant

" is unknown, and it cannot be preventively measured without destroying the information on phase. In this case the only point which does not need any knowledge of the "constant" is the $\phi - \chi = \pi/2$ point, namely just the maximum-derivative zero-current working point. Thus, it is not possible to detect a arbitrary phase difference, the sensitivity depending on the particular input signal, whereas a suitable feedback mechanism is needed to follow the working point. A convenient description of the homodyne detector in view of the above considerations is given in [17], where the zero-point (zero-field) probability distribution is revisited as a sort of a phase probability distribution.

The phase sensitivity of the homodyne detector can be straightforwardly obtained from error propagation calculus using the relation

$$\delta\phi = \sqrt{\langle \Delta \hat{a}_\phi^2 \rangle} \left| \frac{\delta \langle \hat{a}_\phi \rangle}{\delta\phi} \right|^{-1} , \tag{2.47}$$

which is customary in the literature on interferometry. One can see that the $\phi - \chi = \pi/2$ working point minimizes sensitivity.

3 Optimal states for phase measurements

The design a phase measurement needs optimization of both the detection scheme and of the quantum state which carries the phase information. The former is the main task of quantum estimation theory, which leads to an ideal scheme to be compared with the feasible ones. The latter, which is the main concern of this Section, depends on the detection scheme itself, and should account for the actual physical constraints, mainly the total power impinged into the state. Therefore, the problem is that of optimizing the r.m.s sensitivity $\Delta\phi \stackrel{\text{def}}{=} \sqrt{\langle \widehat{\Delta\phi^2} \rangle}$ for fixed average photon number, and depending on the particular detection scheme.

In the following we consider, without loss of generality, a zero average phase state, with real coefficients on the number basis, namely

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad c_n \in \mathbb{R}. \tag{3.1}$$

The state optimization problem is to minimize a quantity of the form

$$\langle \widehat{\Delta\phi^2} \rangle = \frac{\pi^2}{3} + 2 \sum_{n \neq m} A_{n,m} \, c_n c_m , \tag{3.2}$$

with the constraints

$$\sum_{n=0}^{\infty} c_n^2 = 1 , \qquad \sum_{n=0}^{\infty} n \, c_n^2 \equiv \langle n \rangle = \bar{n} . \tag{3.3}$$

The (real symmetric) matrix $\mathbf{A} = \{A_{n,m}\}$ depends on the detection scheme. In particular, for $n \neq m$ one has ($A_{n,n} = 0$)

$$A_{n,m} = \frac{(-)^{n-m}}{(n-m)^2} , \qquad \text{(canonical)} \tag{3.4}$$

$$A_{n,m} = \frac{(-)^{n-m}}{(n-m)^2} \frac{\Gamma(\frac{n+m}{2}+1)}{\sqrt{n!m!}} \qquad \text{(DBH)}. \tag{3.5}$$

The method of Lagrange multipliers reduces the problem to that of minimizing the following expression

$$F(\{c_n\}; \lambda, \beta | \bar{n}) = \frac{\pi^2}{3} + 2 \sum_{n \neq m} A_{n,m} \, c_n c_m + \lambda \left(\sum_{n=0}^{\infty} c_n^2 - 1 \right) + \beta \left(\sum_{n=0}^{\infty} n c_n^2 - \bar{n} \right) \tag{3.6}$$

with respect to $\{c_n\}$, λ and β being the Lagrange multipliers. The variational problem (3.6) is that of a quadratic symmetric form and is equivalent to the eigenvalue problem

$$(\mathbf{M}(\beta) + \lambda \mathbf{I}) \cdot \mathbf{c} = 0 \qquad \mathbf{c} \stackrel{def}{=} (c_0, c_1, \dots) , \tag{3.7}$$

for symmetric matrix $\mathbf{M} = \{M_{nm}\}$ given by

$$M_{n,m} = A_{nm} + \delta_{nm} \beta n . \tag{3.8}$$

Eq.(3.7) for the matrix (3.8) can be numerically solved upon suitable truncation of the Hilbert space $\mathcal{H}_s$. The absolute minimum corresponds to eigenvalue $\lambda = \pi^2/3$, and $\bar{n} \in [0, \dim \mathcal{H}_s/2]$ turns out to be a decreasing function of the running parameter $\beta \in [0,1]$. Notice that one should consider only average values $\bar{n} \ll \dim \mathcal{H}_s/2$, such that the number distribution has vanishing tail at $n = \dim \mathcal{H}_s$, in order to avoid undesired numerical boundary effects.
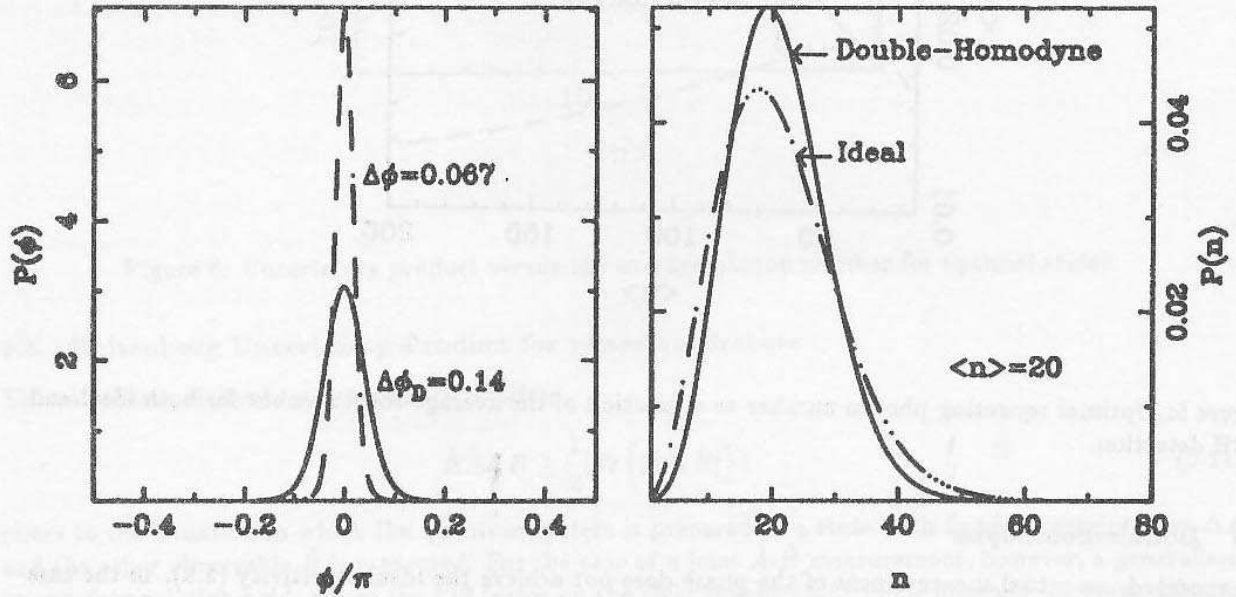

Figure 4: Phase and number probability distributions of an optimal states of $\bar{n} = 20$ for both ideal and DBH detection

3.1 Canonical Measurement

For ideal measurement of the phase, the best phase states obtained through the above optimization procedure, lead to the simple power-law

$$\Delta\phi \sim \frac{1.36 \pm 0.01}{\bar{n}^{1.00 \pm 0.01}} , \tag{3.9}$$

in agreement with results of [18]. The proportionality constant actually increases very slowly as a function of $\bar{n}$, and one has a variation of few percent for two decades of $\Delta\phi$. Eq.(3.9) can be compared with the result of [18], and with the theoretical bound $\Delta\phi \sim 1/(e\bar{n})$ [19] obtained by means of information-theory arguments. One should notice that essentially the same result can be obtained for large $\bar{n}$ ($\bar{n} > 10$) using

squeezed states where the squeezing photon number is optimized as a function of the average total number. It turns out that the optimal states have only $\sim 3.7\%$ of squeezing photons (see Fig. 5). This result is quite different from the customary 50% optimal squeezing number (which also holds true for the homodyne sensitivity of the Mack-Zehnder interferometer [20]).
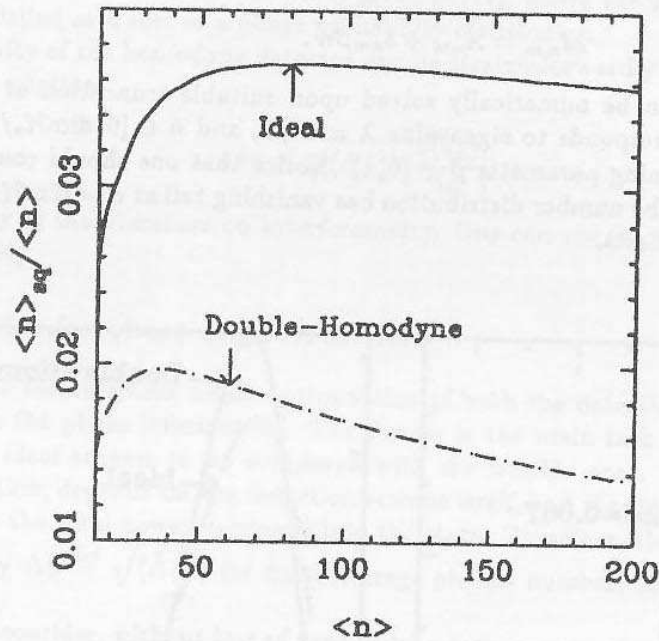

Figure 5: Optimal squeezing photon number as a function of the average total number for both ideal and DBH detection.

3.2 Double-homodyne

As expected, an actual measurement of the phase does not achieve the ideal sensitivity (3.9). In the case of double-homodyne (or equivalently heterodyne) phase detection, the resulting power-law is

$$\Delta\phi_D = \frac{(1.00 \pm 0.01)}{\bar{n}^{0.65\pm0.01}}, \qquad (3.10)$$

which is obtained by numerically solving Eq.(3.7) for matrix M given in Eqs.(3.8) and (3.5). In Fig. 4 the optimized states for both canonical and DBH detection are compared for an equal fixed average photon number $\bar{n} = 20$. One can see that the number and phase probability distributions are qualitatively similar, however the DBH optimum states are slightly sharper in the number distribution and larger in the phase one. Also the best DBH states are essentially indistinguishable from squeezed states which are optimized in the squeezing photon number as a function of the average total number (see Fig. 5). In this case, only less than $\sim 2\%$ of squeezing photons turns out to be optimal.
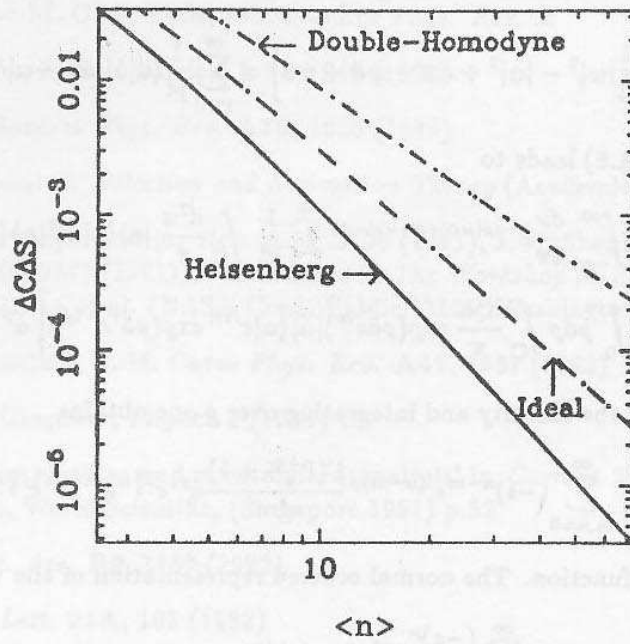
Figure 6: Uncertainty product versus the average photon number for optimal states.

3.3 Heisenberg Uncertainty Product for phase quadrature

The customary Heisenberg uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} |\text{Tr}\{\hat{\rho}[\hat{A}, \hat{B}]\}| \qquad (3.11)$$

refers to the situation in which the quantum system is prepared in a state with fixed uncertainty say ΔA and the other observable $\hat{B}$ is measured. For the case of a joint $\hat{A}$-$\hat{B}$ measurement, however, a generalized uncertainty relation hold, where the 1/2 factor on the right side of Eq.(3.11) is dropped, corresponding to an added noise of 3 dB [21]. For the phase-quadratures one has the commutation relation

$$[\hat{c}, \hat{s}] = -\frac{i}{2} |0\rangle\langle 0|, \qquad (3.12)$$

corresponding to the joint-measurement uncertainty relation

$$\Delta c \Delta s \geq \frac{1}{2} |\langle \psi|0\rangle|^2. \qquad (3.13)$$

In Eqs.(3.11) and (3.13) the uncertainties are defined in the usual way, namely $\Delta O^2 = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$. On the other hand, in the POM approach the correct uncertainty (namely the measured quantity) is defined as $\Delta O^2 = \langle \widehat{O^2} \rangle - \langle \hat{O} \rangle^2$, where $\widehat{O^2} \neq \hat{O}^2$ is defined as in Eq.(2.4). In general, by means of Schwartz inequality, one obtains

$$\langle \widehat{O^2} \rangle - \langle \hat{O} \rangle^2 \geq \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2. \qquad (3.14)$$

Taking the strong LO limit $|z| \to \infty$ and introducing the complex variable $\alpha = \frac{1}{2}(\nu + i\mu)e^{i \arg(z)}$ one gets

$$R = \exp\left[-\frac{1}{2}|u|^2 - \frac{1}{2}|w|^2 - |\alpha|^2 + \alpha\bar{u} - w\bar{\alpha} + w\bar{u}\right] = \sum_{p=0}^{\infty} \frac{1}{p!}\langle u|\hat{a}^\dagger|\alpha\rangle\langle -\alpha|\hat{a}|w\rangle . \tag{A.9}$$

Substituting Eq.(A.9) into Eq.(A.5) leads to

$$
\begin{aligned}
d\hat{\mu}_D(\phi) &= d\phi \int_0^\infty \rho d\rho \int_{-\infty}^\infty \frac{d\mu}{2\pi} \int_{-\infty}^\infty \frac{d\nu}{2\pi} e^{-i\rho(\mu\cos\phi + \nu\sin\phi)} \sum_{p=0}^\infty \frac{1}{p!} \int_C \frac{d^2u}{\pi}|u\rangle\langle u|\hat{a}^{p\dagger}|\alpha\rangle\langle -\alpha|\hat{a}^p \int_C \frac{d^2w}{\pi}|w\rangle\langle w| \\
&= d\phi \sum_{p=0}^\infty \frac{1}{p!}\frac{1}{2\pi}\hat{a}^{p\dagger}\left[2\int_0^\infty \rho d\rho \int_C \frac{d^2\alpha}{\pi}\exp(\rho\hat{a}e^{i\phi})|\alpha\rangle\langle\alpha|e^{i\pi\hat{n}}\exp(\rho\hat{a}^\dagger e^{-i\phi})\right]\hat{a}^p . \tag{A.10}
\end{aligned}
$$

Using the coherent resolution of the identity and integrating over ρ one obtains

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{2\pi}\sum_{p=0}^\infty \frac{1}{p!}\sum_{m,n=0}^\infty (-i)^{n-m}e^{i(n-m)\phi}\frac{\Gamma(\frac{n+m}{2}+1)}{n!m!}\hat{a}^{p\dagger}e^{i\frac{\pi}{2}\hat{n}}\hat{a}^{m\dagger}\hat{a}^n e^{i\frac{\pi}{2}\hat{n}}\hat{a}^p , \tag{A.11}$$

where $\Gamma(z)$ is Eulero's Gamma function. The normal ordered representation of the vacuum state

$$\lim_{\epsilon\to 1}\sum_{p=0}^\infty \frac{(-\epsilon)^p}{p!}\hat{a}^{p\dagger}\hat{a}^p = |0\rangle\langle 0| , \tag{A.12}$$

leads to

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{2\pi}\sum_{m,n=0}^\infty (-i)^{n-m}e^{i(n-m)\phi}\frac{\Gamma(\frac{n+m}{2}+1)}{n!m!}e^{i\frac{\pi}{2}\hat{n}}\hat{a}^{m\dagger}|0\rangle\langle 0|\hat{a}^n e^{i\frac{\pi}{2}\hat{n}} . \tag{A.13}$$

From Eq.(A.13) one obtains the POM of the detector in form of a double series

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{2\pi}\sum_{n,m} e^{i(n-m)\phi}\frac{\Gamma(\frac{n+m}{2}+1)}{\sqrt{n!m!}}|n\rangle\langle m| . \tag{A.14}$$

Alternatively, using the Γ-function integral representation one can write

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{\pi}\int_0^\infty \rho d\rho\, e^{-\rho^2}e^{\rho e^{i\phi}a^\dagger}|0\rangle\langle 0|e^{\rho e^{-i\phi}a} = \frac{d\phi}{\pi}\int_0^\infty \rho d\rho\,|\rho e^{i\phi}\rangle\langle\rho e^{i\phi}| . \tag{A.15}$$

References

[1] G. M. D'Ariano, and M. G. A. Paris, submitted to *Phys. Rev. A*

[2] P. A. M. Dirac. Proc. R. Soc. London A 114 243 (1927)

[3] D. T. Pegg, S. M. Barnett *Phys. Rev.* **A39**, 1665 (1989)

[4] C. W. Helstrom *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976)

[5] J. H. Shapiro, S. R. Shepard *Phys. Rev.* **A43**, 3795 (1991), J. H. Shapiro, S. R. Shepard, N. C. Wong *Phys. Rev. Lett.* **62**, 2377 (1991), J. H. Shapiro in *The Workshop on Squeezed States and Uncertainty Relations* ed. by D.Han et al. (NASA Conf. Public. 3135, Washington DC, 1992), p.107

[6] A. S. Lane, S. Braunstein, C. M. Caves *Phys. Rev.* **A47**, 1667 (1993)

[7] L. Susskind and J. Glogower, *Physics* **1** (1964) 49.

[8] M. Ozawa, 'Operator algebras and nonstandard analysis' in *Current Topics in Operator Algebras* ed. by H. Araki et. al., World Scientific, (Singapore 1991) p.52

[9] M. Ban *J. Opt. Soc. Am.* **B9**, 1189 (1992)

[10] H. P. Yuen *Phys. Lett.* **91A**, 101 (1982)

[11] J. W. Noh, A. Fougeres, L. Mandel *Phys. Rev. Lett.* **67**, 1426 (1991); *Phys. Rev.* **A45**, 424 (1992); *Phys. Rev.* **A46**, 2840 (1992)

[12] Z. Hradil *Quantum Opt.* **4**, 93 (1992)

[13] A. Bandilla, H. Paul, H. H. Ritze *Quantum Opt.* **3**, 267 (1991)

[14] Z. Hradil, J. H. Shapiro *Quantum Opt.* **4**, 31 (1992)

[15] P. Carruthers, M. M. Nieto *Rev. Mod. Phys.* **40**, 411 (1968)

[16] M. J. W Hall, I. G. Fuss *Quantum Opt.* **3**, 147 (1991)

[17] W. Vogel, W. Schleich *Phys. Rev.* **A44**, 7642 (1991)

[18] G. S. Summy, D. T. Pegg, *Opt. Commun.* **77**, 75 (1990)

[19] H. P. Yuen, in *The Workshop on Squeezed States and Uncertainty Relations* ed. by D.Han et al. (NASA Conf. Public. 3135, Washington DC, 1992), p.13

[20] R. S. Bondurant and J. H. Shapiro, Phys. Rev. **A30** 2548 (1984)

[21] E. Arthurs, M. S. Goodman *Phys. Rev. Lett.* **60**, 2447 (1988)