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Amplitude squeezing through operator scaling

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ABSTRACT: A noncanonical entropy-varying operator scaling which produces amplitude squeezing is presented. The mechanism is equivalent to a nonselective measurement of degrees of freedom interacting with radiation.

1. INTRODUCTION

Whenever quantum fluctuations limit the accuracy of measurements, one tries to moderate or suppress their effect by 'squeezing' the state of the field in the variable to be measured. Reduction of quantum noise in the observable of interest $\hat{O}_1$ affects the statistics of a noncommuting observable $\hat{O}_2$ according to the Heisenberg uncertainty principle $\langle\Delta\hat{O}_1^2\rangle\langle\Delta\hat{O}_2^2\rangle \geq \frac{1}{4}\langle\hat{O}_3\rangle^2$, where $[\hat{O}_1, \hat{O}_2] = i\hat{O}_3$ and $\Delta\hat{O} = \hat{O} - \langle\hat{O}\rangle$. The technique of squeezing requires putting the detected field in a minimum-uncertainty state, while simultaneously reducing $\langle\Delta\hat{O}_1^2\rangle$ and increasing $\langle\Delta\hat{O}_2^2\rangle$ by the same factor. The latter is generally written in terms of the squeezing parameter ρ as follows:

$$\langle\Delta\hat{O}_{1,2}^2\rangle = e^{\mp 2\rho}\langle\Delta\hat{O}_{1,2}^2\rangle_{\rho=0} \ . \tag{1}$$

In most experimental situations in quantum optics, the measured observable $\hat{O}_1$ is either the number of photons $\hat{n} = a^\dagger a$ (direct photon counting) or the quadrature component $\hat{\mathcal{E}}_\phi = (e^{i\phi}a + e^{-i\phi}a^\dagger)/\sqrt{2}$ of the electric field at a fixed phase ϕ (homo- or heterodyning detection). The former case is usually referred to as 'amplitude squeezing' (Yamamoto *et al* 1987), whereas the latter as 'quadrature squeezing' or simply 'squeezing' (Yuen 1976).

Here we briefly present a nonunitary operator scaling which produces the amplitude squeezing. The corresponding evolution of the quantum states is entropy-varying and resenbles a squeezing mechanism based on nonselective measurement on degrees of freedom interacting with radiation.

2. THE SCALING TRASFORMATION

When the measured operator is $\hat{O}_1 = \hat{\mathcal{E}}_\phi$, the commutator algebra is very simple: one has $\hat{O}_2 = \hat{\mathcal{E}}_{\phi+\pi/2}$, $\hat{O}_3 = -1$ and $(\hat{O}_1, \hat{O}_2)$ is a couple of conjugated variables, like the position and the momentum of a particle $(\hat{p}, \hat{q})$.

The situation of the number operator $\hat{O}_1 = \hat{n}$ is somewhat more complicated. It turns out that $\hat{n}$ is approximately conjugated to the phase operator $\hat{\Phi}$, which is defined through the relation

$$\hat{E}_{\pm} = e^{\mp i\hat{\Phi}} , \tag{2}$$

$\hat{E}_{\pm}$ denoting the shift operators $\hat{E}_{-} = (a^{\dagger}a + 1)^{-1/2}a$, $\hat{E}_{+} = (\hat{E}_{-})^{\dagger}$ $(\hat{E}_{\pm}|n\rangle = |n \pm 1\rangle)$. As $\hat{\Phi}$ is not Hermitian ($\hat{E}_{\pm}$ are not unitary), one should use the Hermitian operators $\sin\hat{\Phi}$ and $\cos\hat{\Phi}$. However, for highly excited states (i.e. states approximately orthogonal to the vacuum $|0\rangle$) and for small phase uncertainties $\langle\Delta\hat{\Phi}^2\rangle \ll 1$ (such that $\sin\hat{\Phi} \sim \hat{\Phi}$), $\hat{\Phi}$ becomes approximately Hermitian. Furthermore, the asymptotic commutation relation holds

$$[\hat{n}, \hat{\Phi}] \sim i , \tag{3}$$

and $(\hat{n}, \hat{\Phi})$ can be treated as a conjugated pair.
A straightforward way to achieve the squeezing trasformation (1) lies in the realization of a Heisenberg evolution $\mathcal{S}_H$ which rescales the couple $(\hat{O}_1, \hat{O}_2)$ as follows

$$\mathcal{S}_H(\hat{O}_{1,2}) = r^{\mp 1}\hat{O}_{1,2} , \tag{4}$$

the scaling parameter r being connected to the squeezing parameter ρ through the relation $r = \exp\rho$. The corresponding Schrödinger evolution $\mathcal{S}_S$ of the states is defined by the identity

$$\mathrm{Tr}(\varrho\mathcal{S}_H(\hat{O})) = \mathrm{Tr}(\mathcal{S}_S(\varrho)\hat{O}) , \tag{5}$$

where ϱ denotes a general density matrix state. As a result of Eqs.(4) and (5) the state evolution $\mathcal{S}_S$ rescales all the moments in the form

$$\langle\hat{O}^p_{1,2}\rangle \longrightarrow r^{\mp p}\langle\hat{O}^p_{1,2}\rangle , \tag{6}$$

whichever state ϱ is considered. One should notice that, as a consequence of the scaling $\mathcal{S}_S$, the average $\langle\hat{O}_1\rangle$ itself would be reduced. Therefore, in order to preserve the signal, a driving excitation of the state is also needed, which displaces the average value from $\langle\hat{O}_1\rangle$ to $r\langle\hat{O}_1\rangle$, without changing the shape of the probability distribution. The canonical trasformation of Yuen (1976) $\mathcal{S}_H(a) = \mu a + \nu a^{\dagger}$ $(|\mu|^2 - |\nu|^2 = 1)$ provides an example of scaling trasformation for the couple $(\hat{\mathcal{E}}_{\phi}, \hat{\mathcal{E}}_{\phi+\pi/2})$, upon choosing $\mu = (r + r^{-1})/2$ and $\nu = e^{-i\phi}(r^{-1} - r)/2$. Here we show that for the conjugated pair $(\hat{n}, \hat{\Phi})$ the scaling trasformation (4) can be obtained by means of a nonunitary (i.e. noncanonical) evolution. Its general form in the Heisenberg picture can be inferred from the definition of the phase operator $\hat{\Phi}$ in Eq.(2). The latter implies that the rescaling of the phase corresponds to transition from one-particle shift operators $\hat{E}_{\pm}$ to r-particle shift operators $(\hat{E}_{\pm})^r$

$$\mathcal{S}_H(\hat{E}_{\pm}) = (\hat{E}_{\pm})^r , \tag{7}$$

$(\hat{E}_{\pm})^r$ now acting on the Fock space as follows:

$$(\hat{E}_{\pm})^r|n\rangle = |n \pm r\rangle . \tag{8}$$

From Eqs.(7) and (8) it turns out that the scaling of a generic operator $\hat{O}$ has the form

$$\mathcal{S}_H(\hat{O}) = \sum_{\lambda=0}^{r-1} \hat{S}_\lambda^\dagger \hat{O} \hat{S}_\lambda \,, \quad \hat{S}_\lambda = e^{i\phi_\lambda} \sum_{n=0}^{\infty} |n\rangle\langle nr+\lambda| \,. \tag{9}$$

(The phase factors are ineffective in the action (9) and will be dropped in the following). $\hat{S}_\lambda$ are nonunitary operators satisfying the orthogonality and completeness relations

$$\hat{S}_\lambda \hat{S}_\mu^\dagger = \delta_{\lambda\mu} \,, \quad \sum_{\lambda=0}^{r-1} \hat{S}_\lambda^\dagger \hat{S}_\lambda = 1 \,. \tag{10}$$

Although the scaling $\mathcal{S}_H$ is nonunitary, it preserves the operator products, as a consequence of the orthogonality conditions. When applied on the particle operator a the operator scaling gives the result

$$\mathcal{S}_H(a) = \sum_{n=1}^{\infty} |n-1\rangle \sqrt{\left[\frac{n}{r}\right]} \langle n| = \left[\frac{(1+[\hat{n}/r])\hat{n}!}{(\hat{n}+r)!}\right]^{1/2} a^r \equiv b_{(r)} \,, \tag{11}$$

where $[x]$ denotes the maximum integer $\leq x$ (for the creation operator one has $\mathcal{S}_H(a^\dagger) = b_{(r)}^\dagger$). Eqs. (11) show that the scaled particle operator $\mathcal{S}_H(a)$ coincides with the r-boson operator $b_{(r)}$ introduced by Brandt *et al* (1969). The particle operators $b_{(r)}$ and $b_{(r)}^\dagger$ annihilate and create r photons simultaneously and satisfy the commutation relations: $[b_{(r)}, b_{(r)}^\dagger] = 1$, $[\hat{n}, b_{(r)}] = -r b_{(r)}$. The preservation of the operator product implies that the scaling $\mathcal{S}_H$ of a generic operator $\hat{O} = \hat{O}(a, a^\dagger)$ (Hermitian analytic function of a and $a^\dagger$) can simply be obtained substituting a and $a^\dagger$ with $b_{(r)}$ and $b_{(r)}^\dagger$, i.e. $\mathcal{S}_H(\hat{O}) = \hat{O}(b_{(r)}, b_{(r)}^\dagger)$. Therefore, the present operator scaling corresponds to the construction of the r-photon observables of D'Ariano *et al* (1989,1990a).

The scaling of the number operator $\hat{n}$ can be obtained from the defining equation (9). One obtains

$$\mathcal{S}_H(\hat{n}) = [\hat{n}/r] \equiv b_{(r)}^\dagger b_{(r)} \,. \tag{12}$$

It follows that the number operator satisfies the rescaling (4) only asymptotically, in the limit of large mean numbers $\langle \hat{n} \rangle \gg r$

$$\mathcal{S}_H(\hat{n}) \simeq r^{-1} \hat{n} \,. \tag{13}$$

The nonunitary evolution in the Schrödinger picture $\mathcal{S}_S$ is obtained from Eqs.(5) and (9) using the invariance of trace under cyclic permutations

$$\mathcal{S}_S(\varrho) = \sum_{\lambda=0}^{r-1} \hat{S}_\lambda \varrho \hat{S}_\lambda^\dagger \,. \tag{14}$$

One can see that the state evolution (14) does not preserve the Neumann-Shannon entropy $S(\varrho) = -\mathrm{Tr}\varrho \log \varrho$. For example, starting with a pure state $\varrho = |\omega\rangle\langle\omega|$, the mixed state is obtained

$$\mathcal{S}_S(|\omega\rangle\langle\omega|) = \sum_{\lambda=0}^{r-1} |\Omega_\lambda\rangle\langle\Omega_\lambda| \,, \quad |\Omega_\lambda\rangle = \hat{S}_\lambda|\omega\rangle = \sum_{n=0}^{\infty} |n\rangle\langle nr+\lambda|\omega\rangle \,, \tag{15}$$

If the limit $r \to \infty$ is performed while keeping $\langle\hat{n}\rangle$ constant, a pure number state is obtained. Therefore, the entropy is always a decreasing function of r for large r, whereas, for small r, it can be increasing or decreasing, depending on the input state.

3. DISCUSSION

The nonunitary scaling is essentially simulated in the experimental realizations of the amplitude-squeezing based on entropy-varying schemes, where $\hat{n}$-noise reduction is obtained via nonselective measurements of some auxiliary *probe* field quantum-correlated with the radiation *(signal)* field. In the *high-Q micromaser Fock state generation* (Filipowicz et al 1986), for example, the probe is an inverted two-level atom entering the cavity with a well defined velocity. On the other hand, in the *quantum-nondemolition photon-number measurement* and in the *parametrically amplified idler photon counting* approaches proposed by Yamamoto et al (1987) the probe is another electromagnetic wave interacting with the signal mode in a nonlinear medium.

There is no obvious strict comparison between the theoretical scaling (14) and the above mentioned actual realizations of amplitude-squeezing. However, some similarities can be recognized *a posteriori*, upon identifying the dummy variable λ in Eq.(14) with some quantum number of the probe field.

Let us briefly consider the case of the high-Q micromaser Fock state generation. In the generalized scheme of Fu-li Li et al (1990) the probe field is represented by a beam of two-level atoms entering the cavity in clusters of $2J$ atoms, one cluster at a time. One can see that (D'Ariano 1990b) the evolution of the density matrix of the radiation field $\hat{\rho}_N = \sum_{n=0}^{2NJ} \varrho_N(n)|n\rangle\langle n|$ after N clusters passed through the cavity (initially in the vacuum state) simulates the same variation of entropy attained by the scaling $\mathcal{S}_H$ acting on the pure state $|\omega\rangle\langle\omega|$, where $|\omega\rangle = \sum_{n=0}^{r-1}\sqrt{\rho_N(n)}|n(r+1)\rangle$. The role of the scaling factor r is played by the total spin multiplicity $r = 2NJ+1$. The above correspondence can be further carried out when a pure Fock state is reached in the limit $N \to \infty$ (*trapping states*). An asymptotic analysis in the neighborhood of the trapping state $|n_0\rangle\langle n_0|$ for $J = 1/2$ shows that $\langle\hat{n}\rangle \simeq n_0 - \alpha N^{-1}$ ($\alpha = 4(n_0+1)^2\pi^{-2}$), whereas $\langle\Delta\hat{n}^2\rangle \simeq n_0\alpha N^{-1}$. It follows that the Fano factor $F = \langle\Delta\hat{n}^2\rangle/\langle\hat{n}\rangle$ rescales according to the rule $F \sim N^{-1} \sim r^{-1}$, which is in agreement with the scaling (6) and a diriving excitation of the state displacing the average number from $\langle\hat{n}\rangle$ to $r\langle\hat{n}\rangle$.

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