Squeezing versus photon-number fluctuations

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After deriving a general formula for the quantum probability distribution function higher moments, we apply it to the multiphoton squeezed states [the usual Gaussian and the new non-Gaussian Weyl-Heisenberg, SU(2), and SU(1,1)]. The resulting moments are discussed as functions of the photon-number fluctuations. General criteria are considered to determine optimal squeezing properties with respect to photon-number noise. There result interesting generalized uncertainty relations in the form of scaling laws.

I. INTRODUCTION

The possibility of quantum noise reduction in such critical physical situations as those encountered in detection of gravity waves[1] or multiphoton eigenmodes of the electromagnetic field in an optical cavity[2] has recently enhanced the interest of "squeezing"[3,4] in quantized fields.

A field in a squeezed state is characterized by fluctuations in one quadrature component smaller than those in a coherent state, the other quadrature component obviously exhibiting increased fluctuations due to Heisenberg's uncertainty principle.

The field quadrature components are analogous to the canonically conjugate position $\hat{q}$ and momentum $\hat{p}$ variables of the harmonic oscillator. Since knowledge about the oscillator phase is necessary to extract information in a quantum communication channel, the eigenstates of $\hat{q}$ and $\hat{p}$ are not of much use, and one needs to resort to coherent states with a specified oscillator phase. Noise reduction can then be achieved by squeezing the corresponding quantum distribution in one canonical variable.

The usual unsqueezed coherent states have Gaussian distributions in both q and p (Ref. 5). The states commonly referred to as "squeezed states" give rise as well to Gaussian distributions: they are obtained by distorting the vacuum with a unitary operator which is an exponential of a quadratic form in a and $a^\dagger$ (the usual annihilation and creation operators) and then by displacing the unitary operator in the phase space to get the correct average position and momentum values. In this case squeezing is obtained by varying all the distribution moments simultaneously without affecting the Gaussian shape. However, the development of techniques for performing higher-order correlation measurements in quantum optics has stressed the opportunity of squeezing higher-order moments, thus producing non-Gaussian coherent states, as suggested, e.g., in the Hong and Mandel scheme.[6]

The customary Gaussian squeezed states described above can be thought of as generated from an idealized two-photon device. In order to produce non-Gaussian shapes one needs k-photon squeezed states with $k>2$. These may arise, for example, in k-photon parametric amplifiers in which the electromagnetic radiation interacts with nonlinear matter via a kth-order susceptibility.

It turns out, however, that the naive way of generating multiphoton squeezed states by distorting the vacuum with an operator that is the exponential of a linear combination of $a^k, a^{\dagger k}$ leads to formal divergences. These difficulties, first pointed out by Fisher, Nieto, and Sandberg,[7] were recently tackled by a Padé-approximant analysis.[8] Such an analysis has, on the other hand, shown that these states exhibit no squeezing for $k>2$. It will be shown in this paper that this is a special case of a general feature of k-photon states, whereby for a k-photon state only the even-order ($2N$th) moments corresponding to $N>k/2$ (for even k) or $N>k$ (for odd k) can be squeezed.

Resorting to the multi-photon Brandt-Greenberg[9] operators and to the Holstein-Primakoff[10] realization of groups [Weyl-Heisenberg, SU(2), and SU(1,1)] the authors have constructed a number of non-Gaussian squeezed states which allow squeezing of different orders.[11–16]

The dynamical evolution of such states is coherent under the action of the harmonic-oscillator Hamiltonian (one can also consider more complicated coherence-preserving Hamiltonians of the form discussed in Ref. 17).

These states are good test states, interesting in physical applications. They allow a detailed analysis of the quantum noise problem and a deeper understanding of the completely nontrivial connections between the probability distributions of different physical observables.

Focusing, in particular, on the observable number of photons $\hat{n}$, we shall show that the new states correspond to a variety of probability $\hat{n}$ distribution functions: k Poisson, binomial [the SU(2) states correspond to a finite maximum number of photons], and negative binomial.

The squeezing properties of the states will therefore be compared in terms of the number of photon fluctuations. Among the most interesting results of this analysis are (i) the possibility of reducing the number noise with respect to the usual Gaussian states, (ii) the existence of optimal states which allow us to obtain the best squeezing with a bounded noise in $\hat{n}$, and (iii) the appearance of generalized uncertainty relations for both the optimal states and the completely squeezed states, involving the higher $\hat{q}$ distribution moments and the number fluctuations, in the form of scaling laws.

In Sec. II a general formula for the quantum probability distribution moments is derived, which allows a factorization of the moments themselves in two terms accounting, respectively, for the state-vector direction and the structure of the Fock space. In Sec. III the generalized k-photon squeezed states are introduced in a unified formalism, which leads straightforwardly to the probability distribution for $\hat{n}$, and the theorem is proved stating the conditions under which the various moments can be squeezed. Section IV gives an extensive discussion of the squeezing properties of all the k-photon states considered versus $\hat{n}$ fluctuations. In Sec. IV the scaling laws are given based on the numerical analysis permitted by the general formula of Sec. II. A few concluding remarks and a summary of the results are presented in Sec. VI and some details of the analytical calculations are given in Appendixes A and B.

II. A GENERAL FORMULA FOR THE QUANTUM PROBABILITY DISTRIBUTION MOMENTS

In view of the comparison we are interested in, between squeezed states and the customary coherent states (which have a Gaussian distribution for the canonical variables), we construct here a general formula for the quantum probability distribution even moments. We focus our analysis on states $|\omega\rangle$ corresponding to zero-average position and momentum, since such an average can be arbitrarily changed to any desired value by a simple translation

$$|z\rangle_\omega = D(z)|\omega\rangle \ , \tag{2.1}$$

where $D(z)$ is the displacement unitary operator:

$$D(z)=\exp(za^\dagger - z^* a) \ . \tag{2.2}$$

$a^\dagger$ and a denote the usual creation and annihilation operators $[a,a^\dagger]=1$. In fact, for the position $\hat{q}=(1/\sqrt{2})(a+a^\dagger)$, one has

$$_\omega\langle z|\hat{q}|z\rangle_\omega = \sqrt{2}\,\mathrm{Re} z \ . \tag{2.3a}$$

Thus, the generic nth moment is given by

$$_\omega\langle z|(\hat{q}-\langle\hat{q}\rangle)^n|z\rangle_\omega = \langle\omega|\hat{q}^n|\omega\rangle \equiv \chi_\omega^{(n)} \ . \tag{2.3b}$$

Analogous results hold for the momentum operator $\hat{p}=(i/\sqrt{2})(a^\dagger - a)$.

We shall restrict our attention to Fock states $|\omega\rangle$, normalizable in the following sense:

$$|\omega\rangle = \sum_{n=0}^{\infty} \omega_n |n\rangle, \ \langle\omega|\omega\rangle = \|\omega\|^2 = \sum_{n=0}^{\infty} |\omega_n|^2 = 1 \ . \tag{2.4}$$

The aim of the formula derived in this section is simply to provide a change of basis from the number-operator representation to the position (or momentum) representation. There results an interesting factorization of the expression for the moments in two terms, one accounting for the complex ray-vector direction of $|\omega\rangle$ and the other describing the functional structure of the Fock space, reflected in such a change of basis.

The probability distribution in the $\hat{q}$ representation of the state $|\omega\rangle$ is given by

$$Q_\omega(q) = |\langle q|\omega\rangle|^2 = \frac{e^{-q^2}}{\sqrt{\pi}} \sum_{n,m=0}^{\infty} \omega_n \omega_m^* \frac{H_n(q)H_m(q)}{(2^{n+m} n!m!)^{1/2}} \ . \tag{2.5}$$

$H_n(q)$ are Hermite polynomials of order n. The above factorization comes into play if one writes Eq. (2.5) in the form

$$Q_\omega(q) = \mathrm{Tr}[\boldsymbol{\Omega}\mathbb{Q}(q)] \ , \tag{2.6}$$

where

$$\boldsymbol{\Omega} = \{\Omega_{ij}\}, \quad \Omega_{ij} = \omega_i \omega_j^* \ , \tag{2.7}$$

and

$$\mathbb{Q}(q) = \{Q_{ij}(q)\}, \quad Q_{nm}(q) = \frac{e^{-q^2}}{\sqrt{\pi}} \frac{H_n(q)H_m(q)}{(2^{n+m} n!m!)^{1/2}} \ . \tag{2.8}$$

The generating function of the moments is now obtained by Fourier transforming the distribution (2.6):

$$\tilde{Q}_\omega(x) = \mathcal{F}[Q_\omega](x) = \langle\omega|e^{ix\hat{q}}|\omega\rangle = \mathrm{Tr}[\boldsymbol{\Omega}\boldsymbol{\Delta}(x)] \ , \tag{2.9}$$

where

$$\boldsymbol{\Delta}(x) = \mathcal{F}[\mathbb{Q}](x) \ , \tag{2.10a}$$

namely,

$$\boldsymbol{\Delta}(x) = \{\Delta_{ij}(x)\} \ , \tag{2.10b}$$

$$\Delta_{nm}(x) = e^{-x^2/4}\left[\frac{(2\nu)!!}{(2\mu)!!}\right]^{1/2} (ix)^{\mu-\nu} L_\nu^{\mu-\nu}\left[\frac{x^2}{2}\right]$$

with $\nu=\min(n,m)$ and $\mu=\max(n,m)$; $L_\alpha^\beta(x)$ are the usual Laguerre polynomials. The moments (2.3b) are obtained by deriving the generating function (2.9) with respect to x. In particular, for even-order moments we have

$$\chi_\omega^{(2N)}=\mathrm{Tr}[\mathbf{\Omega}\mathbf{\Delta}^{(2N)}] , \tag{2.11}$$

$$\mathbf{\Delta}^{(2N)}=(-)^N\frac{d^{2N}}{dx^{2N}}\Bigg|_{x=0}\mathbf{\Delta}(x) . \tag{2.12}$$

After some nontrivial manipulations, whose details are briefly summarized in Appendix A, one obtains

$$\begin{aligned}\mathbf{\Delta}^{(2N)}&=\{\Delta_{ij}^{(2N)}\},\quad \Delta_{pq}^{(2N)}=\Delta_{qp}^{(2N)},\quad \Delta_{m+2r+1,m}^{(2N)}=0 ,\\ \Delta_{m+2r,m}^{(2N)}&=\frac{\theta(N-r)}{(2r-1)!!}\left[\frac{(m+2r)!}{m!}\right]^{1/2}\begin{bmatrix}N\\ r\end{bmatrix}\\ &\quad\times F(r-N,-m;2r+1;2) ,\end{aligned} \tag{2.13}$$

$\theta(x)$ being the Heaviside function $\theta(x)=1$, $x\geq 0$, $\theta(x)=0$, $x<0$, and $F(h,k,p;x)$ the usual hypergeometric function.

Let us recall, for the sake of reference, that in the number $\hat{n}$ representation, the probability distribution and the generating function of the moments are trivial:

$$N_\omega(n)=|\langle n|\omega\rangle|^2=|\omega_n|^2 , \tag{2.14a}$$

$$\tilde{N}_\omega(w)=e^{-iw\langle n\rangle}\sum_{n=0}^{\infty}e^{iwn}|\omega_n|^2 , \tag{2.14b}$$

where $\langle n\rangle=\langle\omega|\hat{n}|\omega\rangle$.

III. GENERALIZED SQUEEZED STATES

The property that $|\omega\rangle$ is a zero average state is guaranteed if one assumes

$$|\omega\rangle=\hat{S}_\omega|0\rangle , \tag{3.1}$$

where $\hat{S}_\omega$ is a unitary squeezing operator, which is an analytic function of multiparticle operators (i.e., operators that create more than one particle at a time).

Furthermore, as we are interested in even distributions, we need an even number of particle creators. The usual squeezing operator,[4] which gives rise to a Gaussian distribution,

$$\hat{S}(\zeta)_{\mathrm{Gauss}}=\exp[\tfrac{1}{2}(\zeta a^{\dagger 2}-\zeta^* a^2)] , \tag{3.2}$$

satisfies both the above requirements. Fisher, Nieto, and Sandberg[7] have proposed generalizations of the operator (3.2) in the form

$$\hat{S}_k(\zeta)=\exp(\zeta a^{\dagger k}-\zeta^* a^k+h_l) , \tag{3.3}$$

where $h_l=h_l^\dagger$ is a polynomial in a and $a^\dagger$ with powers up to $(k-1)$. It should be remarked that the state corresponding to this operator does not belong to the class of states considered in (2.4), as the vacuum vector does not belong to its domain of analyticity.

In a series of recent papers,[11–16] the authors have proposed a set of generalized multiphoton squeezing operators of the form

$$\hat{S}_k(\zeta;\Gamma,\sigma)=\exp[\zeta E_+^{(k)}(\Gamma,\sigma)-\mathrm{H.c.}] , \tag{3.4}$$

where Γ is a group [the Weyl-Heisenberg group, SU(n), $n\geq 2$, and SU(1,1) were considered], σ is the label of the irreducible representations of Γ, and $E_+^{(k)}$ is the raising operator corresponding to the realization of Γ in the representation σ in terms of generalized k-boson operators. The latter, first introduced by Brandt and Greenberg,[9] are defined by the commutation relations

$$[b_{(k)},b_{(k)}^\dagger]=1 , \tag{3.5}$$

$$[a^\dagger a,b_{(k)}]=-kb_{(k)} , \tag{3.6}$$

showing that $b_{(k)}$ and $b_{(k)}^\dagger$ are annihilation and creation operators of k photons simultaneously, such that $b_{(1)}=a$ but $b_{(k)}\neq a^k$ for $k\geq 2$.

In the Fock space they operate as

$$\begin{aligned}&b_{(k)}|n\rangle=([[n/k]])^{1/2}|n-k\rangle ,\\ &[[x]]=\text{integer part of } x ,\end{aligned} \tag{3.7a}$$

$$b_{(k)}^\dagger|n\rangle=([[n/k]]+1)^{1/2}|n+k\rangle . \tag{3.7b}$$

In the case of the Weyl-Heisenberg (WH) group $E_+^{(k)}=b_{(k)}^\dagger$, whereas for the other groups considered the raising operator is realized in the σ representation according to the Holstein-Primakoff scheme:[10]

$$E_+^{(k)}(\mathrm{SU}(2);\sigma)=(2\sigma+1-b_{(k)}^\dagger b_{(k)})^{1/2}b_{(k)}^\dagger , \tag{3.8a}$$

$$E_+^{(k)}(\mathrm{SU}(1,1);\sigma)=(2\sigma-1+b_{(k)}^\dagger b_{(k)})^{1/2}b_{(k)}^\dagger . \tag{3.8b}$$

Notice also that the usual squeezing operator (3.2) belongs to a representation of SU(1,1) (the $\sigma=\frac{1}{4}$ discrete series), but not in the Holstein-Primakoff realization.

The squeezed states, corresponding to the squeezing operators introduced above [with the obvious exception of (3.3)] can be explicitly written in the Fock basis (2.4) as

$$|\xi\rangle_{\mathrm{Gauss}}=(1-|\xi|^2)^{1/4}\sum_{n=0}^{\infty}\begin{bmatrix}2n\\ n\end{bmatrix}^{1/2}(\tfrac{1}{2}\xi)^n|2n\rangle , \tag{3.9a}$$

$$|\xi;k\rangle_{\mathrm{WH}}=\exp(-\tfrac{1}{2}|\xi|^2)\sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{n!}}|kn\rangle , \tag{3.9b}$$

$$|\xi;k,\sigma\rangle_{\mathrm{SU}(2)}=(1+|\xi|^2)^{-\sigma}\sum_{n=0}^{\infty}\begin{bmatrix}2\sigma\\ n\end{bmatrix}^{1/2}\xi^n|kn\rangle , \tag{3.9c}$$

$$|\xi;k,\sigma\rangle_{\mathrm{SU}(1,1)}=(1-|\xi|^2)^{\sigma}\sum_{n=0}^{\infty}\begin{bmatrix}2\sigma+n-1\\ n\end{bmatrix}^{1/2}\xi^n|kn\rangle . \tag{3.9d}$$

The relation between the parameter ξ labeling the states in (3.9) and the parameter ζ used to define the corresponding squeezing operators (3.2) and (3.4) (which is straightforwardly obtained by using the Baker-Campbell-Hausdorff formula) is reported in the first column of Table I.

Equations (3.9) manifestly show that the squeezed states constructed are multiphoton states: indeed they are

TABLE I. Baker-Hausdorff parameter ξ and its range; and photon-number average, variance, and fluctuation vs ξ, for the different squeezed states considered.

Squeezed state	$\xi=\xi(\zeta)$	Range of $\lvert\xi\rvert$	$\langle n\rangle$	$(\Delta n)^2$	$\delta^{-1}=\frac{\langle n\rangle}{\Delta n}$
Gauss	$\xi=\frac{\zeta}{\lvert\zeta\rvert}\tanh(\lvert\zeta\rvert)$	0–1	$\frac{\lvert\xi\rvert^2}{1-\lvert\xi\rvert^2}$	$2\frac{\lvert\xi\rvert^2}{(1-\lvert\xi\rvert^2)^2}$	$\frac{1}{\sqrt{2}}\lvert\xi\rvert$
WH	$\xi=\zeta$	0–∞	$k\lvert\xi\rvert^2$	$k^2\lvert\xi\rvert^2$	$\lvert\xi\rvert$
SU(2)	$\xi=\frac{\zeta}{\lvert\zeta\rvert}\tan(\lvert\zeta\rvert)$	0–∞	$2k\sigma\frac{\lvert\xi\rvert^2}{1+\lvert\xi\rvert^2}$	$2k^2\sigma\frac{\lvert\xi\rvert^2}{(1+\lvert\xi\rvert^2)^2}$	$\sqrt{2\sigma}\lvert\xi\rvert$
SU(1,1)	$\xi=\frac{\zeta}{\lvert\zeta\rvert}\tanh(\lvert\zeta\rvert)$	0–1	$2k\sigma\frac{\lvert\xi\rvert^2}{1-\lvert\xi\rvert^2}$	$2k^2\sigma\frac{\lvert\xi\rvert^2}{(1-\lvert\xi\rvert^2)^2}$	$\sqrt{2\sigma}\lvert\xi\rvert$

of the general form (2.4) with $\omega_n\neq 0$ only if n is a multiple of k with $k>2$ [$k=2$ in (3.9a)].

The moments of such multiphoton states satisfy the following theorem.

For a k-photon state $|\omega\rangle$, *only the moments* $\chi_\omega^{(2N)}$ *corresponding to* $2N\geq k$ *can be squeezed for even k,* $N\geq k$ *for odd k.*

The proof of this theorem is based on the observation that upon dividing the contributions to $\mathrm{Tr}[\Omega\Delta^{(2N)}]$ on the right-hand side (RHS) of (2.11) into a "diagonal" part $\sum_{i=0}^{\infty}\Omega_{ii}\Delta_{ii}^{(2n)}$ and an "off-diagonal" part

$$\sum_{\substack{ij=0\\ i\neq j}}^{\infty}\Omega_{ij}\Delta_{ji}^{(2N)}\ ,$$

the former is always $>1=\chi_{\mathrm{Gauss}}^{(2N)}$ [as one can check from (2.13) with $r=0$, and (2.7) with $i=j$]. Thus, in order to have squeezing, i.e., $\chi^{(2N)}<1$, the latter must be strictly negative. A necessary condition for this is that $\theta(N-r)=1$ in (2.13), namely, $N\geq r$ or $2N\geq k$ for even k. For odd k, the final condition in (2.13), further limits squeezing to higher moments in that $2r$ should also be an integer multiple of k, then $N\geq k$. It is interesting that such a property holds also for the states obtained with the squeezing operator (3.3) even though they do not belong to the class of normalizable states considered here. This was verified numerically using Padé approximants.[8] By inspection of (3.9) one can check that ω_{kn} is proportional to ξ^n; hence, (2.7), (2.11), and (2.13) imply that the best squeezing can be obtained for real negative ξ for each moment.

The probability distribution (2.14) of the number operator for the states (3.9) is given by

$$N_{\mathrm{Gauss}}^{(\xi)}(2n)=(1-|\xi|^2)^{1/2}\begin{bmatrix}2n\\ n\end{bmatrix}(\tfrac{1}{2}|\xi|)^{2n}\ , \tag{3.10a}$$

$$N_{\mathrm{WH}}^{(\xi,k)}(kn)=e^{-|\xi|^2}|\xi|^{2n}/n!\ , \tag{3.10b}$$

$$N_{\mathrm{SU(2)}}^{(\xi,k,\sigma)}(kn)=(1+|\xi|^2)^{-2\sigma}\begin{bmatrix}2\sigma\\ n\end{bmatrix}|\xi|^{2n}\ , \tag{3.10c}$$

$$N_{\mathrm{SU(1,1)}}^{(\xi,k,\sigma)}(kn)=(1-|\xi|^2)^{2\sigma}\begin{bmatrix}2\sigma+n-1\\ n\end{bmatrix}|\xi|^{2n}\ , \tag{3.10d}$$

$$N_{\mathrm{Gauss}}^{(\xi)}(2n+1)=0\ , \tag{3.10e}$$

$$N_{\mathrm{WH}}^{(\xi,k)}(p)=N_{\mathrm{SU(2)}}^{(\xi,k,\sigma)}(p)=N_{\mathrm{SU(1,1)}}^{(\xi,k,\sigma)}(p)=0,\quad p\neq kn\ . \tag{3.10f}$$

Equations (3.10b)–(3.10d) represent, respectively, the Poisson, binomial, and negative-binomial distributions in the many-photon variable kn.

IV. SQUEEZING VERSUS PHOTON-NUMBER FLUCTUATIONS

In this section we focus our attention on the photon-number operator and, in particular, on its probability distribution. It appears that, contrary to the case of position and momentum operators, the latter depends on the displacement characteristic of the state [see Eqs. (2.1) and (2.2)].

We shall, however, still perform our analysis in the zero average position and momentum case, which physically corresponds to a weak signal limit. This not only permits us to simplify the numerical analysis, but describes as well the most critical (and therefore most interesting for applications) situation of noise superimposed on a weak signal. It turns out that in order to compare the statistical properties inherent in the different states, the squeezing parameter ξ introduced in Eqs. (3.9) does not lend itself to a transparent physical interpretation and appears therefore somewhat ambiguous. On the other hand, one can see from Table I that the quantity describing the fluctuation

$$\delta=\Delta n/\langle n\rangle \tag{4.1}$$

of the number operator is inversely proportional to $|\xi|$ with coefficients depending on the group representation. We therefore adopt δ^{-1} as a good independent variable to compare different kinds of squeezing.

Figure 1 reports the second moments $\chi^{(2)}$ for the various two-photon squeezed states as functions of δ^{-1}. The states considered correspond to Weyl-Heisenberg and to SU(2) and SU(1,1) groups (the latter two in the $\sigma=3$ representation). For the sake of comparison, the results for Gaussian states are also shown.[4] One can notice that among all states, the Gaussian ones exhibit the best squeezing for a fixed value of δ^{-1}. However, they cannot attain a fluctuation in the observable number of photons lower than $\sqrt{2}$; in other words, the Gaussian states are *photon noisy*. Furthermore, as $\chi_{\mathrm{Gauss}}^{(2)}$ is a monotonic decreasing function of δ^{-1}, the best squeezing corresponds to the lowest $\hat{n}$ fluctuation. On the other hand,

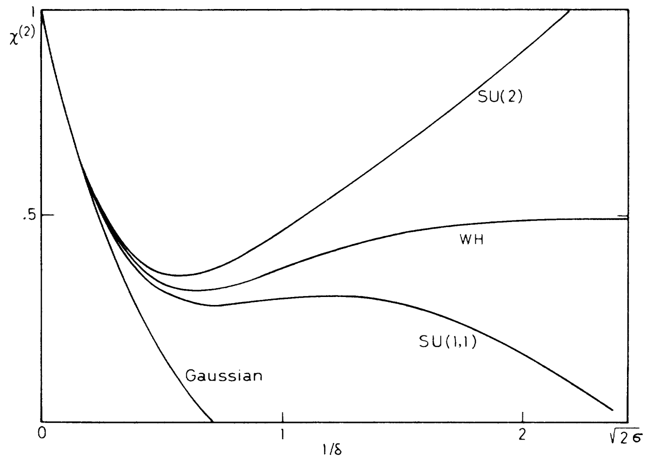

FIG. 1. Squeezing (i.e., second moment for negative squeezing parameter) vs the inverse $\hat{n}$ fluctuation parameter δ^{-1}, for the various two-photon squeezed states; SU(2) and SU(1,1) states correspond to the $\sigma=3$ representation.

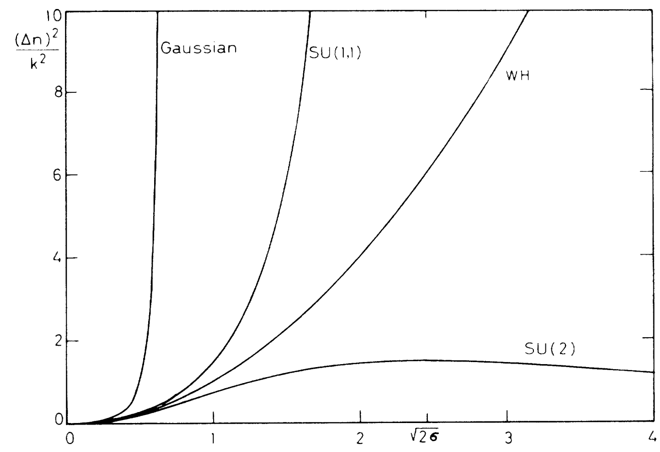

FIG. 2. Reduced $\hat{n}$ square variance vs the inverse $\hat{n}$-fluctuation parameter δ^{-1} for the same two-photon squeezed states of Fig. 1.

all the other non-Gaussian states give rise to functions $\chi^{(2)}$ (δ^{-1}) which are not monotonic but exhibit a local minimum. Among them only the SU(1,1) states can be completely squeezed ($\chi^{(2)}=0$).

One can also notice that, in general, non-Gaussian states can attain a photon-number fluctuation smaller than those of the Gaussian states. In particular, the Weyl-Heisenberg states can have an arbitrary small photon noise, but they are *squeezing limited* in that the second moment $\chi^{(2)}_{\rm WH}$ exhibits an *absolute* minimum $\chi^{(2)}_{\rm WH,min}=0.31744$ corresponding to $\delta^{-1}=0.64675$. The SU(1,1) states can be squeezed to zero second moment in correspondence to the optimal value $\delta^{-1}=\sqrt{2\sigma}$ (an explicit analytic determination of this zero absolute minimum for $\chi^{(2)}_{\rm SU(1,1)}$ is done in Appendix B). Therefore, one can simultaneously reduce to zero both $\hat{n}$ noise and $\hat{q}$ noise in the limit $\sigma\rightarrow\infty$. It is worth pointing out that whereas for WH states the local minimum is also a global one, for the SU(1,1) states the absolute minimum does not coincide with the relative minimum (numerical values of such minima for large σ are given in Ref. 13). Finally, the SU(2) states are no longer squeezed ($\chi^{(2)}_{\rm SU(2)}>1$) for small $\hat{n}$ fluctuations.

Figure 2 shows the reduced absolute fluctuations $(\Delta n/k)^2$ vs δ^{-1} for the same states considered in Fig. 1. It appears from this figure, comparing it with the previous one, that the better the squeezing the higher the photon-number fluctuations. In particular, the Gaussian states exhibit the highest photon noise.

In the limit of squeezing to zero second moment, $\delta\rightarrow\sqrt{2}$ for Gaussian states or $\delta\rightarrow 1/\sqrt{2\sigma}$ for SU(1,1) states, the $\hat{n}$ variance increases asymptotically to infinity for both states. ${(\Delta n/k)_{\rm WH}}^2$ grows parabolically with δ^{-1}, whereas ${(\Delta n/k)_{\rm SU(2)}}^2$ shows a maximum, and decreases to zero as δ^{-1} tends to infinity, as $\sim(\delta^{-1})^{-2}$.

From Figs. 1 and 2, one can then conclude that the local minimum of $\chi^{(2)}$ for non-Gaussian states can be considered an optimum situation as it provides a good compromise between the requisite of the maximum squeezing and that of minimum absolute noise in the photon number.

V. SCALING LAWS

The existence of the two vertical asymptotes for Gaussian and SU(1,1) states in Fig. 2, corresponding to the vanishing of $\chi^{(2)}$, suggests that we look at the dependence of $\chi^{(2)}$ vs $(\Delta n/k)^2$. One expects a scaling relation, that in the limit of large $(\Delta n)^2$ should give a generalized uncertainty relation in the form

$$\chi^{(2)}(\Delta n)^{2\gamma}\sim C\ . \tag{5.1}$$

Figure 3 shows the log-log plot of squeezing versus reduced absolute photon-number fluctuation for all the two-photon states of Fig. 1. One can notice that $\gamma=\frac{1}{2}$ for both Gaussian and SU(1,1) states, in the latter case γ being independent of the value of σ, provided it is finite (see Appendix B for some details of this analysis). The constant C depends on both the state [Gaussian or SU(1,1)] and the representation (σ). Thus, the parameter γ can be thought of as a universal scale exponent. One should notice that, considering the WH states as the $\sigma\rightarrow\infty$ limit of SU(1,1), as discussed in Ref. 13, the universal behavior is broken in the same limit and we have $\gamma=0$.

Scaling laws analogous to (5.1) can be found for higher-order moments as well. Somewhat unexpectedly, scaling laws for second- and higher-order moments appear for all states corresponding to the local minima of the moments themselves versus δ^{-1}. In this case, the parameter whereby the two uncertainties $\chi^{(2N)}$ and $(\Delta n)^2$ can be connected is the representation label σ, which is the only remaining free variable. Therefore we continue now our analysis of the *optimal squeezing properties* of

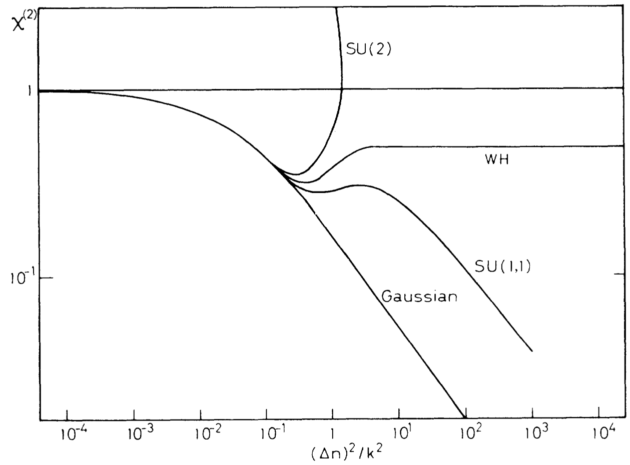

FIG. 3. Log-log plot of squeezing vs reduced absolute $\hat{n}$ fluctuation for the same two-photon squeezed states of Figs. 1 and 2.

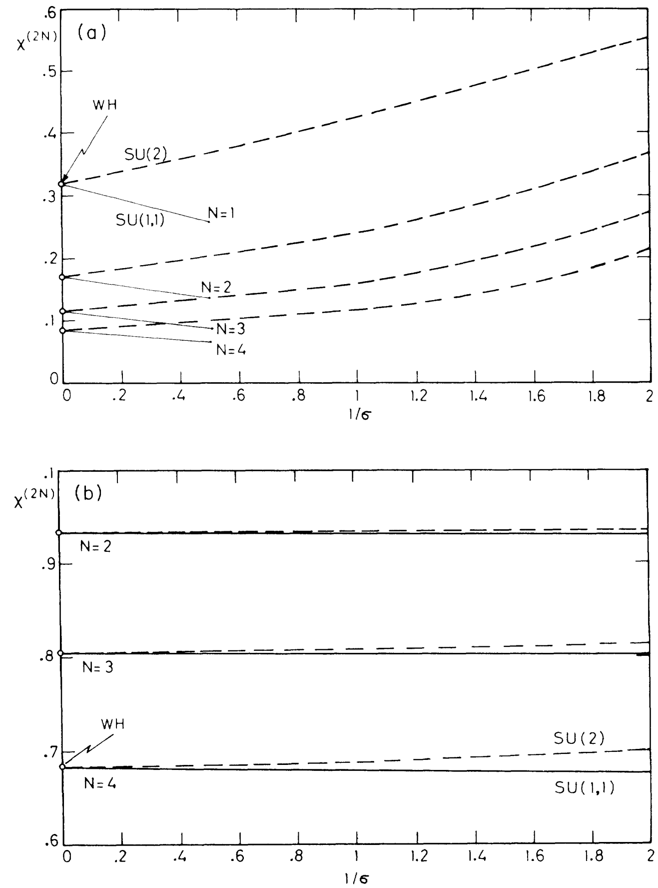

FIG. 4. Generalized squeezing for the $2N$th moments vs $1/\sigma$ for k-photon states at the local minimum in δ^{-1}: (a) $k=2$, $N=1$–4; (b) $k=4$, $N=2$–4.

the non-Gaussian states, by studying numerically the minimal moments $\chi_{\min}^{(2N)}$ for even k ($k=2,4$) and order $2N$ ($N=1$–4) vs σ^{-1}.

Figure 4 shows the resulting plots (for simplicity we write $\chi^{(2N)}$ instead of $\chi_{\min}^{(2N)}$). One can notice how the curves corresponding to SU(1,1) and SU(2) converge for $\sigma\to\infty$ to the Weyl-Heisenberg values, with opposite limiting derivatives. A similar limiting behavior can be found in the plot of the square variance $(\Delta n)_{\min}{}^2$ [namely, $(\Delta n)^2$ corresponding to the local minimum of $\chi^{(2N)}$ vs δ^{-1}, which we simply denote as $(\Delta n)^2$] vs σ^{-1} of Fig. 5.

Generalized scaling laws of the form

$$\chi_{(k)}^{(2N)}(\Delta n)^{2\gamma_k(N)} \sim C_k(N) \tag{5.2}$$

can be obtained by eliminating σ^{-1} between corresponding curves of Figs. 4 and 5.

Figure 6 gives the log-log plots of the minimal moments $\chi^{(2N)}$ versus the corresponding $\hat{n}$ variance, which manifestly exhibit a power-law behavior of the form (5.2). It is interesting to point out how, in this representation, all states [WH, SU(2), and SU(1,1)] lie on the same straight lines. The exponents $\gamma_k(N)$ are positive numbers less than 1, whose dependence on N and k is shown in Fig. 7(a). Figure 7(b) shows the dependence of the constants $C_k(N)$ on the same integer parameters. Notice that whereas $\gamma_k(N)$ is monotonically increasing with N and decreasing with k, $C_k(N)$ is decreasing with N and increasing with k.

For $k=2$ and $N=1$ we have, from Fig. 7(a), $\gamma\sim\frac{1}{3}$ which is lower than the value $\gamma=\frac{1}{2}$ that we found at the absolute minimum and is very close to the value reported in Ref. 13 in a slightly different, but related context.

VI. CONCLUSIONS

A numerical analysis of the squeezing and higher-order moments features of the k-photon squeezed states has been performed, based on the general formula for the probability distribution moments derived in Sec. II.

Two interesting results emerge from such analysis. One can attain good squeezing without unbounded increase of the fluctuations in the number of photons. In fact, there exist states giving a stable minimum squeezing with a limited photon-number noise. Moreover, both these states and the states corresponding to complete squeezing are characterized by new uncertaintylike relations in the form of scaling laws connecting the $\hat{q}$-distribution higher moments with the number fluctuations. In view of the generality of the $\hat{n}$ distributions involved as well as of the flexibility of all the k-photon states, we are led to conjecture that our results represent characteristic features of squeezing in general.

A rigorous proof of this conjecture would require a general definition of squeezed state, probably using nontrivial methods of functional analysis. Work is in progress along these lines.

APPENDIX A

The generating function $\tilde{Q}_\omega(x)$ for the moments $\chi_\omega^{(2N)}$ has been given in Eq. (2.9) in the form of the trace of a product of two infinite-dimensional matrices:

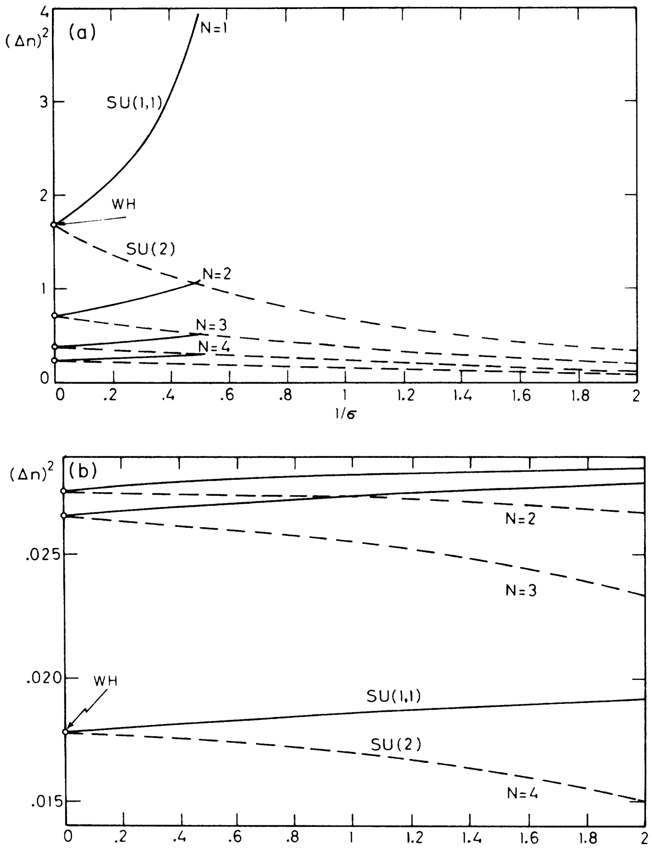

FIG. 5. Absolute $\hat{n}$ fluctuation vs $1/\sigma$ for k-photon states at the $2N$th moment local minimum: (a) $k=2$, $N=1$–4; (b) $k=4$, $N=2$–4.

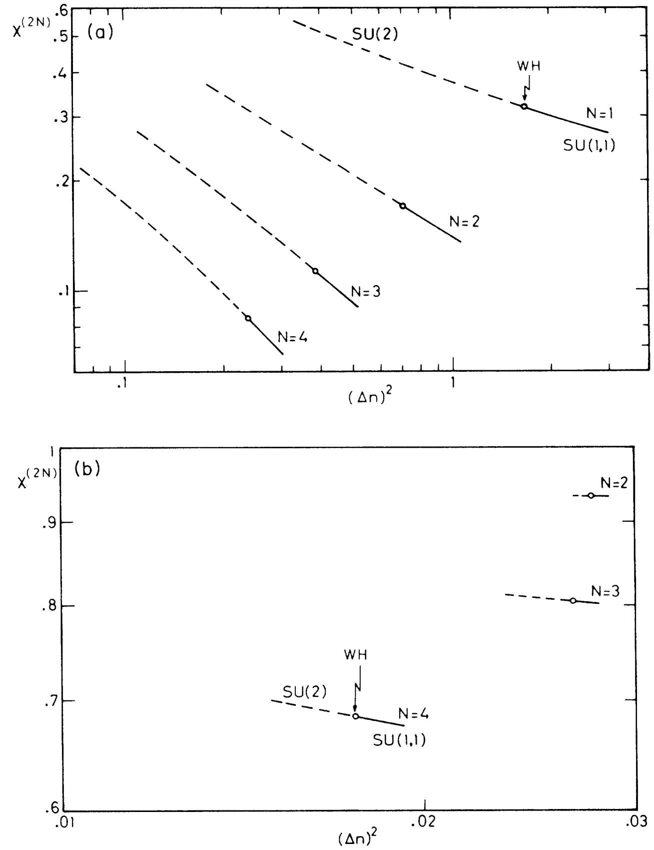

FIG. 6. Generalized squeezing for the $2N$th moments vs absolute $\hat{n}$ fluctuation at the local minimum (log-log plot): (a) $k=2$, $N=1$–4; (b) $k=4$, $N=2$–4.

$$\tilde{Q}_\omega(x)=\mathrm{Tr}[\boldsymbol{\Omega}\boldsymbol{\Delta}(x)] , \tag{A1}$$

where the matrix $\boldsymbol{\Omega}$ is given by Eq. (2.7) and depends only on the state $|\omega\rangle$, and $\boldsymbol{\Delta}(x)$, independent of $|\omega\rangle$, is linked only to the change of representation from the number observable $\hat{n}$ to the position $\hat{q}$. $\boldsymbol{\Delta}(x)$ can be derived by Fourier transforming the matrix given by Eq. (2.8) and using the identity

$$\int_{-\infty}^{\infty} dq\, e^{-q^2+ixq}H_n(q)H_m(q) = \sqrt{\pi}e^{-x^2/4}2^\mu \nu! \left[\frac{ix}{2}\right]^{\mu-\nu} L_\nu^{\mu-\nu}\left[\frac{x^2}{2}\right] , \tag{A2}$$

$$\nu=\min(n,m), \quad \mu=\max(n,m) .$$

One obtains Eq. (2.10), which gives the matrix elements $\Delta_{nm}(x)$ of $\boldsymbol{\Delta}(x)$ as

$$\Delta_{nm}(x)=e^{-x^2/4}\left[\frac{(2\nu)!!}{(2\mu)!!}\right]^{1/2}(ix)^{\mu-\nu}L_\nu^{\mu-\nu}\left[\frac{x^2}{2}\right] . \tag{A3}$$

The even-order moments are obtained by computing the derivative

$$\chi_\omega^{(2N)}=\mathrm{Tr}[\boldsymbol{\Omega}\boldsymbol{\Delta}^{(2N)}], \quad \boldsymbol{\Delta}^{(2N)}=(-)^N\frac{d^{2N}}{dx^{2N}}\bigg|_{x=0}\boldsymbol{\Delta}(x) . \tag{A4}$$

Using the identities

$$\frac{d^{2l}}{dz^{2l}}\bigg|_{z=0} e^{z^2/4}=2^{-l}(2l-1)!! , \tag{A5}$$

$$\frac{d^{2l}}{dz^{2l}}\bigg|_{z=0} L_\beta^\alpha\left[-\frac{z^2}{2}\right]=(2l-1)!!\begin{bmatrix}\alpha+\beta\\ \beta-l\end{bmatrix} , \tag{A6}$$

one obtains the intermediate formula

$$\frac{d^{2l}}{dz^{2l}}\bigg|_{z=0} e^{z^2/4}L_\beta^\alpha\left[-\frac{z^2}{2}\right] =2^{-l}(2l-1)!!\begin{bmatrix}\alpha+\beta\\ \alpha\end{bmatrix}F(-l,-\beta;\alpha+1;2) , \tag{A7}$$

where $F(n,m,\alpha;x)$ is the usual hypergeometric function.

Equation (A7) together with

$$\frac{d^{2l}}{dz^{2l}}\bigg|_{z=0} z^p=\delta_{p,2l}(2l)! \tag{A8}$$

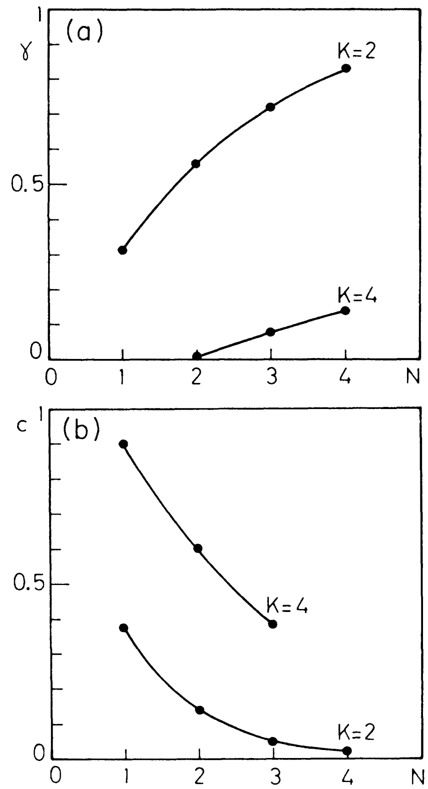

FIG. 7. (a) The exponent $\gamma_k(N)$ and (b) the constant $C_k(N)$ of the scaling laws (5.2), corresponding to plots of Fig. 6, vs N for $k=2,4$.

allow us to compute the matrix derivative in (A4) giving the result (2.13):

$$\Delta^{(2N)}_{m+2r,m}=\frac{\theta(N-r)}{(2r-1)!!}\left[\frac{(m+2r)!}{m!}\right]^{1/2}\begin{bmatrix}N\\ r\end{bmatrix}\times F(r-N,-m;2r+1;2)\,,$$

$$\Delta^{(2N)}_{m+2r+1,m}=0,\quad \Delta^{(2N)}_{nm}=\Delta^{(2N)}_{mn}\,. \tag{A9}$$

APPENDIX B

In this appendix the zero absolute minimum of the squeezing for $k=2$ SU(1,1) states is analytically checked and the scaling law of squeezing versus $\hat{n}$ variance is obtained near such a minimum. The basic formulas in computing the squeezing are the expectation values in the $k=2$ SU(1,1) state ($\xi=\rho e^{i\phi}$):

$$\langle a^\dagger a\rangle=\frac{4\sigma\rho^2}{1-\rho^2}\,, \tag{B1}$$

$$\langle (a^\dagger)^2\rangle=2\xi^*(1-\rho^2)^{2\sigma}\times\sum_{n=0}^{\infty}\rho^{2n}\begin{bmatrix}n+2\sigma-1\\ n\end{bmatrix}[(n+\tfrac{1}{2})(n+2\sigma)]^{1/2}\,, \tag{B2}$$

which can be simply obtained from Eq (3.9d).

Equations (B1) and (B2) allow us to compute the second moment in the squeezing case, corresponding to $\phi=\pi$:

$$2\chi^{(2)}=1+2[\langle a^\dagger a\rangle-|\langle (a^\dagger)^2\rangle|\,. \tag{B3}$$

One needs to study the asymptotic behavior of the series in (B2) for $\rho\to 1$ and compare its singular part with the divergence in Eq. (B1). It is convenient to rewrite Eq. (B2) as

$$\langle (a^\dagger)^2\rangle=\frac{2\xi^*}{(2\sigma-1)!}(1-\rho^2)^{2\sigma}F_\sigma(\rho^2)\,, \tag{B4}$$

where the auxiliary function $F_\sigma(x)$ is given by

$$F_\sigma(x)=\frac{d^{2\sigma}}{dx^{2\sigma}}G_\sigma(x),\quad G_\sigma(x)=\sum_{n=0}^{\infty}x^n\left[\frac{n+\frac{1}{2}}{n+2\sigma}\right]^{1/2}. \tag{B5}$$

To get an asymptotic expansion for $G_\sigma(x)$ and, hence, for $F_\sigma(x)$ one needs to sum $G_\sigma(x)$ resorting to its integral representation. This can be done using the identity

$$\left[\frac{n+\alpha-\beta}{n+\alpha+\beta}\right]^{1/2}=n\int_0^\infty dt\,e^{-nt}\left[e^{-\alpha t}I_0(\beta t)+(\alpha-\beta)\int_0^t du\,e^{-\alpha u}I_0(\beta u)\right]\,, \tag{B6}$$

where $I_\nu(x)$ are the customary modified Bessel functions. After an integration by parts one obtains

$$G_\sigma(x)=\frac{1}{1-x}-f_\sigma(\infty)+\int_0^\infty\frac{dt}{1-xe^{-t}}g_\sigma(t)\,, \tag{B7}$$

where

$$g_\sigma(t)=\frac{df_\sigma}{dt},\quad f_\sigma(t)=e^{-(\sigma+1/4)t}I_0[(\sigma-\tfrac{1}{4})t]+\tfrac{1}{2}\int_0^t du\,e^{-(\sigma+1/4)u}I_0[(\sigma-\tfrac{1}{4})u]\,. \tag{B8}$$

The singular part of the integral on the right-hand side of Eq. (B7) can be extracted through the intermediate steps

$$\int_0^\infty \frac{dt}{1-xe^{-t}} g_\sigma(t) = \sum_{n=0}^{\infty} (-)^n C_n (\sigma-\tfrac{1}{4})^{n+1} \frac{d^n}{dy^n}\bigg|_{y=\sigma+1/4} - \left\{ \frac{1}{x^y} \left[\ln(1-x) + \sum_{p=0}^{\infty} \begin{bmatrix} y-1 \\ p+1 \end{bmatrix} \frac{(-)^{p+1}}{p+1} [(1-x)^{p+1}-1] \right] \right\}$$
$$= \sum_{n=0}^{\infty} (-)^n C_n (\sigma-\tfrac{1}{4})^{n+1} \frac{d^n}{dy^n}\bigg|_{y=\sigma+1/4} \left[-\frac{1}{x^y}\ln(1-x) + \text{reg}(1-x) \right], \qquad \text{(B9)}$$

where reg$(1-x)$ denotes the regular part of the integral and C_n are the coefficients of the expansion

$$I_0'(t) - I_0(t) = \sum_{n=0}^{\infty} C_n t^n . \qquad \text{(B10)}$$

It follows that the asymptotic behavior of $F_\sigma(x)$ for $x \to 1$ is given by

$$F_\sigma(x) = \frac{(2\sigma)!}{(1-x)^{2\sigma+1}} - (\sigma-\tfrac{1}{4}) \frac{d^{2\sigma}}{dx^{2\sigma}} \left[x^{\sigma-1/4} \sum_{n=0}^{\infty} C_n (\sigma-\tfrac{1}{4})^n (\ln x)^n \ln(1-x) + \text{reg}(1-x) \right] . \qquad \text{(B11)}$$

Using Eqs. (B1)–(B5) and (B11) one finally obtains squeezing near the $\rho=1$ absolute minimum:

$$2\chi^{(2)} = 1 - \frac{8\sigma\rho}{1+\rho} - \frac{4\sigma-1}{(2\sigma-1)!} \rho (1-\rho^2)^{2\sigma} \left[\sum_{n=0}^{\infty} C_n (\sigma-\tfrac{1}{4})^n \frac{d^{2\sigma}}{dx^{2\sigma}}\bigg|_{x=\rho^2} x^{\sigma-1/4} (\ln x)^n \ln(1-x) \right] + \text{reg}(1-x)$$
$$= 1 - 4\sigma - \frac{4\sigma-1}{(2\sigma-1)!}(2\sigma-1)![-1+O(1-\rho)] + O(1-\rho) \sim 1-\rho . \qquad \text{(B12)}$$

From the last equation one can see that the squeezing goes to zero near $\rho=1$ as $1-\rho$. From Table I we extract the singular behavior of the variance $\langle \Delta n^2 \rangle$ near $\rho=1$:

$$\langle \Delta n^2 \rangle \sim (1-\rho)^2 . \qquad \text{(B13)}$$

Comparing the asymptotic behavior of $\chi^{(2)}$ and $\langle \Delta n^2 \rangle$ one finally obtains the scaling law

$$\chi^{(2)} \sim \langle \Delta n^2 \rangle^{-1/2} . \qquad \text{(B14)}$$

For Gaussian states the same scaling law follows directly from Table I and the explicit expression of the second moment:

$$\chi^{(2)}_{\text{Gauss}} = \frac{1-|\xi|}{1+|\xi|} \sim 1-|\xi| \sim [(\Delta n)^2]^{-1/2} . \qquad \text{(B15)}$$

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[1]V. B. Braginskii, Y. I. Vorontsov, and V. D. Krivchenkov, Zh. Eksp. Teor. Fiz. **68**, 55 (1975) [Sov. Phys. JETP **41**, 28 (1975)]; C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).

[2]H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[3]J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).

[4]D. Stoler, Phys. Rev. D **1**, 3217 (1970).

[5]R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[6]C. K. Hong and L. Mandel, Phys. Rev. Lett. **54**, 323 (1985).

[7]R. A. Fisher, M. M. Nieto, and V. D. Sandberg, Phys. Rev D **29**, 1107 (1984).

[8]S. L. Braunstein, and R. I. McLachlan, Phys. Rev. A **35**, 1659 (1987).

[9]R. A. Brandt and O. W. Greenberg, J. Math. Phys. **10**, 1168 (1969).

[10]T. Holstein and H. Primakoff, Phys. Rev. **58**, 1048 (1940).

[11]G. D'Ariano, M. Rasetti, and M. Vadacchino, Phys. Rev. D **32**, 1034 (1985).

[12]G. D'Ariano and M. Rasetti, Phys. Rev. D **35**, 1239 (1987).

[13]J. Katriel, A. I. Solomon, G. D'Ariano, and M. Rasetti, Phys. Rev. D **34**, 2332 (1986).

[14]J. Katriel, M. Rasetti, and A. I. Solomon, Phys. Rev. D **35**, 1248 (1987).

[15]J. Katriel, M. Rasetti, and A. I. Solomon, Phys. Rev. D **35**, 2601 (1987).

[16]J. Katriel, A. I. Solomon, G. D'Ariano, and M. Rasetti, J. Opt. Soc. (to be published).

[17]G. D'Ariano, M. Rasetti, and M. Vadacchino, J. Phys. A **18**, 1295 (1985).