Multiphoton squeezed states

J. Katriel*

Department of Physics, Drexel University, Philadelphia, Pennsylvania 19104

A. I. Solomon

Faculty of Mathematics, The Open University, Milton Keynes, United Kingdom

G. D'Ariano

Dipartimento di Fisica A. Volta, 27100 Pavia, Italy

M. Rasetti

Dipartimento di Fisica del Politecnico, 10129 Torino, Italy

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The multiphoton squeezed states defined in this paper are generalizations of the conventional coherent (Glauber) and squeezed (Yuen) states previously discussed by many authors. We define multiphoton generalizations of the latter by a unified class of states that includes the Holstein–Primakoff realizations of SU(2) and SU(1, 1) as well as the standard harmonic oscillator coherent states (Weyl–Heisenberg group) and squeezed states in a general framework that allows also non-Hermitian realizations. We determine the squeezing properties of these states in a unified formalism and study numerically their dependence on the parameter classifying the states.

1. INTRODUCTION

Customary Glauber coherent states are eigenstates of the annihilation operator with the property that the uncertainty product for the position and momentum canonical variables attains its minimum value when the two standard variances are equal. Squeezed states are characterized by the property that one of the uncertainties is smaller than that in a coherent state (naturally at the expense of the other, because of Heisenberg's principle). Squeezed states may therefore prove to be useful in low-noise detection experiments. The original form in which squeezed states were written by Stoler,[1] Yuen,[2] and others quite naturally identifies them with generalized coherent states of the group SU(1, 1). A straightforward generalization of such a form, however, runs into difficulties, which appear as nonanalyticities of the vacuum vector, as pointed out by Fisher *et al.*[3] In order to overcome this difficulty the simple one-photon coherent states were first generalized to k-photon states by using the generalized k-boson operators first defined by Brandt and Greenberg.[4] Similarly squeezed states were generalized by considering Holstein–Primakoff representations of a desired Lie group [the generalization was performed not only for SU(1, 1) but for SU(2) as well] expressed in terms of the k-boson creation and annihilation operators. It was shown that in all cases improved squeezing can be obtained.

All this information prompts the question of whether the various generalizations that have come into play could indeed be handled on a common ground and be united in a common structure from which the various cases could be obtained by suitable limiting procedures. In the present paper we discuss this unified treatment of coherent and squeezed states, using non-Hermitian realizations of the generators of the relevant algebras in terms of multiboson operators. Each realization is labeled by a pair of parameters (besides the photon number) that may be varied in such a way as to connect all previously defined squeezed states in a unique scheme. It is not clear at the present time which of the states in the scheme may be obtained experimentally. The new non-Hermitian cases put into play show interesting features of their own, which also allows us, incidentally, to cast new light on the appearance of the vacuum singularities pointed out by Fisher *et al.*[3]

The non-Hermitian realization of the k-boson operators is introduced in Section 2. In Section 3 such a realization is utilized to construct general Weyl–Heisenberg-group coherent states. Analogous procedures lead, in Sections 4 and 5, to the construction of coherent squeezed states for non-Hermitian SU(2) and SU(1, 1), respectively.

All cases are illustrated by a detailed numerical analysis of the squeezing properties, which clarifies the connections among different situations as well as the role of the parameter controlling the non-Hermiticity of the operators. A few concluding remarks are made in Section 6.

2. NON-HERMITIAN MULTIBOSON REALIZATION

The physical properties of photon states are derived from the algebraic commutation relation for the photon-annihilation and photon-creation operators a, $a^\dagger$:

$$[a, a^\dagger] = I. \tag{2.1}$$

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From the work of Brandt and Greenberg[4] we may define k-photon operators $A_{(k)}$ and $A_{(k)}^\dagger$ by

$$A_{(k)} \equiv a^k f_k(\hat{n}), \qquad A_{(k)}^\dagger \equiv f_k(\hat{n})(a^\dagger)^k, \tag{2.2}$$

$$f_k(\hat{n}) \equiv [[\![\hat{n}/k]\!](\hat{n}-k)!/\hat{n}!]^{1/2}. \tag{2.3}$$

In Eq. (2.3) the symbol $[\![x]\!]$ means the greatest integer less than or equal to x, k is a positive integer, and functions of the operators $\hat{n} \equiv a^\dagger a$ are evaluated in eigenstates of $\hat{n}$ as the functions of the corresponding eigenvalue. The generalized boson operators $A_{(k)}$, $A_{(k)}^\dagger$ satisfy the usual boson commutation relations

$$[A_{(k)}, A_{(k)}^\dagger] = I. \tag{2.4}$$

We may generalize Eqs. (2.2) as follows. Define

$$A_{(k)}^{(\eta)+} \equiv \Phi_{(k)}^{(\eta)}(\hat{n})(a^\dagger)^k,$$
$$A_{(k)}^{(\eta)-} \equiv a^k \Phi_{(k)}^{(1-\eta)}(\hat{n}), \tag{2.5}$$

where

$$\Phi_{(k)}^{(\eta)} \equiv [f_k(\hat{n})]^{2\eta}. \tag{2.6}$$

The commutation relation [Eq. (2.4)] is preserved:

$$[A_{(k)}^{(\eta)-}, A_{(k)}^{(\eta)+}] = I. \tag{2.7}$$

Further,

$$A_{(k)}^{(\eta)-} = [A_{(k)}^{(1-\eta)+}]^\dagger. \tag{2.8}$$

However, in general, $A_{(k)}^{(\eta)+} \neq (A_{(k)}^{(\eta)-})^\dagger$, and so we no longer have Hermiticity. For $\eta = 1/2$ we recover the Hermitian realization of Eqs. (2.2). The case $\eta = 0$, which is always non-Hermitian unless $k = 1$, is reminiscent of a realization of SU(2) first considered by Dyson in the context of spin waves.[5]

For $k = 1$ we recover the usual boson operators a and $a^\dagger$ for all η.

3. WEYL–HEISENBERG-GROUP COHERENT STATES

Conventional coherent states may be defined, up to normalization, by the exponentiated action of the photon creation operator on the vacuum, thus $e^{\zeta a^\dagger}|0\rangle$. Using the non-Hermitian multiphoton operators defined in Section 2, we may attempt to generalize this to

$$|\zeta; k, \eta\rangle = \mathcal{N}^{-1} \exp[\zeta A_{(k)}^{(\eta)+}]|0\rangle. \tag{3.1}$$

In Eq. (3.1) the factor $\mathcal{N}^{-1}$ is a normalization coefficient, when $|\zeta; k, \eta\rangle$ is normalizable. For the conventional coherent states $|\zeta; 1, \eta\rangle$ and squeezed states[1] $|\zeta; 2, 0\rangle$ such normalization is possible; naive extensions of the same analysis to states of this form with $k \geq 3$, $|\zeta; k, 0\rangle$ fail[3] because the vacuum $|0\rangle$ is not an analytical vector of the operator $A_{(k)}^{(0)+} = (a^\dagger)^k$ ($k \geq 3$). (Braunstein and McLachlan[6] have recently studied this problem thoroughly by using an accurate Padé approximant analysis instead of the usual Taylor-series expansion.)

By using Eqs. (2.5) we can write Eq. (3.1) as

$$|\zeta; k, \eta\rangle = \mathcal{N}^{-1} \sum_{m=0}^{\infty} \frac{\zeta^m}{\sqrt{m!}} \left[\frac{(km)!}{m!}\right]^{(1/2)-\eta} |km\rangle. \tag{3.2}$$

We evaluate formally the normalization coefficient, indicating explicitly its dependence on ζ, k, and η, as

$$\mathcal{N}^2(\zeta; k, \eta) = \sum_{m=0}^{\infty} |\zeta|^{2m} \frac{[(km)!]^{1-2\eta}}{(m!)^{2-2\eta}}. \tag{3.3}$$

We note some values of this coefficient:

(1) Conventional coherent states: $\mathcal{N}^2(\zeta; 1, \eta) = \exp(|\zeta|^2)$.
(2) Generalized coherent states: $\mathcal{N}^2(\zeta; k, 1/2) = \exp(|\zeta|^2)$.
(3) Conventional squeezed states: $\mathcal{N}^2(\zeta; 2, 0) = (1 - 4|\zeta|^2)^{-1/2}$.

The generalized coherent states mentioned above in Section 2 were derived in Refs. 7–9. The region of convergence of the normalization coefficient in the (k, η) plane is given in Fig. 1. This figure illustrates the impossibility of naïve generalization of the usual squeezed states, referred to by Fisher et al.[3] Their states lie along the $\eta = 0$ axis, and the boundary of convergence occurs for $k = 2$. The nonnormalizability of the state $|\zeta; k, 0\rangle$ rigorously implies the nonnormalizability of the state $\exp[\alpha(a^\dagger)^k - \bar{\alpha}a^k]|0\rangle$ of Ref. 3. The Hermitian multiphoton (Brandt–Greenberg) states of Refs. 7 and 8 lie on the $\eta = 1/2$ asymptote and so encounter no convergence boundary. For values of η that are intermediate between 0 and 1/2, the maximum allowed value of k is given by $[\![(2-2\eta)/(1-2\eta)]\!]$. We now evaluate the quantum uncertainties $(\Delta x)^2 \equiv \langle x^2\rangle - \langle x\rangle^2$ and $(\Delta p)^2 \equiv \langle p^2\rangle - \langle p\rangle^2$, where $x = (1/\sqrt{2})(a + a^\dagger)$, $p = (i/\sqrt{2})(a^\dagger - a)$, and $\langle\cdot\rangle = \langle\zeta; k, \eta| \cdot |\zeta; k, \eta\rangle$.

These uncertainties are given by

$$(\Delta x)^2 = 1/2 + \langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle + \mathrm{Re}\,[\langle(a^\dagger)^2\rangle - \langle a^\dagger\rangle^2],$$
$$(\Delta p)^2 = 1/2 + \langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle - \mathrm{Re}[\langle(a^\dagger)^2\rangle - \langle a^\dagger\rangle^2]. \tag{3.4}$$

The expectations $\langle a^n\rangle$ are nonzero only for $k \leq n$. Since $k = 1$ gives the conventional coherent state for which there is no squeezing $[(\Delta x)^2 = (\Delta p)^2 = 1/2]$, we shall treat only the case in which $k = 2$. We obtain the following expectations:

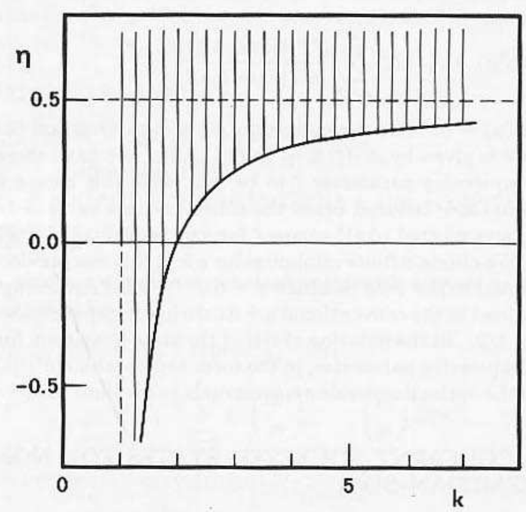

Fig. 1. Region of convergence $\{\eta \geq 1/2 - 1/[2(k-1)]\}$ of the normalization for the non-Hermitian multiboson Weyl–Heisenberg-group coherent states.

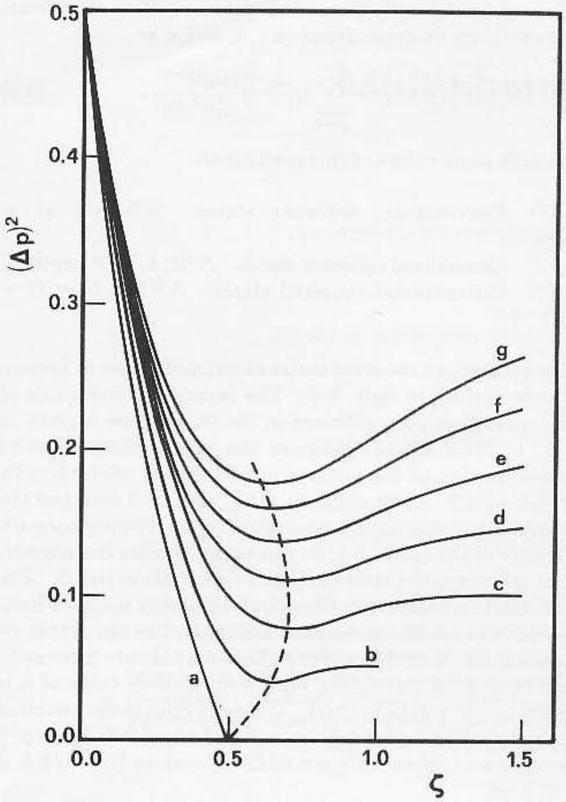

Fig. 2. Momentum uncertainties for the two-boson non-Hermitian Weyl–Heisenberg-group coherent states ($\eta = 0$ to 0.6 in steps of 0.1, from a to g).

$$\langle a^2\rangle = \mathcal{N}^{-2}\sqrt{2}\zeta \sum_{m=0}^{\infty} \frac{\zeta^{2m}}{m!}\left\{\frac{[2(m+1)]!}{(m+1)!}\,\frac{(2m)!}{m!}\right\}^{(1/2)-\eta} \times \sqrt{2m+1}, \tag{3.5}$$

$$\langle a^\dagger a\rangle = \mathcal{N}^{-2}2\zeta^2 \sum_{m=0}^{\infty} \frac{\zeta^{2m}}{m!}\left\{\frac{[2(m+1)]!}{(m+1)!}\right\}^{1-2\eta}, \tag{3.6}$$

and $\langle a\rangle = 0$. The normalization $\mathcal{N}^2$ in Eqs. (3.5) and (3.6) above is given by $\mathcal{N}^2(\zeta; 2, \eta)$ in Eq. (3.3). We have chosen the squeezing parameter ζ to be real; with this choice we obtain $(\Delta p)^2$ reduced below the coherent-state value of 1/2. We have plotted $(\Delta p)^2$ versus ζ for various values of η (Fig. 2). We obtain a finite minimum for $\eta > 0$; this was previously noted in the case in which $\eta = 0.5$.[7,8] Most squeezing is obtained in the conventional $\eta = 0$ case in correspondence to $|\zeta| = 1/2$. In the notation of Ref. 3 the state is written, for a real squeezing parameter, in the form $\exp[\frac{1}{2}\tanh r(a^\dagger)^2]|0\rangle$, and the optimal squeezing corresponds to the limit as $r \to \infty$.

4. COHERENT SQUEEZED STATES FOR NON-HERMITIAN SU(2)

The starting point for the non-Hermitian realization of SU(2) that we adopt in this section is the Holstein–Primakoff scheme[10]:

$$J_- = a(2\sigma + 1 - \hat{n})^{1/2},$$
$$J_+ = (2\sigma + 1 - \hat{n})^{1/2}a^\dagger = J_-^\dagger,$$
$$J_3 = \tfrac{1}{2}[J_+, J_-] = \hat{n} - \sigma. \tag{4.1}$$

The normalized states $|N\rangle = (N!)^{-1/2}(a^\dagger)^N|0\rangle$, $N = 0, 1, \ldots, 2\sigma$ form a basis for the $(2\sigma+1)$-dimensional representation of SU(2). We may generalize the realization (4.1) to a multiboson realization as follows:

$$J_-^{(k)} = a^k f_{k,\sigma}(\hat{n}),$$
$$J_+^{(k)} = f_{k,\sigma}(\hat{n})(a^\dagger)^k = [J_-^{(k)}]^\dagger,$$
$$J_3^{(k)} = [\![\hat{n}/k]\!] - \sigma, \tag{4.2}$$

where

$$f_{k,\sigma} = \left[(2\sigma + 1 - [\![\hat{n}/k]\!])\,\frac{(\hat{n}-k)!\,[\![\hat{n}/k]\!]}{\hat{n}!}\right]^{1/2}. \tag{4.3}$$

The squeezing properties of these operators were previously investigated.[9,11] We now generalize Eqs. (4.2) in the spirit of Section 2. An appropriate generalization is

$$J_{(\eta)-}^{(k)} = a^k\Phi_{k,\sigma}^{(1-\eta)}(\hat{n}),$$
$$J_{(\eta)+}^{(k)} = \Phi_{k,\sigma}^{(\eta)}(\hat{n})(a^\dagger)^k,$$
$$J_{(\eta)3}^{(k)} = [\![\hat{n}/k]\!] - \sigma, \tag{4.4}$$

where

$$\Phi_{k,\sigma}^{(\eta)}(\hat{n}) = [f_{k,\sigma}(\hat{n})]^{2\eta}. \tag{4.5}$$

We note that this is a non-Hermitian realization; in general $J_{(\eta)+}^{(k)} \neq [J_{(\eta)-}^{(k)}]^\dagger = J_{(1-\eta)+}^{(k)}$, unless $\eta = 1/2$ when we recover Eqs. (4.2). The original Dyson realization of Ref. 5 corresponds to $k = 1$ and $\eta = 1$. In determining the squeezing properties of these operators, as in Eq. (3.4), that involve only first- and second-moment calculations, we need consider only the cases of $k = 1$ and $k = 2$,

SU(2), $k = 1$ Realization

$$J_{(\eta)+}^{(1)} = a^\dagger(2\sigma - a^\dagger a)^\eta,$$
$$J_{(\eta)-}^{(1)} = (2\sigma - a^\dagger a)^{1-\eta}a,$$
$$J_{(\eta)3}^{(1)} = a^\dagger a - \sigma. \tag{4.6}$$

We define the corresponding group-theoretical coherent state by

$$|\zeta\rangle_1 = \mathcal{N}_1^{-1}\exp(\zeta J_{(\eta)+}^{(1)})|0\rangle \tag{4.7}$$

$$= \mathcal{N}_1^{-1}\sum_{m=0}^{2\sigma}\zeta^m\frac{\binom{2\sigma}{m}^\eta}{(m!)^{(1/2)-\eta}}|m\rangle. \tag{4.8}$$

The normalization factor $\mathcal{N}_1$ for this state is given by

$$\mathcal{N}_1^2 = \sum_{m=0}^{2\sigma}\frac{|\zeta|^{2m}}{(m!)^{1-2\eta}}\binom{2\sigma}{m}^{2\eta}, \tag{4.9}$$

where there is no convergence difficulty, since Eq. (4.9) is a finite sum. The requisite expectations with respect to the state described by Eq. (4.8) are given by

$$\langle a\rangle_1 = \mathcal{N}_1^{-2}\zeta\sum_{m=0}^{2\sigma-1}\frac{|\zeta|^{2m}}{(m!)^{1-2\eta}}\binom{2\sigma}{m}^{2\eta}(2\sigma - m)^\eta, \tag{4.10}$$

$$\langle a^2\rangle_1 = \mathcal{N}_1^{-2}\zeta^2 \sum_{m=0}^{2\sigma-1} \frac{|\zeta|^{2m}}{(m!)^{1-2\eta}}\binom{2\sigma}{m}^{2\eta} \times [(2\sigma - m)(2\sigma - m - 1)]^{\eta}, \tag{4.11}$$

$$\langle a^{\dagger}a\rangle_1 = \mathcal{N}_1^{-2} \sum_{m=1}^{2\sigma} m \frac{|\zeta|^{2m}}{(m!)^{1-2\eta}}\binom{2\sigma}{m}^{2\eta}. \tag{4.12}$$

When ζ is chosen to be real, $(\Delta x)^2$ is squeezed in Eqs. (3.4); it is convenient to define a squeezing parameter $\rho = (2\sigma)^{\eta}\zeta$, since, for large σ, $|\zeta\rangle_1$ behaves like a Glauber coherent state with parameter ρ, by inspection of Eqs. (4.6) and (4.7). If η and σ are fixed, then there is an optimally squeezed value of $(\Delta x)^2$ as ρ varies; we have plotted this minimal $(\Delta x)^2$ versus $1/\sigma$ for various values of η in Fig. 3. The value of $(\Delta x)^2$ minimized with respect to both ρ and η is shown in Fig. 4(a), the corresponding optimal value of η is shown in Fig. 4(b). We note from the figures that as $\sigma \to \infty$ the minimum value $(\Delta x)^2(\eta_{\min}, \rho_{\min})$ of the $(\Delta x)^2$ variance (minimized with respect to both η and ρ) tends to zero. The limit is approached in a rather singular manner; the numerical results suggest that

$$(\Delta x)_{\min}{}^2 \sim 0.326\sigma^{-1/3},$$
$$\eta_{\min} \sim 0.867\sigma^{-1/5}.$$

For fixed η, we would obtain the (nonsingular) limit corresponding to a standard coherent state, i.e., $(\Delta x)^2 = (\Delta p)^2 =$ 0.5. Note that we obtain the same asymptotic behavior $(\Delta x)^2 \sim \sigma^{-1/3}$ noted in Ref. 11 for single-photon SU(2) squeezing. Since σ gives an upper bound to the number of photons N_{ph} available, this asymptotic behavior corresponds to $(\Delta x)^2 \geqslant C N_{\mathrm{ph}}{}^{-1/3}$, where C is some constant of the order of unity.

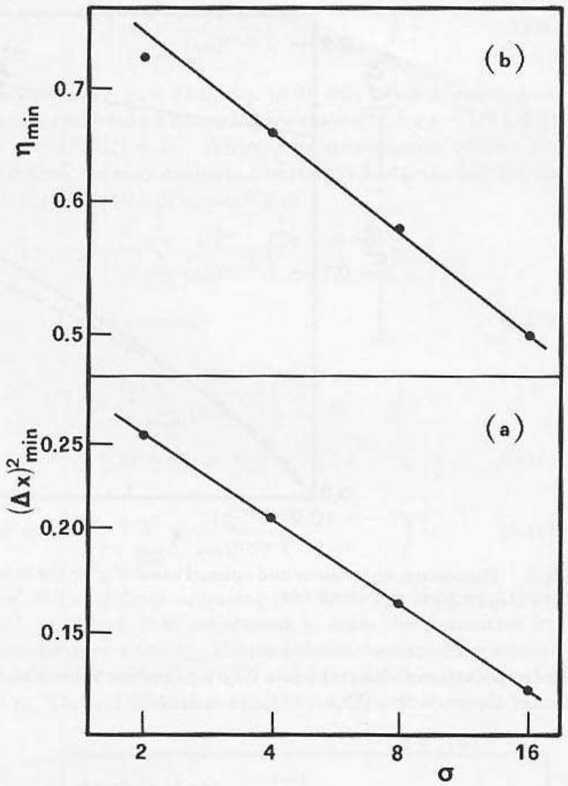

Fig. 4. One-boson non-Hermitian SU(2) coherent states (log–log plot): (a) $(\Delta x)^2$ minimized with respect to both ρ and η (the line describes the power law $(\Delta x)_{\min}{}^2 \sim 0.326\sigma^{-1/3}$); (b) The optimal value of η (the line describes the power law $\eta_{\min} \sim 0.867\sigma^{-1/5}$).

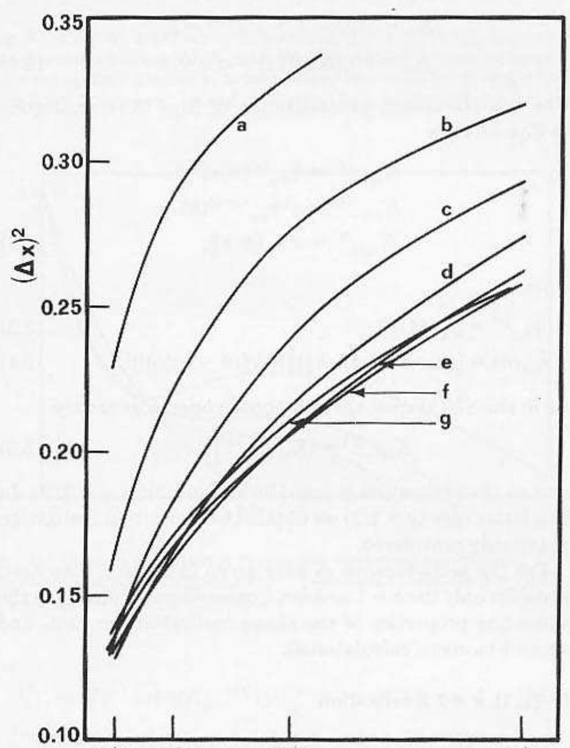

Fig. 3. Positon uncertainty, optimized with respect to the coherence parameter ρ, for one-boson non-Hermitian SU(2) coherent states, (η = 0.2 to 0.8 in steps of 0.1, from a to g).

SU(2), $k = 2$ Realization

$$J_{(\eta)+}{}^{(2)} = (a^{\dagger})^2 \{([\![\hat{n}/2]\!] + 1)(2\sigma - [\![\hat{n}/2]\!])/[(\hat{n}+1)(\hat{n}+2)]\}^{\eta},$$
$$J_{(\eta)-}{}^{(2)} = \{([\![\hat{n}/2]\!] + 1)(2\sigma - [\![\hat{n}/2]\!])/[(\hat{n}+1)(\hat{n}+2)]\}^{1-\eta}a^2,$$
$$J_{(\eta)3}{}^{(2)} = [\![a^{\dagger}a/2]\!] - \sigma. \tag{4.13}$$

As above, we define a coherent state with respect to this realization by

$$|\zeta\rangle_2 = \mathcal{N}_2^{-1}\exp(\zeta J_{(\eta)+}{}^{(2)})|0\rangle = \mathcal{N}_2^{-1}\sum_{m=0}^{2\sigma} \zeta^m \binom{2m}{m}^{(1/2)-\eta}\binom{2\sigma}{m}^{\eta}|2m\rangle. \tag{4.14}$$

The normalization coefficient $\mathcal{N}_2$ for this state is given by

$$\mathcal{N}_2{}^2 = \sum_{m=0}^{2\sigma} |\zeta|^{2m}\binom{2m}{m}^{1-2\eta}\binom{2\sigma}{m}^{2\eta}. \tag{4.15}$$

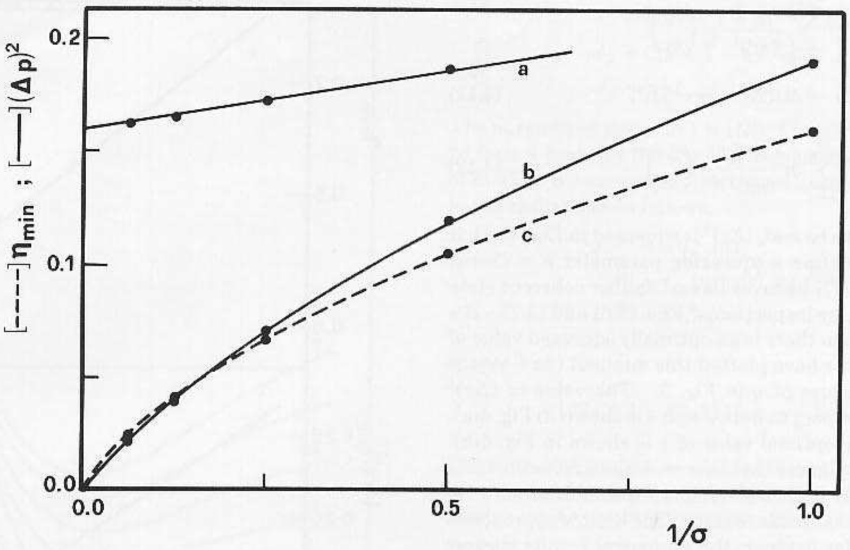

Fig. 5. Momentum uncertainty and optimal value of η for the two-boson non-Hermitian SU(2) coherent states: a, $(\Delta p)^2[\rho_{\min}]$ for $\eta = 0.5$; b, $(\Delta p)^2[\rho_{\min}, \eta_{\min}]$; c, $\eta_{\min}$ versus $1/\sigma$.

The expectations with respect to $|\zeta\rangle_2$ required for the evaluation of the variances [Eqs. (3.4)] are as follows:

$$\langle a\rangle_2 = 0,$$

$$\langle a^2\rangle_2 = 2^{1-\eta}\mathcal{N}_2^{-2}\zeta\sum_{m=0}^{2\sigma-1}|\zeta|^{2m}\binom{2m}{m}^{1-2\eta}\binom{2\sigma}{m}^{2\eta} \times (2m+1)^{1-\eta}(2\sigma-m)^{\eta}, \tag{4.16}$$

$$\langle a^\dagger a\rangle_2 = 2\mathcal{N}_2^{\,2}\sum_{m=1}^{2\sigma} m|\zeta|^{2m}\binom{2m}{m}^{1-2\eta}\binom{2\sigma}{m}^{2\eta}. \tag{4.17}$$

In Fig. 5 the squeezing of $(\Delta p)^2$ is plotted. For $\eta = 0.5$ we recover the result of the conventional Hermitian Holstein–Primakoff approach[11] (Fig. 3); we also show the minimum value of $(\Delta p)^2$ as a function of both ρ and η. For this two-photon SU(2) case, the asymptotic behavior suggested by the numerical results is

$$(\Delta p)^2 \sim 0.2\sigma^{-0.8}$$

and

$$\eta_{\min} \sim 0.16\sigma^{-0.67},$$

again implying a bound to the amount of squeezing possible with a finite number of photons (here, too, $N_{\mathrm{ph}} \sim \sigma$).

5. COHERENT (SQUEEZED) STATES FOR NON-HERMITIAN SU(1, 1)

Since the conventional squeezing operator,[1] $S(\zeta) = \exp[\frac{1}{2}\zeta(a^\dagger)^2 - \frac{1}{2}\zeta^* a^2]$ is an element of the representation of the group SU(1, 1), it comes as no surprise that realizations of SU(1, 1) play an important role in squeezing phenomena. In this section we write a non-Hermitian nonlinear realization of SU(1, 1) that is analogous to that given in Section 4 for SU(2). The genesis of this realization lies in the Holstein–Primakoff realization of SU(1, 1), analogous to Eqs. (4.1),[10,11]

$$K_+ = (2\sigma - 1 + a^\dagger a)^{1/2}a^\dagger,$$
$$K_- = a(2\sigma - 1 + a^\dagger a)^{1/2} = K_+^{\dagger},$$
$$K_3 = -\tfrac{1}{2}[K_+, K_-] = a^\dagger a + \sigma. \tag{5.1}$$

The non-Hermitian generalization of Eqs. (5.1), analogous to Eqs. (4.4), is

$$K_{(\eta)+}^{(k)} = \tilde{\Phi}_{k,\sigma}^{(\eta)}(\hat{n})(a^\dagger)^k,$$
$$K_{(\eta)-}^{(k)} = a^k\tilde{\Phi}_{k,\sigma}^{(1-\eta)}(\hat{n}),$$
$$K_{(\eta)3}^{(k)} = \sigma + [\![\hat{n}/k]\!], \tag{5.2}$$

where

$$\tilde{\Phi}_{k,\sigma}^{(\eta)} = [\tilde{f}_{k,\sigma}(\hat{n})]^{2\eta}, \tag{5.3}$$

$$\tilde{f}_{k,\sigma}(\hat{n}) = [(2\sigma - 1 + [\![\hat{n}/k]\!])[\![\hat{n}/k]\!](\hat{n}-k)!/\hat{n}!]^{1/2}. \tag{5.4}$$

As in the $SU(2)$ case, these k-photon operators satisfy

$$K_{(\eta)+}^{(k)} = [K_{(1-\eta)-}^{(k)}]^\dagger, \tag{5.5}$$

and so the realization is non-Hermitian unless $\eta = 1/2$. In this latter case ($\eta = 1/2$) we obtain the Hermitian realization previously considered.

For the same reasons as were given in Section 4 we need consider only the $k = 1$ and $k = 2$ cases when determining the squeezing properties of the above realization (in first- and second-moment calculations).

SU(1, 1), $k = 1$ Realization

$$K_{(\eta)+}^{(1)} = (2\sigma - 1 + \hat{n})^{\eta}a^\dagger = a^\dagger(2\sigma + \hat{n})^{\eta},$$
$$K_{(\eta)-}^{(1)} = (2\sigma + \hat{n})^{1-\eta}a,$$
$$K_{(\eta)3}^{(1)} = \sigma + \hat{n}. \tag{5.6}$$

The appropriate squeezed state is defined to be

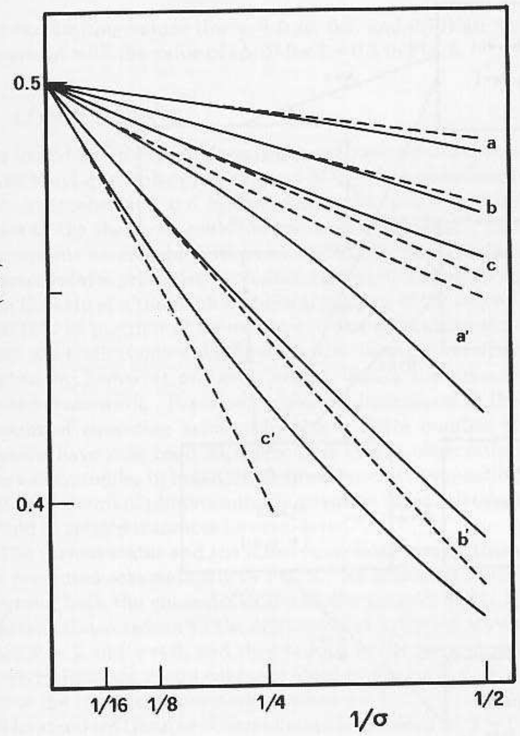

Fig. 6. Positon uncertainty at constant $\rho = \zeta(2\sigma)^\eta$ for the one-boson non-Hermitian SU(2) (solid lines) and SU(1, 1) (dashed lines) coherent states. Curves a, b, and c refer to $\eta = 0.25$, 0.5, and 0.75 with $\rho = 0.5$; curves a′, b′, and c′ refer to the same η's with $\rho = 1$.

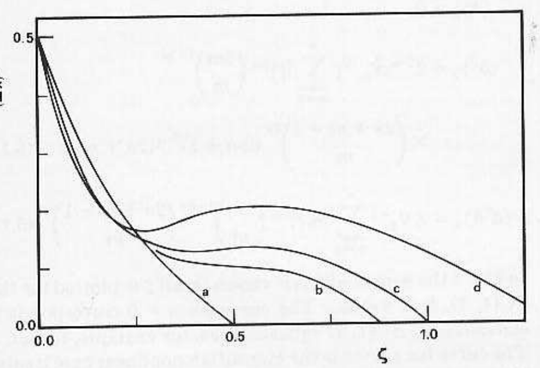

Fig. 7. Positon uncertainty versus the squeezing parameter ζ for the two-boson non-Hermitian SU(1, 1) coherent state, for $\sigma = 2$ and a, $\eta = 0$, b, $\eta = 0.4$; c, $\eta = 0.5$; d, $\eta = 0.7$.

$$|\tilde{\zeta}\rangle_1 = \tilde{\mathcal{N}}_1^{-1} \exp(\zeta K_{(\eta)+}{}^{(1)})|0\rangle \tag{5.7}$$

$$= \tilde{\mathcal{N}}_1^{-1} \sum_{m=0}^{\infty} \frac{\zeta^m}{(m!)^{(1/2)-\eta}} \binom{2\sigma + m - 1}{m}^{\eta} |m\rangle. \tag{5.8}$$

We may formally express the normalization constant $\tilde{\mathcal{N}}_1$ of this state as

$$\tilde{\mathcal{N}}_1{}^2 = \sum_{m=0}^{\infty} \frac{|\zeta|^{2m}}{(m!)^{1-2\eta}} \binom{2\sigma + m - 1}{m}^{2\eta}. \tag{5.9}$$

However, we note that Eq. (5.9) only gives a convergent series, and hence a normalizable state $|\tilde{\zeta}\rangle_1$, for $\eta < 1/2$ (all ζ) or $\eta = 1/2$ ($|\zeta| < 1$). When these convergence criteria are satisfied, we may evaluate the required expectation values for the obtention of squeezing as

$$\langle \tilde{a} \rangle_1 = \tilde{\mathcal{N}}_1{}^{-2} \zeta \sum_{m=0}^{\infty} \frac{|\zeta|^{2m}}{(m!)^{1-2\eta}} \binom{2\sigma + m - 1}{m}^{2\eta} \times (2\sigma + m)^{\eta}, \tag{5.10}$$

$$\langle \tilde{a}^2 \rangle_1 = \tilde{\mathcal{N}}_1{}^{-2} \zeta^2 \sum_{m=0}^{\infty} \frac{|\zeta|^{2m}}{(m!)^{1-2\eta}} \binom{2\sigma + m - 1}{m}^{2\eta} \times [(2\sigma + m)(2\sigma + m + 1)]^{\eta}, \tag{5.11}$$

$$\langle \tilde{a}^\dagger \tilde{a} \rangle_1 = \tilde{\mathcal{N}}_1{}^{-2} \sum_{m=1}^{\infty} m \frac{|\zeta|^{2m}}{(m)!^{1-2\eta}} \binom{2\sigma + m - 1}{m}^{2\eta}. \tag{5.12}$$

We obtain optimal squeezing [in $(\Delta p)^2$] by choosing ζ to be real; as before, it is convenient to scale the parameter by introducing $\rho \equiv (2\sigma)^\eta \zeta$. Figure 6 shows the squeezing attainable for two values of ρ (0.5 and 1.0) and three values of η (0.25, 0.5, and 0.75) as a function of $1/\sigma$. For comparison,

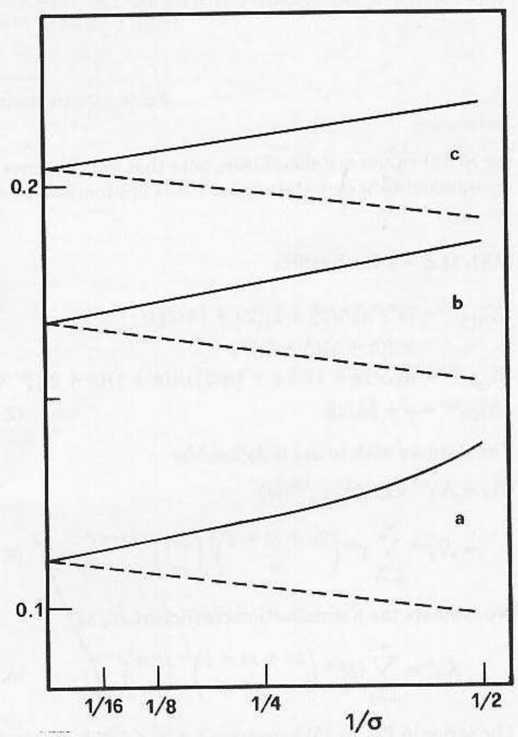

Fig. 8. Momentum uncertainty versus $1/\sigma$, at constant $\rho = \zeta(2\sigma)^\eta = 0.5$, for the two-boson non-Hermitian SU(2) (solid lines) and SU(1, 1) (dashed lines) coherent states and a, $\eta = 0.25$; b, $\eta = 0.5$; c, $\eta = 0.75$.

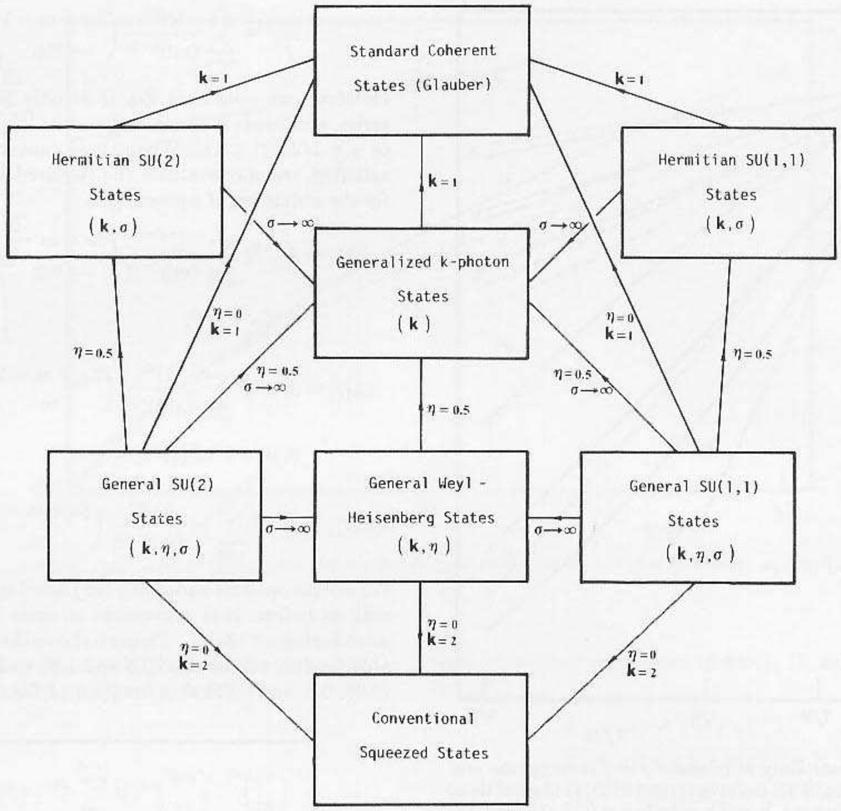

Fig. 9. Diagrammatic interconnections between states.

the SU(2) values are also shown; note that in both cases the unsqueezed-coherent-state value 0.5 is approached as $\sigma \to \infty$.

SU(1, 1), $k = 2$ Realization

$$
\begin{aligned}
K_{(\eta)+}{}^{(2)} &= (a^\dagger)^2\{(\llbracket \hat{n}/2 \rrbracket + 1)(2\sigma + \llbracket \hat{n}/2 \rrbracket)/ \\
&\quad \times [(\hat{n}+1)(\hat{n}+2)]\}^\eta, \\
K_{(\eta)-}{}^{(2)} &= \{(\llbracket \hat{n}/2 \rrbracket + 1)(2\sigma + \llbracket \hat{n}/2 \rrbracket)/[(\hat{n}+1)(\hat{n}+2)]\}^{1-\eta} a^2, \\
K_{(\eta)3}{}^{(2)} &= \sigma + \llbracket \hat{n}/2 \rrbracket.
\end{aligned} \tag{5.13}
$$

The state we wish to use is defined by

$$
\begin{aligned}
|\tilde{\zeta}\rangle_2 &= \tilde{\mathcal{N}}_2^{-1} \exp(\zeta K_{(\eta)+}{}^{(2)})|0\rangle \\
&= \tilde{\mathcal{N}}_2^{-1} \sum_{m=0}^{\infty} \zeta^m \binom{2\sigma + m - 1}{m}^\eta \binom{2m}{m}^{(1/2)-\eta} |2m\rangle.
\end{aligned} \tag{5.14}
$$

We evaluate the normalization coefficient $\tilde{\mathcal{N}}_2$ as

$$
\tilde{\mathcal{N}}_2{}^2 = \sum_{m=0}^{\infty} |\zeta|^{2m} \binom{2\sigma + m - 1}{m}^{2\eta} \binom{2m}{m}^{1-2\eta}. \tag{5.15}
$$

The series in Eq. (5.15) converges for $|\zeta| < 2^{2\eta-1}$; in terms of the parameter $\rho \equiv (2\sigma)^\eta \zeta$, this criterion is $\rho < \frac{1}{2}(8\sigma)^\eta$. The relevant expectations for use in Eqs. (3.4) are

$$
\langle \tilde{a} \rangle_2 = 0,
$$

$$
\begin{aligned}
\langle \tilde{a}^2 \rangle_2 &= 2^{1-\eta} \tilde{\mathcal{N}}_2{}^{-2} \zeta \sum_{m=0}^{\infty} |\zeta|^{2m} \binom{2m}{m}^{1-2\eta} \\
&\quad \times \binom{2\sigma + m - 1}{m}^{2\eta} (2m+1)^{1-\eta}(2\sigma + m)^\eta,
\end{aligned} \tag{5.16}
$$

$$
\langle \tilde{a}^\dagger \tilde{a} \rangle_2 = 2\mathcal{N}_2{}^{-2} \sum_{m=1}^{\infty} m |\zeta|^{2m} \binom{2m}{m}^{1-2\eta} \binom{2\sigma + m - 1}{m}^\eta. \tag{5.17}
$$

In Fig. 7 the squeezed $(\Delta x)^2$ versus (real) ζ is plotted for this SU(1, 1), $k = 2$ case. The curve for $\eta = 0$ corresponds to conventional SU(1, 1) squeezing, as, for example, in Ref. 3. The curve for $\eta = 0.5$ is the Hermitian nonlinear case treated previously by us[11]; note that in all cases the minimal squeezing goes to zero (this may be shown analytically[9]). The $\eta = 0.5$ and $\eta = 0.7$ curves show a local minimum; in terms of physically desirable states, this point exhibits minimal quantum photon noise[9] and so may be considered optimal. [It is in this optimal-minimum (local-minimum), and not global-minimum, sense that Figs. 3 and 4 of Ref. 11 should be interpreted.] In Fig. 8, the squeezed values of $(\Delta p)^2$ as a function of $1/\sigma$ (for $\rho = 0.5$) are compared with the values obtained for the SU(2) case. These tend to a common limit as $\sigma \to \infty$. The two-boson generalized-coherent-state limit and the limiting values (for $\eta = 0.25$, 0.5, and 0.75) are in agreement with the value of $(\Delta p)^2$ for $\zeta = 0.5$ in Fig. 2.

6. CONCLUSIONS

The introduction of non-Hermitian multiboson realizations of the Weyl–Heisenberg, SU(2), and SU(1, 1) algebras leads to a comprehensive and unified treatment of the various facets of the theory of coherent and squeezed states. The connections among the different generalized coherent and squeezed states previously introduced were pointed out here with the help of a thorough numerical analysis of the second moments of position or momentum. The difficulties that these generalizations encounter in the formulation first studied by Fisher *et al.*[3] are clarified within the present unified framework. Previously observed limitations to the amount of squeezing achievable with a finite number of photons have now been supplemented by the observation that a compromise between the desired amount of squeezing and the amount of photon-number quantum noise tolerated should in certain instances be considered.[9]

The various states and the interconnections among them are presented schematically in Fig. 9. As indicated in the diagram, both the general SU(2) and the general SU(1, 1) coherent states reduce to the conventional squeezed states when $k = 2$ and $\eta = 0$, and they reduce to the generalized Weyl–Heisenberg group coherent state in the limit $\sigma \to \infty$ and to the Hermitian counterpart when $\eta = 1/2$.

The standard Glauber coherent state is obtained for $k = 1$, by either setting $\eta = 0$ or by taking the limit as $\sigma \to \infty$.

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G. D'Ariano and M. Rasetti are also affiliated with Gruppo Nazionale di Struttura della Materia of the Consiglio Nazionale delle Ricerche, Italy, and Istituto Nazionale di Fisica Nucleare, Italy.

* Permanent address, Department of Chemistry, Technion—Israel Institute of Technology, Haifa, Israel.

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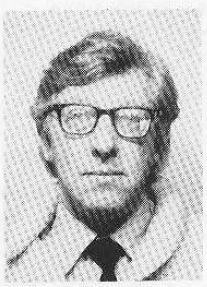
J. Katriel

J. Katriel was born in Israel in 1945. He received the B.Sc., M.Sc., and D.Sc. degrees in chemistry from the Technion—Israel Institute of Technology, Haifa, Israel, in 1967, 1970, and 1972, respectively. From 1970 to 1973 he was a member of the research staff at the Nuclear Research Centre, Negev, Beer Sheva, Israel. In 1973 he joined the Department of Chemistry, Technion, where he is now a professor. He was a Joint Institute for Laboratory Astrophysics Visiting Fellow in Boulder, Colorado, in 1978 and 1979, was a visiting professor at the University of Washington, Seattle, in 1979 and 1980, and is now a visiting professor at Drexel University, Philadelphia. He has held shorter visiting appointments at the Universities of Oxford, Paris-Sud (Orsay), North Carolina (Chapel Hill), Waterloo, Geneva, and Lyon and at the Institute for Scientific Interchange (Torino). His current research interests involve several aspects of the quantum theory of matter, including nonlinear evolution equations and inverse-scattering methods in quantum mechanics, group theoretical aspects of the many-body problem, and the theory of nonlinear-optical phenomena and optical bistability.

A. I. Solomon



A. I. Solomon graduated from Trinity College, Dublin, Ireland, in mathematics and physics, did postgraduate work in Cambridge, England, and received his doctorate in quantum field theory from the University of Paris (1963). After a period in the United States (Republic Aviation Corporation, Polytechnic Institute of Brooklyn), he worked at Tel Aviv University and the Institute for Advanced Studies, Ireland, of which he is a research associate. He has published over 50 papers in the applications of group theory to many-body physics and quantum optics. He is currently Reader in Mathematical Physics at the Open University, England, and is on the Committees of the Mathematical Physics Group (United Kingdom) and the International Colloquium on Group Theoretical Methods in Physics.

G. D'Ariano

G. D'Ariano received the *Laurea* degree in Physics from the University of Pavia, Italy, in 1978. His first researches included nuclear magnetic resonance investigations of phase transitions and critical phenomena. Later he was involved in exactly solvable models in statistical mechanics and group-theoretical approach to quantum coherence, including an extensive study of new squeezed states. He is currently a research associate at the University of Pavia, a member of the Gruppo Nazionale di Struttura della Materia and of the Gruppo Nazionale di Fisica Matematica of the National Research Council, and affiliated with the Istituto Nazionale di Fisica Nucleare.

M. Rasetti

M. Rasetti graduated from the Politecnico, Torino, Italy, in 1969 and the University of Torino in 1971 and did postgraduate work in Göteborg, Sweden, and in Coral Gables, Florida, in collaboration with L. Onsager. He worked for extended periods of time at the Institute for Advanced Study, Princeton, New Jersey, at the University of California (Berkeley, Santa Cruz, and Davis), and at the Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette, France. He has published over 80 papers in different areas of theoretical and mathematical physics: superfluidity, quantization of vortices, the theory of phase transitions in particular for pseudo-spin systems, exactly solvable models in statistical mechanics, coherent states, and quantum optics. In all of these he has emphasized the use of group-theoretical, algebraic, and topological methods. He is currently a full professor of statistical mechanics at the Politecnico and director of the Institute for Scientific Interchange in Torino. He is a member of the Torino Academy of Sciences.