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The Feynman problem and fermionic entanglement: Fermionic theory versus qubit theory

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The present paper is both a review on the Feynman problem, and an original research presentation on the relations between Fermionic theories and qubits theories, both regarded in the novel framework of operational probabilistic theories. The most relevant results about the Feynman problem of simulating Fermions with qubits are reviewed, and in the light of the new original results, the problem is solved. The answer is twofold. On the computational side, the two theories are equivalent, as shown by Bravyi and Kitaev [S. B. Bravyi and A. Y. Kitaev, *Ann. Phys.* **298**, 210 (2002)]. On the operational side, the quantum theory of qubits and the quantum theory of Fermions are different, mostly in the notion of locality, with striking consequences on entanglement. Thus the emulation does not respect locality, as it was suspected by Feynman [R. Feynman, *Int. J. Theor. Phys.* **21**, 467 (1982)].

Keywords: Fermionic computation; fermionic entanglement; monogamy of entanglement; superselection rules; operational probabilistic theories; local tomography.

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Contents

1. Introduction

In the last three decades, the relation between Fermionic systems and other quantum systems has been throughly investigated from both the computational and the physical point of view. In particular the puzzling anticommuting nature of the Fermionic systems casts a shadow on the possibility of simulating the physical evolution of a bunch of Fermionic systems by means of commuting quantum systems — say *qubits*. This issue was raised by Feynman in 1982,[2] when in his seminal work on physical computation, he wondered about the possibility of simulating Fermions by local quantum systems in interaction — what we would call nowadays a *quantum computer*:

> *Could we imitate every quantum mechanical system which is discrete and has a finite number of degrees of freedom? I know, almost certainly, that we could do that for any quantum mechanical system which involves Bose particles. I'm not sure whether Fermi particles could be described by such a system. So I leave that open.*

The problem is that of encoding the evolution of Fermionic fields onto localized quantum systems. A well-known encoding of N Fermionic systems into N qubits is given by the *Jordan–Wigner transform* (JWT).[3] Such an encoding, based on the identification between the Fock space of N Fermions and the Hilbert space of N qubits, provides a $*$-algebra isomorphism between the Fermionic anticommuting algebra and the commuting algebra of qubits. Such a correspondence has been a valuable instrument in modern solid state physics for solving the one-dimensional XY spin-chains[4,5] and then for the understanding of superconductivity and quantum

Hall effect. Moreover, a *time-adaptive* JWT has been introduced in Ref. 6, which allows to contract Fermionic unitary circuits with the same complexity as for the corresponding spin model. In quantum information science the JWT has been used to extend to the Fermionic case notions as entanglement,[7] entropic area law,[8] and universal computation.[1] More recently, the JWT which originally regards one-dimensional chains of spin-$\frac{1}{2}$ systems, has been generalized to any spin[9] and lattice[10] dimension.

Despite its computational power, the JWT fails to solve completely the issue established by Feynman: physically local Fermionic operations are mapped into nonlocal quantum ones and vice versa. As noticed by many authors, this can lead to ambiguities in defining the partial trace,[11–14] and in assessing the local nature of operations.[15]

Independently on the JWT, the Fermionic systems are usually assumed to obey the *Wigner superselection rule*. Based on the simple argument of the impossibility of discriminating a 2π rotation from the identity,[16,17] this superselection rule corresponds to an inhibition to the superposition rule and forbids superpositions among states with an odd number and an even number of Fermionic excitations. Such a constraint on the admitted states for a set of Fermionic systems avoids the ambiguities connected to the JWT,[7] but it has never been shown to promote the Jordan–Wigner isomorphism to a "physical isomorphism" — i.e. preserving some sort of locality of the Fermionic operations through the encoding.

In this paper, we tackle the issue of retaining locality of Fermionic operations through a qubit simulation in a novel way, namely considering the Fermionic modes as the elementary systems of an *operational probabilistic theory* (OPT). The context of OPTs provides a unified framework for studying and comparing properties of different probabilistic models, such as locality. Well-known examples of OPTs are: (i) quantum theory (QT) (recently axiomatized within the operational framework[18–20]), (ii) the classical information theory,[20] (iii) the box-world[21] and (iv) the real quantum theory (RQT).[22,23] In Sec. 2, we review the operational framework and present the recent results of Ref. 24, where the superselection rule has been formalized in the general context of OPTs.

In Sec. 3, we build up the largest OPT corresponding to the Fermionic computation. We write all possible events (states, transformations, effects) of the theory achieved with the anticommuting algebra of the Fermionic field and assuming operations involving fields on some Fermionic modes to be local on those modes. Locality here is meant in the operational sense, namely operations on systems that are not causally connected must commute. The derivation leads naturally to the *Wigner superselection rule*. Since there is not a unique OPT respecting such a superselection rule, we then look for the largest theory compatible with the locality of Fermionic operations, here denoted Fermionic quantum theory (FQT).

In the second part of the paper (see Sec. 4), we study the operational consequences of superselection. Unlike QT, FQT does not satisfy *local tomography*, i.e. the possibility of discriminating between two nonlocal states using only local

measurements. After proving the correspondence between Fermionic and qubit local operations with classical communication (LOCC), we study the emerging notion of entanglement for Fermionic systems, an issue addressed in Ref. 7 for the first time. Here we will identify nonseparability as the unique notion of entanglement in FQT. Upon defining the Fermionic *entanglement of formation* and *concurrence*, we see that in FQT, there are states with maximal entanglement of formation that are mixed and that Fermionic entanglement does not satisfy *monogamy*, i.e. the limitation on the sharing of entanglement between many parties. Moreover, the notion of maximally entangled state must be replaced with the one of *maximally entangled set* (MES)[25] also in the bipartite case, unlike QT. Interestingly, while in QT, a simple linear criterion for full separability of states is lacking we will see that FQT allows for it.

It is worth mentioning that FQT is only a special example of superselected QT, while the notion of superselection of Ref. 24 allows for many other theories. Among them, we will discuss briefly the case of RQT — which also lacks local tomography[23] and monogamy of entanglement[26] — and the theory with *number superselection* — which only admits superposition of states having the same particle occupation number.

A computational model based on Fermionic systems has already been proposed by Bravyi and Kitaev in Ref. 1. They showed that such a model supports *universal computation* and that it is equivalent to the qubit computational one. The computational model of Ref. 1 is just the FQT with the additional constraint given by the *conservation of parity*; as a consequence the resulting sets of transformations are strictly included in the FQT's ones. In Subsec. 4.4, we compare QT and FQT from the point of view of computational complexity, and exploiting the results of Bravyi and Kitaev[1] (here reviewed), we show the equivalence of the two theories and that even FQT supports universal computation.

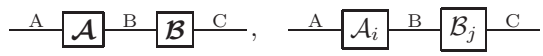
2. Operational Probabilistic Theories

Before starting we need to review the basic definitions and notations for Operational Probabilistic Theories (OPT). For a detailed discussion, see Ref. 19. The fundamental notion in the operational framework is that of *test*, which is the abstract element of the framework corresponding to a (single use) of a physical device. In more details, a test $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_i\}_{i\in\eta}$ describes an elementary operation that usually produces an outcome i belonging to the set η of all the possible outcomes. The readout of the outcome i specifies the occurrence of the physical circumstance identified by the *event* $\mathcal{A}_i$. Tests are also specified by an input and an output label — e.g. A, B — that identify the *system types* (*systems*, for short). The test $\boldsymbol{\mathcal{A}}$ and its building events $\mathcal{A}_i \in \boldsymbol{\mathcal{A}}$ can also be represented in the following pictorial way:

$$\boldsymbol{\mathcal{A}} \equiv \overset{\text{A}}{\text{---}}\boxed{\boldsymbol{\mathcal{A}}}\overset{\text{B}}{\text{---}}\,, \quad \mathcal{A}_i \equiv \overset{\text{A}}{\text{---}}\boxed{\mathcal{A}_i}\overset{\text{B}}{\text{---}}\,.$$

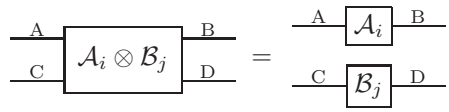
If an event $\mathcal{A}$ belongs to a singleton test $\boldsymbol{\mathcal{A}}$ — i.e. $\boldsymbol{\mathcal{A}} = \{\mathcal{A}\}$ — we say that $\mathcal{A}$ is *deterministic*.

Physical devices can be connected in sequence, as long as the output system type of each device is the same as the input system type of the next one. So do tests: two tests $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_i\}_{i\in\eta}$, $\boldsymbol{\mathcal{B}} = \{\mathcal{B}_j\}_{j\in\chi}$ can be connected in sequence as long as the output wire of the first one in the sequence (say $\boldsymbol{\mathcal{A}}$) is of the same type as that of the input wire of the last one (say $\boldsymbol{\mathcal{B}}$), thus giving the *sequential composition* $\boldsymbol{\mathcal{B}} \circ \boldsymbol{\mathcal{A}} \coloneqq \{\mathcal{B}_j \circ \mathcal{A}_i\}_{(i,j)\in\eta\times\chi}$; pictorially

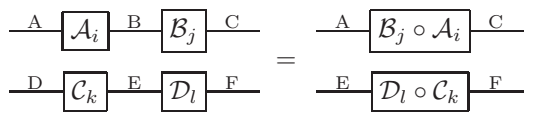


The labels of the input and output systems provide rules for connecting tests in sequences. Notice that the input/output relation has no causal connotation, and it does not entail an underlying "time arrow." As we will see shortly, only in a causal OPT it is possible to understand the input/output relation as a time direction.

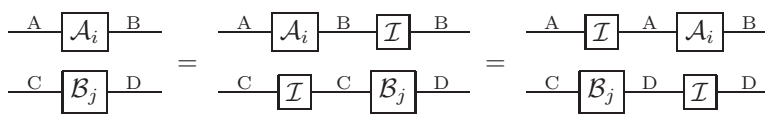
For every system A, there exists a unique singleton test $\{\mathcal{I}_\mathrm{A}\}$ such that $\mathcal{I}_\mathrm{B} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{I}_\mathrm{A}$ for every event $\mathcal{A}$ with input A and output B. For every couple of systems A, B, we can form the composite system C $\coloneqq$ AB, on which we can perform tests $\boldsymbol{\mathcal{A}} \otimes \boldsymbol{\mathcal{B}}$ with events $\mathcal{A}_i \otimes \mathcal{B}_j$ in *parallel composition*, represented as follows



and satisfying the condition



in formulae $(\mathcal{B}_j \otimes \mathcal{D}_l) \circ (\mathcal{A}_i \otimes \mathcal{C}_k) = (\mathcal{B}_j \circ \mathcal{A}_i) \otimes (\mathcal{D}_l \circ \mathcal{C}_k)$. Notice that the symbol $\otimes$ is just a formal way to identify the parallel composition among tests (and events), and it is not the usual tensor product of linear spaces. Moreover, the previous property implies commutation of tests on different systems, i.e. for every couple of events $\mathcal{A}_i$, $\mathcal{B}_j$ it is



(1)

There is a special system type I, the *trivial system*, such that AI = IA = A. The tests with input system I and output A are called *preparation-tests* of A, while the tests with input system A and output I are called *observation-tests* of A. Preparation-events of A are denoted by the symbols $|\rho)_\mathrm{A}$ or $\overset{}{(\rho|}^{\mathrm{A}}$, and observation-events by $(c|_\mathrm{A}$ or $\overset{\mathrm{A}}{|c)}$.

An arbitrary complex test obtained by parallel and/or sequential composition of "elementary tests" is called *circuit*. An *operational theory* is a collection of systems closed under composition, and a collection of tests closed under parallel and

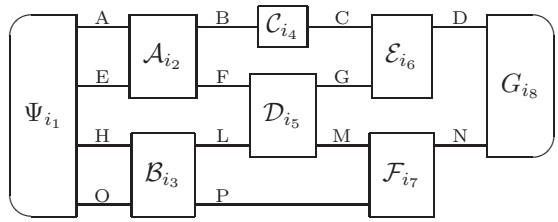


Fig. 1. The closed circuit in the figure represents the joint probability $\Pr[i_1, i_2, \ldots, i_8|\boldsymbol{\Psi}, \boldsymbol{\mathcal{A}}, \ldots, \boldsymbol{G}]$ of outcomes $i_1, i_2, \ldots, i_8$ conditioned by the choice of tests $\boldsymbol{\Psi}, \boldsymbol{\mathcal{A}}, \ldots, \boldsymbol{G}$. Since the output of the event $\mathcal{A}_{i_2}$ is connected to the input of the event $\mathcal{D}_{i_5}$ through the system F, the event $\mathcal{A}_{i_2}$ immediately precedes the event $\mathcal{D}_{i_5}$ ($\mathcal{A}_{i_2} \prec_1 \mathcal{D}_{i_5}$). Similarly, since between the event $\mathcal{B}_{i_3}$ and the event $\mathcal{E}_{i_6}$, there is $\mathcal{D}_{i_5}$ such that $\mathcal{B}_{i_3} \prec_1 \mathcal{D}_{i_5} \prec_1 \mathcal{E}_{i_6}$, the event $\mathcal{B}_{i_3}$ precedes the event $\mathcal{E}_{i_6}$ ($\mathcal{B}_{i_3} \prec \mathcal{E}_{i_6}$). If the closed circuit of the figure belongs to a causal theory, we have that the marginal probability of the event $\mathcal{D}_{i_5} \in \boldsymbol{\mathcal{D}}$ cannot depend on the choice of any test $\boldsymbol{\mathcal{X}}$ such that $\boldsymbol{\mathcal{X}} \not\prec \boldsymbol{\mathcal{D}}$, i.e. $\Pr[i_5|\boldsymbol{\Psi}, \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}, \boldsymbol{G}] = \Pr[i_5|\boldsymbol{\Psi}, \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}]$.

sequential composition, i.e. every circuit belongs to the theory. Given a circuit, we say that an event $\mathcal{H}$ *is immediately connected to the input of* $\mathcal{K}$, and write $\mathcal{H} \prec_1 \mathcal{K}$, if there is an output system of $\mathcal{H}$ that is connected with an input system of $\mathcal{K}$; e.g. in Fig. 1, $\mathcal{A}_{i_2} \prec_1 \mathcal{D}_{i_5}$. Moreover, we can introduce the transitive closure $\prec$ of the relation $\prec_1$, and we say that $\mathcal{H}$ *is connected to the input of* $\mathcal{K}$ if $\mathcal{H} \prec \mathcal{K}$ (e.g. $\mathcal{B}_{i_3} \prec \mathcal{E}_{i_6}$). The two relations $\prec_1$ and $\prec$ can be extended to tests trivially.

A circuit is *closed* if its overall input and output systems are the trivial ones. Figure 1 is an example of a closed circuit. An OPT is an operational theory where every closed circuit represents a probability distribution; e.g. the closed circuit in Fig. 1 represents the probability $\Pr[i_1, i_2, \ldots, i_8|\boldsymbol{\Psi}, \boldsymbol{\mathcal{A}}, \ldots, \boldsymbol{G}]$ of outcomes $i_1, i_2, \ldots, i_8$ conditioned by the choice of tests $\boldsymbol{\Psi}, \boldsymbol{\mathcal{A}}, \ldots, \boldsymbol{G}$. In probabilistic theories, we can quotient the set of preparation-events of A by the equivalence relation $|\rho)_{\mathrm{A}} \sim |\sigma)_{\mathrm{A}} \Leftrightarrow$ *the probability of preparing* $|\rho)_{\mathrm{A}}$ *and measuring* $(c|_{\mathrm{A}}$ *is the same as that of preparing* $|\sigma)_{\mathrm{A}}$ *and measuring* $(c|_{\mathrm{A}}$ *for every observation-event* $(c|_{\mathrm{A}}$ *of* A (and similarly for observation-events). The equivalence classes of preparation-events and observation-events of A will be denoted by the same symbols as their elements $|\rho)_{\mathrm{A}}$ and $(c|_{\mathrm{A}}$, respectively, and will be called *state* $|\rho)_{\mathrm{A}}$ for system A, and *effect* $(c|_{\mathrm{A}}$ for system A. For every system A, we will denote by St(A), Eff(A) the sets of states and effects, respectively. States are real-valued functionals over the effects, and vice versa; thus they can be embedded respectively in the real vector spaces $\mathrm{St}_{\mathbb{R}}(\mathrm{A})$, $\mathrm{Eff}_{\mathbb{R}}(\mathrm{A})$. $\mathrm{St}_{\mathbb{R}}(\mathrm{A})$ is the dual space of $\mathrm{Eff}_{\mathbb{R}}(\mathrm{A})$, and vice versa since the dimension $D_{\mathrm{A}} := \dim \mathrm{Eff}_{\mathbb{R}}(\mathrm{A})$ is assumed to be finite. The application of the effect $(c_i|_{\mathrm{A}}$ on the state $|\rho)_{\mathrm{A}}$ is written as $(c_i|\rho)_{\mathrm{A}}$ and corresponds to the closed circuit $\boxed{\rho}\!\!-\!\!\overset{\mathrm{A}}{}\!\!-\!\!\boxed{c_i}$, denoting therefore the probability of the ith outcome of the observation-test $\boldsymbol{c} = \{(c_i|_{\mathrm{A}}\}_{i\in\eta}$ performed on the state ρ of system A, i.e. $(c_i|\rho)_{\mathrm{A}} := \Pr[c_i|\boldsymbol{\rho}, \boldsymbol{c}]$.

Any event with input system A and output system B induces a collection of linear mappings from $\mathrm{St}_{\mathbb{R}}(\mathrm{AC})$ to $\mathrm{St}_{\mathbb{R}}(\mathrm{BC})$, for varying system C. Such a collection

is called *transformation* from A to B. The set of transformations from A to B will be denoted by $\mathrm{Transf}(\mathrm{A},\mathrm{B})$, and its linear span by $\mathrm{Transf}_{\mathbb{R}}(\mathrm{A},\mathrm{B})$. The symbols $\mathcal{A}$ and  denoting the event $\mathcal{A}$ will be also used to represent the corresponding transformation.

One usually requires that an experimentalist can randomize the choice of the devices in an experiment with arbitrary probabilities. This implies that, for every system, all the set of states, effects, and transformations of an OPT are convex. The extremal points of the convex set of the deterministic states (and similarly for effects and transformations) correspond to the so-called *atomic states*, also known as *pure states* since they cannot be seen as convex combinations of other deterministic states.

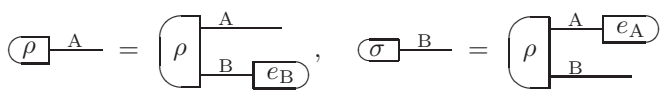
An OPT can satisfy many different properties;[20] among the most important ones there is the property of *causality*.

Definition 2.1. An OPT is *causal* if for every preparation-test $\boldsymbol{\rho} = \{|\rho_i)\}_{i\in\eta}$ and any two observation-tests $\boldsymbol{a} = \{(a_j|\}_{j\in\chi}$ and $\boldsymbol{b} = \{(b_j|\}_{j\in\xi}$, one has $\sum_{j\in\chi}(a_j|\rho_i) = \sum_{k\in\xi}(b_k|\rho_i)$, $\forall i \in \eta$, namely the probability of the preparation is independent of the choice of observation.

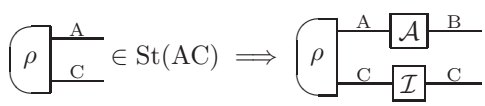
Causality is equivalent to the so-called *no-backward signaling*,[27] namely within a closed circuit, the marginal probabilities of the outcomes of an arbitrary test $\boldsymbol{\mathcal{H}}$ do not depend on the choice of any test $\boldsymbol{\mathcal{K}} \not\prec \boldsymbol{\mathcal{H}}$. For example, in the circuit of Fig. 1 causality implies

$$\Pr[i_5|\boldsymbol{\Psi},\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}},\boldsymbol{\mathcal{C}},\boldsymbol{\mathcal{D}},\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{F}},\boldsymbol{G}] = \Pr[i_5|\boldsymbol{\Psi},\boldsymbol{\mathcal{A}},\boldsymbol{\mathcal{B}}]\,.$$

The present notion of causality is a rigorous definition in the operational framework of the so-called *Einstein causality*. Indeed, a corollary of *no-backward signaling* is the *no-signaling without interaction*.[19] In an OPT, a condition equivalent to causality is that of uniqueness of the deterministic effect[19] (usually denoted by $(e|$). Notice that given a bipartite state $|\rho)_{\mathrm{AB}}$, the deterministic effects $(e|_{\mathrm{B}}$ and $(e|_{\mathrm{A}}$ allow one to evaluate the marginal states (e.g. partial trace in QT) on the component systems A and B



Another property is the so-called *no-restriction hypothesis*. We say that a linear map $\mathcal{A} \in \mathrm{Transf}_{\mathbb{R}}(\mathrm{A},\mathrm{B})$ is *admissible* if it locally preserves the set of states $\mathrm{St}(\mathrm{AC})$ for every ancillary system C; namely



The no-restriction hypothesis requires that every admissible map in $\mathrm{Transf}_{\mathbb{R}}(\mathrm{A},\mathrm{B})$ actually belongs to $\mathrm{Transf}(\mathrm{A},\mathrm{B})$.[a] Notice that an OPT satisfying the no-restriction hypothesis is completely determined by its systems and the respective set of states, since even the effects — being particular kind of transformations — are all the admissible ones. We can therefore say that a no-restricted OPT is simply the collection $\Theta \coloneqq \{(\mathrm{X},\mathrm{St}(\mathrm{X}))\}_{\mathrm{X}\in\eta}$ for varying system X.

2.1. *Local, bilocal, . . . , n-local tomography*

A common assumption in the literature of probabilistic theories is the so-called *local tomography* (also called by some authors *local discriminability* or *local distinguishability*); namely the possibility of distinguishing two different bipartite states, by means of local devices.

Definition 2.2. A theory enjoys local tomography if for any $|\rho)$, $|\sigma) \in \mathrm{St}(\mathrm{AB})$, we have

$$|\rho) \neq |\sigma) \implies \exists (a| \in \mathrm{Eff}(\mathrm{A})\,,$$

$$(b| \in \mathrm{Eff}(\mathrm{B}) \text{ such that } \begin{array}{|l} \rho \\ \end{array}\!\!\begin{array}{l} \mathrm{A}\ \boxed{a} \\ \mathrm{B}\ \boxed{b} \end{array} \neq \begin{array}{|l} \sigma \\ \end{array}\!\!\begin{array}{l} \mathrm{A}\ \boxed{a} \\ \mathrm{B}\ \boxed{b} \end{array}\,.$$

An OPT with local tomography allows to perform tomography on multipartite states with only local measurements. Indeed, in such a scenario every bipartite effect $(c|_{\mathrm{AB}}$ can be written as linear combination of product effects, therefore every probability $(c|\rho)_{\mathrm{AB}}$ can be computed as a linear combination of the probabilities $((a|_{\mathrm{A}} \otimes (b|_{\mathrm{B}})|\rho)_{\mathrm{AB}}$ arising from a finite set of product effects. In other words, we have the property that the linear space of effects of a composite system is actually the tensor product of the linear spaces of effects of the component systems, i.e. $\mathrm{Eff}_{\mathbb{R}}(\mathrm{AB}) \equiv \mathrm{Eff}_{\mathbb{R}}(\mathrm{A}) \otimes \mathrm{Eff}_{\mathbb{R}}(\mathrm{B})$. Since $\mathrm{St}_{\mathbb{R}}(\mathrm{AB}) = \mathrm{Eff}_{\mathbb{R}}(\mathrm{AB})^*$, we have that the same result holds also for the linear space of states. Thus, in a local-tomographic OPT the parallel composition of two states (effects) denoted by the symbol $\otimes$ can be in fact understood as a tensor product, moreover the following relation between the dimension of the set of states/effects holds: $D_{\mathrm{AB}} = D_{\mathrm{A}} D_{\mathrm{B}}$.

Remark 2.1. An important consequence of local tomography is that a transformation $\mathcal{T} \in \mathrm{Transf}(\mathrm{A},\mathrm{B})$ is completely specified by its action on $\mathrm{St}(\mathrm{A})$:[19]

$$\mathcal{C}|\rho) = \mathcal{C}'|\rho) \quad \forall |\rho) \in \mathrm{St}(\mathrm{A}) \Rightarrow \mathcal{C} = \mathcal{C}'\,.$$

One can imagine to relax the property of local tomography in many different ways; the most general scenario is given by the *n-local tomography*.[23] First, we

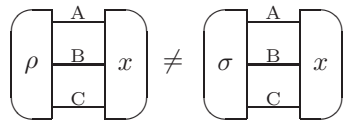
[a]In previous literature,[19] the same nomenclature has been used for a different concept: for every system A, the convex cone generated by Eff(A) coincides with the dual convex cone generated by the set of states St(A).

define an effect to be *n-local*, if it can be written as a conic combination of tensor products of effects that are at most n-partite.

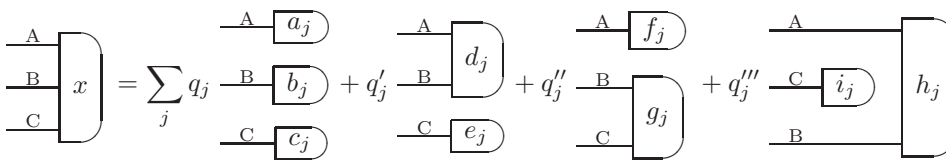
Definition 2.3. A theory enjoys n-local tomography if whenever two states $|\rho)$, $|\sigma)$ are different, there is a n-local effect $(a|$ such that $(a|\rho) \neq (a|\sigma)$.

Clearly, local tomography is the particular case of n-local tomography with $n = 1$. Given a n-local-tomographic theory with $n > 1$, for an arbitrary bipartite system AB one has $D_{\mathrm{AB}} \geq D_{\mathrm{A}} D_{\mathrm{B}}$, since in general $\mathrm{St}_{\mathbb{R}}(\mathrm{AB}) = \mathrm{St}_{\mathbb{R}}(\mathrm{A}) \otimes \mathrm{St}_{\mathbb{R}}(\mathrm{B}) \oplus \mathrm{St}^{\mathrm{NL}}_{\mathbb{R}}(\mathrm{AB})$, where $\mathrm{St}^{\mathrm{NL}}_{\mathbb{R}}(\mathrm{AB}) \coloneqq (\mathrm{St}_{\mathbb{R}}(\mathrm{A}) \otimes \mathrm{St}_{\mathbb{R}}(\mathrm{B}))^{\perp}$ is the subspace where the nonlocal components of the bipartite states live. By definition, a n-local-tomographic theory is also $(n+1)$-local-tomographic, since a n-local effect is also $(n+1)$-local. We are interested in OPTs that are *strictly n-local-tomographic*, namely n-local-tomographic OPTs that are not $(n-1)$-local-tomographic.

Another case already studied in literature is *bilocal tomography*,[23] namely two-local tomography. In particular, for such a case we have that for every couple of different tripartite states $|\rho), |\sigma) \in \mathrm{St}(\mathrm{ABC})$ there exist a two-local effect $(x| \in \mathrm{Eff}(\mathrm{ABC})$ such that



Notice that, since $(x|$ is two-local, it can be written as the following conic combination



with q_j, q'_j, q''_j, $q'''_j \geq 0$ and $(d_j| \in \mathrm{Eff}^{\mathrm{NL}}_{\mathbb{R}}(\mathrm{AB})$ (and similarly for $(g_j|$, $(h_j|)$. For a bilocal-tomographic theory, we have therefore

$$D_{\mathrm{AB}} \geq D_{\mathrm{A}} D_{\mathrm{B}}\,, \tag{2}$$

$$D_{\mathrm{ABC}} \leq D_{\mathrm{A}} D_{\mathrm{B}} D_{\mathrm{C}} + D_{\mathrm{A}} \tilde{D}_{\mathrm{BC}} + D_{\mathrm{B}} \tilde{D}_{\mathrm{AC}} + D_{\mathrm{C}} \tilde{D}_{\mathrm{BC}}\,, \tag{3}$$

where $\dim(\mathrm{Eff}^{\mathrm{NL}}_{\mathbb{R}}(\mathrm{AB})) =: \tilde{D}_{\mathrm{AB}} = D_{\mathrm{AB}} - D_{\mathrm{A}} D_{\mathrm{B}}$. A strictly bilocal-tomographic theory has the first bound tight, moreover if the upper bound is saturated, we say that the OPT is *maximally bilocal-tomographic*, since it requires all the two-local effects to discriminate multipartite states.

2.2. *Superselected operational probabilistic theories*

A superselection rule σ on a theory Θ corresponds to a linear section of all sets of transformations for each multipartite system, which under the no-restriction

hypothesis reduces to sectioning linearly just the sets of states. We can give the following formal definition of superselection rule:

Definition 2.4. A superselection rule σ is a map from an OPT Θ to another OPT $\bar{\Theta}$,

$$\sigma : \quad \Theta \to \bar{\Theta}\,, \quad (\mathrm{A}, \mathrm{St}(\mathrm{A})) \mapsto \sigma((\mathrm{A}, \mathrm{St}(\mathrm{A}))) =: (\bar{\mathrm{A}}, \mathrm{St}(\bar{\mathrm{A}}))\,,$$

such that, for every system A, $\mathrm{St}(\bar{\mathrm{A}})$ is a linear section of $\mathrm{St}(\mathrm{A})$, i.e.

$$\mathrm{St}(\bar{\mathrm{A}}) \coloneqq \{\rho \in \mathrm{St}(\mathrm{A}) \mid (s_i^\sigma|\rho) = 0,\ i = 1, \ldots, V_{\mathrm{A}}^\sigma\}\,,$$

where $(s_i^\sigma| \in \mathrm{Eff}_{\mathbb{R}}(\mathrm{A})$ are V_{A}^σ linear independent constraints.

For consistency, the superselection map σ must commute with system composition, forcing the definition of composition for the constrained theory as $\sigma(\mathrm{A})\sigma(\mathrm{B}) \coloneqq \sigma(\mathrm{AB})$. Notice that, being linear σ preserves convexity of the theory, i.e. all the sets $\mathrm{St}(\bar{\mathrm{A}})$, $\mathrm{Eff}(\bar{\mathrm{A}})$, $\mathrm{Transf}(\bar{\mathrm{A}}, \bar{\mathrm{B}})$, for every system $\bar{\mathrm{A}}$, $\bar{\mathrm{B}}$ of the constrained theory are convex. For instance, this means that in a QT with superselection, states from different sectors cannot be superimposed, but can be mixed. From the definition, it follows immediately $\mathrm{St}(\bar{\mathrm{A}}) \subseteq \mathrm{St}(\mathrm{A})$, $\mathrm{Eff}(\bar{\mathrm{A}}) \subseteq \mathrm{Eff}(\mathrm{A})$, and $D_{\bar{\mathrm{A}}} = D_{\mathrm{A}} - V_{\mathrm{A}}^\sigma$.

The number V_{A}^σ of linearly independent constraints on a system A cannot be arbitrary, for example consider the trivial bound $V_{\mathrm{A}}^\sigma \leq D_{\mathrm{A}}$. In fact, one has other more interesting bounds due to the system composition.

Proposition 2.1. *Let $\bar{\Theta}$ be the superselected* OPT *build from the* OPT Θ *by means of the superselection map σ. Then the following bounds hold*:

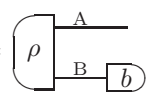
$$V_{\mathrm{AB}}^\sigma \geq D_{\mathrm{A}} V_{\mathrm{B}}^\sigma + D_{\mathrm{B}} V_{\mathrm{A}}^\sigma - 2 V_{\mathrm{A}}^\sigma V_{\mathrm{B}}^\sigma\,, \tag{4}$$

$$V_{\mathrm{AB}}^\sigma \leq D_{\mathrm{A}} V_{\mathrm{B}}^\sigma + D_{\mathrm{B}} V_{\mathrm{A}}^\sigma - V_{\mathrm{A}}^\sigma V_{\mathrm{B}}^\sigma + D_{\mathrm{AB}} - D_{\mathrm{A}} D_{\mathrm{B}}\,. \tag{5}$$

Proof. The upper bound of Eq. (5) is easily proven upon noticing that for an arbitrary OPT it always happens that $\mathrm{St}_{\mathbb{R}}(\mathrm{AB}) \supseteq \mathrm{St}_{\mathbb{R}}(\mathrm{A}) \otimes \mathrm{St}_{\mathbb{R}}(\mathrm{B})$, and thus $D_{\mathrm{AB}} \geq D_{\mathrm{A}} D_{\mathrm{B}}$. Hence, one has $D_{\bar{\mathrm{A}}\bar{\mathrm{B}}} \geq D_{\bar{\mathrm{A}}} D_{\bar{\mathrm{B}}}$, and using $D_{\bar{\mathrm{B}}} = D_{\mathrm{B}} - V_{\mathrm{B}}^\sigma$ and $D_{\bar{\mathrm{B}}} = D_{\mathrm{B}} - V_{\mathrm{B}}^\sigma$, we get Eq. (5).

The lower bound of Eq. (4) is proved by showing that all the local constraints on the component systems A and B are also constraints of the composite system AB, namely for any $(b| \in \mathrm{Eff}(\bar{\mathrm{B}})$ and any $i = 1, \ldots, V_{\mathrm{A}}^\sigma$, one has that $(s_i^\sigma| \otimes (b| \in \mathrm{Eff}_{\mathbb{R}}(\mathrm{AB})$ is a constraint for $\bar{\mathrm{A}}\bar{\mathrm{B}}$. Indeed, suppose by contradiction that $(s_i^\sigma \otimes b|\rho) \neq 0$ for some $\rho \in \mathrm{St}(\bar{\mathrm{A}}\bar{\mathrm{B}})$, and $i \in \{1, \ldots, V_{\mathrm{AB}}^\sigma\}$. Since

$$|\rho_b) \coloneqq$$



is a valid state for the system $\bar{\mathrm{A}}$, we have $(s_i^\sigma|\rho_b) \neq 0$ against the hypothesis. The same argument holds reversing the roles of the subsystems $\bar{\mathrm{A}}$ and $\bar{\mathrm{B}}$, so we conclude

that $V^\sigma_{\bar{\mathrm{A}}\mathrm{B}}$ shall be at least $D_{\bar{\mathrm{A}}}V^\sigma_{\mathrm{B}} + D_{\bar{\mathrm{B}}}V^\sigma_{\mathrm{A}}$, which gives the lower bound of Eq. (4) using $D_{\bar{\mathrm{B}}} = D_{\mathrm{B}} - V^\sigma_{\mathrm{B}}$ and $D_{\bar{\mathrm{B}}} = D_{\mathrm{B}} - V^\sigma_{\mathrm{B}}$. □

Given an OPT Θ, one can build "bottom-up" a superselected theory $\bar{\Theta}$ by defining the constraints only for the elementary systems (the ones that cannot be obtained by composition of other systems) and taking the minimal number of linear constraints (4) on the composite ones. We call such superselected OPTs *minimally superselected*.

Definition 2.5. A superselected OPT is minimally superselected if it saturates the lower bound of Eq. (4).

In a minimally superselected OPT the only constraints on bipartite systems are $(r^\sigma_i| \otimes (b|$ and $(a| \otimes (s^\sigma_j|$, with $a \in \mathrm{Eff}(\bar{\mathrm{A}})$, $b \in \mathrm{Eff}(\bar{\mathrm{B}})$, $r^\sigma_i \in \mathrm{Eff}_{\mathbb{R}}(\mathrm{A})$, $r^\sigma_j \in \mathrm{Eff}_{\mathbb{R}}(\mathrm{B})$.

On the other hand, the saturation of the upper bound of Eq. (5) leads to a *maximally superselected* OPT:

Definition 2.6. A superselected OPT is maximally superselected if it saturates the upper bound of Eq. (5).

Since enforcing superselection constraints on a OPT leads to a change of the structure of the set of states, effects, and transformations, we shall expect a change also in the properties satisfied by the resulting theory. Indeed, while a causal theory retains causality once superselected, the converse is not true. Moreover, a local-tomographic theory is in general no more local-tomographic upon superselection, as the following proposition shows.

Proposition 2.2. *Let $\bar{\Theta}$ be a superselection of a local-tomographic theory Θ. Then*:

(i) *Minimal superselection $\Rightarrow$ $\bar{\Theta}$ maximally bilocal-tomographic*,
(ii) *Maximal superselection $\Rightarrow$ $\bar{\Theta}$ local-tomographic.*

Proof. Let us prove the first implication. The superselected theory $\bar{\Theta}$ is maximally bilocal-tomographic if it saturates the bound of Eq. (3), namely

$$D_{\bar{\mathrm{A}}\bar{\mathrm{B}}\bar{\mathrm{C}}} = D_{\bar{\mathrm{A}}}D_{\bar{\mathrm{B}}}D_{\bar{\mathrm{C}}} + D_{\bar{\mathrm{A}}}\tilde{D}_{\bar{\mathrm{B}}\bar{\mathrm{C}}} + D_{\bar{\mathrm{B}}}\tilde{D}_{\bar{\mathrm{A}}\bar{\mathrm{C}}} + D_{\bar{\mathrm{C}}}\tilde{D}_{\bar{\mathrm{B}}\bar{\mathrm{C}}}\,.$$

We prove this equality evaluating the left-hand side (LHS) and the right-hand side (RHS) of the equation and enforcing the minimal superselection given by the lower bound of Eq. (4)

$$V^\sigma_{\bar{\mathrm{A}}\mathrm{B}} = D_{\mathrm{A}}V^\sigma_{\mathrm{B}} + D_{\mathrm{B}}V^\sigma_{\mathrm{A}} - 2V^\sigma_{\mathrm{A}}V^\sigma_{\mathrm{B}}\,. \tag{6}$$

LHS: we have $D_{\overline{\mathrm{ABC}}} = D_{\mathrm{ABC}} - V^\sigma_{\mathrm{ABC}}$; taking the partition $\bar{\mathrm{A}}\bar{\mathrm{B}}\bar{\mathrm{C}} = \bar{\mathrm{A}}(\bar{\mathrm{B}}\bar{\mathrm{C}})$, the requirement of minimal superselection gives

$$D_{\bar{\mathrm{A}}\bar{\mathrm{B}}\bar{\mathrm{C}}} = D_{\mathrm{ABC}} - \left(D_{\mathrm{A}}V^\sigma_{\mathrm{BC}} + D_{\mathrm{BC}}V^\sigma_{\mathrm{A}} - 2V^\sigma_{\mathrm{A}}V^\sigma_{\mathrm{BC}}\right).$$

Using again the minimal superselection requirement, we expand V^{σ}_{BC} and V^{σ}_{BC} getting

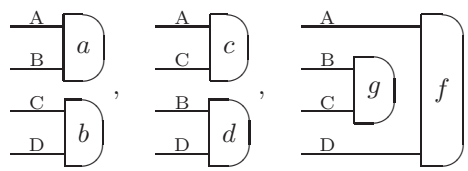
$$D_{\bar{\mathrm{A}}\bar{\mathrm{B}}\bar{\mathrm{C}}} = D_{\mathrm{A}}D_{\mathrm{B}}D_{\mathrm{C}} - \big(D_{\mathrm{A}}D_{\mathrm{B}}V^{\sigma}_{\mathrm{C}} + D_{\mathrm{A}}D_{\mathrm{C}}V^{\sigma}_{\mathrm{B}} + D_{\mathrm{B}}D_{\mathrm{C}}V^{\sigma}_{\mathrm{A}}\big) + 2\big(D_{\mathrm{A}}V^{\sigma}_{\mathrm{B}}V^{\sigma}_{\mathrm{C}} + D_{\mathrm{A}}V^{\sigma}_{\mathrm{C}}V^{\sigma}_{\mathrm{B}} + D_{\mathrm{B}}V^{\sigma}_{\mathrm{C}}V^{\sigma}_{\mathrm{A}}\big) - 4V^{\sigma}_{\mathrm{A}}V^{\sigma}_{\mathrm{B}}V^{\sigma}_{\mathrm{C}}\,,$$

where we used the identity $D_{\mathrm{AB}} = D_{\mathrm{A}}D_{\mathrm{B}}$ since the OPT Θ is local-tomographic.

RHS: We use the identities $\tilde{D}_{\bar{\mathrm{X}}\bar{\mathrm{Y}}} = D_{\bar{\mathrm{X}}\bar{\mathrm{Y}}} - D_{\bar{\mathrm{X}}}D_{\bar{\mathrm{Y}}}$, $D_{\bar{\mathrm{X}}\bar{\mathrm{Y}}} = D_{\mathrm{XY}} - V^{\sigma}_{\mathrm{XY}}$, and the requirement of minimal superselection of Eq. (6). Finally, the local tomography condition $D_{\mathrm{XY}} = D_{\mathrm{X}}D_{\mathrm{Y}}$ for the OPT Θ gives the same expression of the LHS.

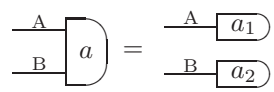
Let us now prove the second implication, namely that equality in Eq. (5) implies $D_{\bar{\mathrm{A}}\bar{\mathrm{B}}} = D_{\bar{\mathrm{A}}}D_{\bar{\mathrm{B}}}$. Expanding $D_{\bar{\mathrm{A}}\bar{\mathrm{B}}}$ as $D_{\mathrm{AB}} - V^{\sigma}_{\mathrm{AB}}$, using Eq. (5), and remembering that the OPT Θ is local-tomographic, we get $D_{\bar{\mathrm{A}}\bar{\mathrm{B}}} = D_{\mathrm{A}}D_{\mathrm{B}} - D_{\mathrm{A}}V_{\mathrm{B}} - D_{\mathrm{B}}V_{\mathrm{A}} + V^{\sigma}_{\mathrm{A}}V^{\sigma}_{\mathrm{B}} = (D_{\mathrm{A}} - V_{\mathrm{A}})(D_{\mathrm{B}} - V_{\mathrm{B}}) = D_{\bar{\mathrm{A}}}D_{\bar{\mathrm{B}}}$. □

In general, in a bilocal-tomographic theory, two different states ρ and σ of the four-partite system ABCD can be discriminated by the following classes of effects



(7)

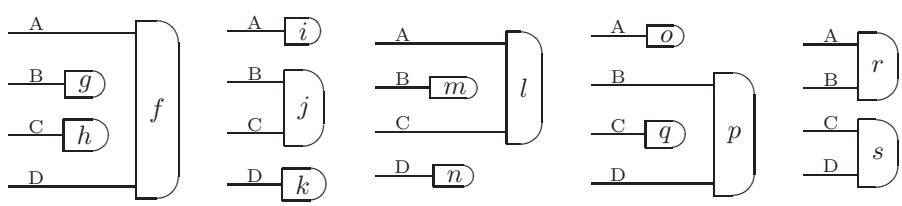
where $(a|$, $(b|$, $(c|$, $(d|$, or $(f|$ can also be local, e.g.



(8)

A remarkable feature of maximally bilocal-tomographic theories is given by the following theorem, which reduces the number of the above classes.

Theorem 2.1. *Let Θ be a maximally bilocal-tomographic theory. Then, for any four-partite system* ABCD, *the following classes of effects is sufficient in order to discriminate two different states ρ and σ*



(i) (ii) (iii) (iv) (v)

Proof. First, notice that every class (i), (ii), (iii), (iv), (v) spans a linear space of effects of dimension

$$\underset{\text{(i)}}{D_{\mathrm{AD}}D_{\mathrm{B}}D_{\mathrm{C}}} \qquad \underset{\text{(ii)}}{D_{\mathrm{A}}D_{\mathrm{BC}}D_{\mathrm{D}}} \qquad \underset{\text{(iii)}}{D_{\mathrm{AC}}D_{\mathrm{B}}D_{\mathrm{D}}} \qquad \underset{\text{(iv)}}{D_{\mathrm{A}}D_{\mathrm{BD}}D_{\mathrm{C}}} \qquad \underset{\text{(v)}}{D_{\mathrm{AB}}D_{\mathrm{CD}}} \,.$$

All such linear spaces have a common linear subspace identified by the local effects belonging to the class $(a|_{\mathrm{A}} \otimes (b|_{\mathrm{B}} \otimes (c|_{\mathrm{C}} \otimes (d|_{\mathrm{D}}$. Having this subspace dimension $D_{\mathrm{A}}D_{\mathrm{B}}D_{\mathrm{C}}D_{\mathrm{D}}$, we have that the span of all the classes (i)–(v) taken together is

$$\begin{aligned} &D_{\mathrm{AD}}D_{\mathrm{B}}D_{\mathrm{C}} + D_{\mathrm{A}}D_{\mathrm{BC}}D_{\mathrm{D}} + D_{\mathrm{AC}}D_{\mathrm{B}}D_{\mathrm{D}} + D_{\mathrm{A}}D_{\mathrm{BD}}D_{\mathrm{C}} \\ &\quad + D_{\mathrm{AB}}D_{\mathrm{CD}} - 4D_{\mathrm{A}}D_{\mathrm{B}}D_{\mathrm{C}}D_{\mathrm{D}} \,. \end{aligned} \tag{9}$$

We recall here the definition of maximally bilocal-tomographic theory, given in terms of the following dimensional relation

$$D_{\mathrm{ABC}} = D_{\mathrm{A}}D_{\mathrm{B}}D_{\mathrm{C}} + D_{\mathrm{A}}\tilde{D}_{\mathrm{BC}} + D_{\mathrm{B}}\tilde{D}_{\mathrm{AC}} + D_{\mathrm{C}}\tilde{D}_{\mathrm{AB}} \,. \tag{10}$$

Let consider the tripartition (AB)CD. Applying the property of maximal bilocal-tomography of Eq. (10), we have

$$D_{\mathrm{ABCD}} = D_{(\mathrm{AB})\mathrm{CD}} = D_{\mathrm{AB}}D_{\mathrm{C}}D_{\mathrm{D}} + D_{\mathrm{AB}}\tilde{D}_{\mathrm{CD}} + D_{\mathrm{C}}\tilde{D}_{(\mathrm{AB})\mathrm{D}} + D_{\mathrm{D}}\tilde{D}_{(\mathrm{AB})\mathrm{C}} \,.$$

By definition, we have $D_{\mathrm{AB}} = D_{\mathrm{A}}D_{\mathrm{B}} + \tilde{D}_{\mathrm{AB}}$, and by Eq. (10)

$$\tilde{D}_{(\mathrm{AB})\mathrm{C}} = D_{\mathrm{ABC}} - D_{\mathrm{AB}}D_{\mathrm{C}} = D_{\mathrm{A}}\tilde{D}_{\mathrm{BC}} + D_{\mathrm{B}}\tilde{D}_{\mathrm{AC}} \,.$$

Thus, we conclude

$$\begin{aligned} D_{\mathrm{ABCD}} &= D_{\mathrm{AB}}\big(D_{\mathrm{C}}D_{\mathrm{D}} + \tilde{D}_{\mathrm{CD}}\big) + D_{\mathrm{C}}\tilde{D}_{(\mathrm{AB})\mathrm{D}} + D_{\mathrm{D}}\tilde{D}_{(\mathrm{AB})\mathrm{C}} \\ &= D_{\mathrm{AB}}D_{\mathrm{CD}} + D_{\mathrm{A}}D_{\mathrm{BD}}D_{\mathrm{C}} + D_{\mathrm{AD}}D_{\mathrm{B}}D_{\mathrm{C}} \\ &\quad + D_{\mathrm{A}}D_{\mathrm{BC}}D_{\mathrm{D}} + D_{\mathrm{AC}}D_{\mathrm{B}}D_{\mathrm{D}} - 4D_{\mathrm{A}}D_{\mathrm{B}}D_{\mathrm{C}}D_{\mathrm{D}} \,, \end{aligned}$$

namely the dimension given by Eq. (9). □

As we will discuss later, the last theorem has important consequences on the notion of entanglement for Fermionic computation.

2.3. *Quantum theory as an operational probabilistic theory*

It has been shown recently in Ref. 20 that QT (in finite dimension) can be regarded as an OPT satisfying six properties: the already mentioned causality and local tomography, along with *perfect distinguishability*, *pure conditioning*, *ideal compression*, and *purification*. Thus, all the operational notions introduced in Sec. 2, can be specified in the case of QT. In details, a quantum system A is specified by a Hilbert space $\mathscr{H}_{\mathrm{A}}$ with $\dim \mathscr{H}_{\mathrm{A}} = d_{\mathrm{A}} < +\infty$; so $\mathscr{H}_{\mathrm{A}} = \mathbb{C}^{d_{\mathrm{A}}}$. The deterministic states

(usually called normalized states) of the system A are the positive semidefinite operators over $\mathscr{H}_{\rm A}$ with trace 1. On the other hand, the linear set of states $\mathrm{St}_{\mathbb{R}}(\mathrm{A})$ is the whole space $\mathrm{Herm}(\mathscr{H}_{\rm A})$ of Hermitian operators over $\mathscr{H}_{\rm A}$ with dimension $D_{\rm A} = d_{\rm A}^2$. A nondeterministic preparation test $\boldsymbol{\rho} = \{\rho_i\}_{i\in\eta}$ is a collection of deterministic states $\{\tilde{\rho}_i\}$ along with a collection of probabilities $\{p_i\}_{i\in\eta}$ such that $\rho_i = p_i\tilde{\rho}_i$ and $\sum_{i\in\eta}\mathrm{Tr}[\rho_i] = 1$. A deterministic state of A is a rank one projector $|\varphi\rangle\langle\varphi|$ if it is pure, while it is a full rank density matrix when it is completely mixed (e.g. $I_{\rm A}/d_{\rm A}$ with $I_{\rm A}$ the identity operator on $\mathscr{H}_{\rm A}$). Accordingly the whole set of states $\mathrm{St}(\mathrm{A})$ of system A is the set of all unnormalized density matrices ρ, namely $\rho \geq 0$, $\mathrm{Tr}[\rho] \leq 1$.

Since the effects on A are linear functionals over the set of states we have that the linear space of effects $\mathrm{Eff}_{\mathbb{R}}(\mathrm{A})$ is the space $\mathrm{Herm}(\mathscr{H}_{\rm A})$ of Hermitian operators over $\mathscr{H}_{\rm A}$. The actual set of effects $\mathrm{Eff}(\mathrm{A})$ is made of the positive semidefinite operators bounded from above by the identity, namely $\mathrm{Eff}(\mathrm{A}) = \{P \in \mathrm{Herm}(\mathscr{H}_{\rm A}) \mid P \geq 0, P \leq I_{\rm A}\}$. An observation test $\boldsymbol{P}$ is given by a Positive Operator Valued Measure (POVM), namely a collection of effects $\{P_i\}_{i\in\eta}$ such that $\sum_{i\in\eta} P_i = I_{\rm A}$. Again, an atomic effect is simply a rank-one projector.

The probability resulting from the pairing between a state $|\rho)$ and an effect $(P|$ of the system A is given in QT by the *Born rule*, i.e. $(P|\rho) \equiv \mathrm{Tr}[\rho P^\dagger]$.

A transformation $\mathcal{C}$ between the systems A and B is given by a *quantum operation*, namely a completely positive trace nonincreasing linear map from $\mathrm{Herm}(\mathscr{H}_{\rm A})$ to $\mathrm{Herm}(\mathscr{H}_{\rm B})$. Notice that a quantum operation $\mathcal{C}$ always admit the *Kraus decomposition* $\mathcal{C}(\cdot) = \sum_{i\chi} C_i \cdot C_i^\dagger$ for suitable bounded operators C_i. A transformation test $\boldsymbol{\mathcal{C}} \subseteq \mathrm{Transf}(\mathrm{A},\mathrm{B})$ is a collection of quantum operations $\{\mathcal{C}_i\}_{i\in\eta}$ such that $\sum_{i\in\eta}\mathcal{C}_i$ is a deterministic transformation, namely a *quantum channel*, i.e. a trace preserving completely positive map. A unitary transformation — e.g. the Schrödinger evolution — is a deterministic test made of a single quantum operation with a Kraus decomposition made of a single Kraus operator.

3. The Fermionic Quantum Theory

In this section, we construct an OPT whose systems are the composition of the so-called *local Fermionic modes*. There is not a unique way for realizing a Fermionic OPT. The one presented here, denoted *Fermionic Quantum Theory* (FQT), stems from simple assumptions on the states/effects of the Fermionic systems and on the local nature of the Fermionic operations, and is the least constrained theory satisfying these assumptions. The resulting FQT corresponds to a superselected version of the QT of qubits with the superselection rule derived from the consistency of local Fermionic operations in an operational framework. A crucial assumption will be that of locality for the Fermionic theory, and it is related to considering the operator φ_i as the Kraus operator of an atomic local transformation.

In order to proceed with the construction, first we have to introduce the concept of *Fermionic algebra*.

3.1. *The Fermionic algebra*

The algebra $\mathcal{F}(N)$ of an arbitrary number $N < \infty$ of *local Fermionic modes* (LFMs) is generated by Fermionic operators $\{\varphi_i, \varphi_i^\dagger : i \in J_N\}$ with $J_N \coloneqq \{1, \ldots, N\}$ satisfying the *canonical anticommutation relation* (CAR)

$$\left\{\varphi_i, \varphi_i^\dagger\right\} = \delta_{ij} I\,, \qquad \{\varphi_i, \varphi_j\} = 0\,, \quad 1 \leq i,\ j \leq N\,. \tag{11}$$

Due to the CAR, the positive operators $\varphi_i^\dagger \varphi_i$ have spectrum $\mathsf{S} = \{0, 1\}$. The operators φ_i and $\varphi_i^\dagger$ act respectively as *lowering* and *raising* operators for $\varphi_i^\dagger \varphi_i$, namely if $|\Phi\rangle$ is an eigenvector of $\varphi_i^\dagger \varphi_i$ with eigenvalue 1, then $\varphi_i|\Phi\rangle$ is an eigenvector with eigenvalue 0, and $\varphi_i^\dagger|\Phi\rangle = 0$, while if $|\Phi\rangle$ is an eigenvector with eigenvalue 0 then $\varphi_i|\Phi\rangle = 0$ and $\varphi_i^\dagger|\Phi\rangle$ is an eigenvector with eigenvalue 1.

The operators $\varphi_i^\dagger \varphi_i$ form a set of mutually commuting positive operators and we call *vacuum eigenvector*, denoted $|\Omega\rangle$, a simultaneous eigenvector with eigenvalue 0 for all i. A vacuum eigenvector of the Fermionic algebra corresponds to all the LFMs unoccupied and it is annihilated by the lowering operators:

$$\varphi_i|\Omega\rangle = 0 \quad \forall i\,.$$

In general the vacuum $|\Omega\rangle$ is not unique. However, we can always restrict to the unique case, corresponding to having a trivial multiplicity, with a vacuum vector space where the field operators act identically. From now on, we will consider the vacuum as unique.

By raising $|\Omega\rangle$ in all possible ways we get the 2^N orthonormal vectors forming the *Fock basis* in the occupation number representation

$$|s_1, \ldots, s_N\rangle_{\mathrm{F}} \coloneqq (\varphi_1^\dagger)^{s_1} \cdots \left(\varphi_N^\dagger\right)^{s_N} |\Omega\rangle\,, \quad s_i \in \{0, 1\}\,, \tag{12}$$

with s_i corresponding to the occupation number at the ith site, i.e. the expectation value of the operator $\varphi_i^\dagger \varphi_i$. We call *total occupation number* of the vector $|s_1, \ldots, s_N\rangle_{\mathrm{F}}$ the sum $\sum_i s_i$. The linear span of these vectors corresponds to the *antisymmetric Fock space* $\mathscr{F}_N$ of dimension 2^N.

3.2. *Assumptions*

The assumptions are the following:

(i) the FQT is causal;
(ii) the states of N LFMs are represented by density matrices on the antisymmetric Fock space $\mathscr{F}_N$;
(iii) the transformations on N LFMs are represented by linear Hermitian preserving maps;
(iv) the map $\mathcal{X}_i$ with Kraus operators $X_i \coloneqq \varphi_i + \varphi_i^\dagger$ is physical;
(v) for a composite systems A made of N LFMs, transformations with Kraus operators in the algebra of field operators φ_i, $\varphi_i^\dagger$ with $i \in \chi \subset J_N$ are local on the subsystem B of the LFMs associated to χ;

(vi) local transformations on a system retain the same Kraus representation when other systems are added or discarded;
(vii) the pairing between states and effects is given by the Born rule $(a|\rho) \coloneqq \mathrm{Tr}[\rho a]$;
(viii) on a single LFM the pairing with the deterministic effect is represented by $(e|\rho) \coloneqq \mathrm{Tr}[\rho]$.

Notice that since the projection on the vacuum eigenvector has field representation $|\Omega\rangle\langle\Omega| = \prod_{i=1}^{N} \varphi_i \varphi_i^\dagger$, then any state can be written as

$$\rho \coloneqq \sum_j K_j |\Omega\rangle\langle\Omega| K_j^\dagger = \sum_j K_j \left(\prod_{i=1}^{N} \varphi_i \varphi_i^\dagger \right) K_j^\dagger$$

for some collection of operators

$$K_j \coloneqq \sum_{s^{(j)}} \alpha_{s^{(j)}} {\varphi_1^\dagger}^{s_1^{(j)}} \cdots {\varphi_N^\dagger}^{s_N^{(j)}}, \quad \alpha_{s^{(j)}} \in \mathbb{C}.$$

Hence, a state ρ of N LFMs can be written equivalently as

$$\rho = \sum_{st} \rho_{st} \prod_{i=1}^{N} {\varphi_i^\dagger}^{s_i} \varphi_i \varphi_i^\dagger {\varphi_i}^{t_i}, \quad \rho_{st} \in \mathbb{C}, \tag{13}$$

where s is a binary string $s_1 \cdots s_N$ (and similarly for t).

Moreover, from assumptions (iv), (vii) and (viii) the following proposition holds.

Proposition 3.1. *In a system* A *made of* N LFM*s for every* i *the map* $\mathcal{X}_i$ *is deterministic and* $\mathcal{X}_i^2 = \mathcal{I}$.

Proof. First notice that from the CAR relations we have $X_i^\dagger X_i = X_i^2 = \left(\varphi_i^\dagger \varphi_i + \varphi_i \varphi_i^\dagger\right) = I$, and thus $\mathcal{X}_i^2 = \mathcal{I}$. Moreover, we have $\mathrm{Tr}[\mathcal{X}_i(\rho)] = \mathrm{Tr}\left[X_i \rho X_i^\dagger\right] = \mathrm{Tr}\left[\rho X_i^\dagger X_i\right] = \mathrm{Tr}[\rho] = 1$. □

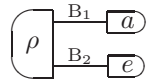
3.3. *Discarding of a subsystem*

We derive now the simple rule for discarding a subsystem in the FQT. First, we need two lemmas that can be derived by the assumptions.

Lemma 3.1. *Consider a system* $\mathrm{A} = \mathrm{B}_1\mathrm{B}_2$ *made of* $N = N_1 + N_2$ LFMs, *and let* B_1 *be made of* N_1 LFMs *corresponding to* $\chi_1 \subset J_N$. *Then a transformation* $\mathcal{T} \in \mathrm{Transf(A)}$ *is local on* B_1 *if and only if it can be expressed in terms of Kraus operators belonging to the algebra generated by field operators* φ_i *and* $\varphi_i^\dagger$ *for* $i \in \chi_1$.

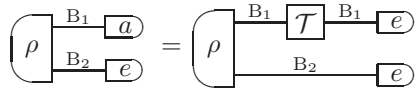
Proof. By assumption (iii) a transformation $\mathcal{T}$ on B_1 has Kraus operators in the algebra of fields φ_i, $\varphi_i^\dagger$ with $i \in J_{N_1}$. By assumption (vi), if we now consider the composite system $\mathrm{A} = \mathrm{B}_1\mathrm{B}_2$ the local transformations on B_1 have Kraus operators in the algebra generated by the field operators φ_i, $\varphi_i^\dagger$ with $i \in J_{N_1}$. On the other hand, by assumption (v), also the converse is true. □

Lemma 3.2. *The parallel composition of the effect* $(a| \in \mathrm{Eff}(\mathrm{B}_1)$ *and the deterministic effect* $(e| \in \mathrm{Eff}(\mathrm{B}_2)$ *is represented by*



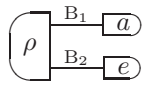
$$= \mathrm{Tr}[\rho a]\,, \quad \forall |\rho) \in \mathrm{St}(\mathrm{B}_2\mathrm{B}_2)\,.$$

Proof. Since in a causal theory $-\boxed{a} = -\boxed{\mathcal{T}}-\boxed{e}$ for some transformation $\mathcal{T}$, we have



(14)

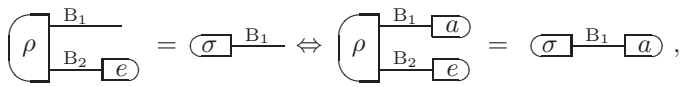
$\mathcal{T}$, being local on the subsystem B_1, has Kraus form $\mathcal{T}(\sigma) \coloneqq \sum_i s_i K_i \sigma K_i^\dagger$ where K_i is in the algebra of the field operators acting on B_1. By assumption (vi), the map retains the same Kraus expression when extended on a system $\mathrm{B}_1\mathrm{B}_2$, so



$$= \sum_j s_j \,\mathrm{Tr}\left[\rho K_j^\dagger K_j\right] = \mathrm{Tr}[\rho a]\,. \tag{15}$$

□

Consider now the system $\mathrm{A} = \mathrm{A}_1 \ldots \mathrm{A}_N$ made of N LFMs, and the bipartition $\mathrm{A} = \mathrm{B}_1\mathrm{B}_2$, corresponding to the disjoint partition $\{\chi_1, \chi_2\}$ of S. Since by assumption (i) the FQT is causal, the marginal state σ of system B_1 for an arbitrary state $\rho \in \mathrm{St}(\mathrm{A})$ is defined by the following implicit equation



for any effect $a \in \mathrm{Eff}(\mathrm{B}_1)$ on the complementary system of B_2. In formula we write

$$\mathrm{Tr}[\sigma a] \coloneqq \mathrm{Tr}[\rho a]\,, \quad \forall a \in \mathrm{Eff}(\mathrm{B}_1)\,. \tag{16}$$

Let $|\rho) \in \mathrm{St}(\mathrm{B}_1\mathrm{B}_2)$ be as in Eq. (13), we can perform the following swapping of the field operators

$$\begin{aligned}\rho &= \sum_{st} \rho_{st} \prod_{i=1}^{N} {\varphi_i^\dagger}^{s_i} \varphi_i \varphi_i^\dagger \varphi_i^{t_i} \\ &= \sum_{st} (-1)^{f(s,t)} \rho_{st} \prod_{k\in\chi_2} \left({\varphi_k^\dagger}^{s_k} \varphi_k \varphi_k^\dagger \varphi_k^{t_k}\right) \prod_{i\in\chi_1} \left({\varphi_i^\dagger}^{s_i} \varphi_i \varphi_i^\dagger \varphi_i^{t_i}\right),\end{aligned}$$

where $f(s,t)$ is the function evaluating the number of swaps needed to perform the reordering, which is given by

$$f(s,t) \coloneqq \sum_{k\in\chi_2} (s_k \oplus t_k) \Sigma_{\substack{i\in\chi_1 \\ i<k}} (s_i \oplus t_i)\,.$$

The RHS of Eq. (16) then becomes

$$\sum_{st} \rho_{st} \operatorname{Tr}\left[\prod_{i=1}^{N} \left(\varphi_i^{\dagger s_i} \varphi_i \varphi_i^{\dagger} \varphi_i^{t_i}\right) a\right]$$

$$= \sum_{st} (-1)^{f(s,t)} \rho_{st} \operatorname{Tr}\left[\prod_{i\in\chi_1} \left(\varphi_i^{\dagger s_i} \varphi_i \varphi_i^{\dagger} \varphi_i^{t_i}\right) a \prod_{k\in\chi_2} \left(\varphi_k^{\dagger s_k} \varphi_k \varphi_k^{\dagger} \varphi_k^{t_k}\right)\right]$$

$$= \sum_{st} (-1)^{f(s,t)} \rho_{st} \operatorname{Tr}\left[\prod_{i\in\chi_1} \left(\varphi_i^{\dagger s_i} \varphi_i \varphi_i^{\dagger} \varphi_i^{t_i}\right) a \prod_{k\in\chi_2} \delta_{s_k,t_k}\right].$$

Since $f(s,t) = 0$ whenever for $k \in \chi_2$ it happens $s_k = t_k$, the previous equation shows that the marginal state on subsystem B_1 of a state $\rho \in \mathrm{St}(\mathrm{B}_1\mathrm{B}_2)$ is given by

$$\sigma := \operatorname{Tr}_{\mathrm{B}_2}[\rho] = \operatorname{Tr}_{\mathrm{B}_2}\left[\sum_{s,t} \rho_{st} \prod_{i=1}^{N} \varphi_i^{\dagger s_i} \varphi_i \varphi_i^{\dagger} \varphi_i^{t_i}\right]$$

$$= \sum_{\substack{s,t \text{ with} \\ s_k = t_k,\ k\in\chi_2}} \rho_{st} \prod_{i\in\chi_1} \varphi_i^{\dagger s_i} \varphi_i \varphi_i^{\dagger} \varphi_i^{t_i}, \tag{17}$$

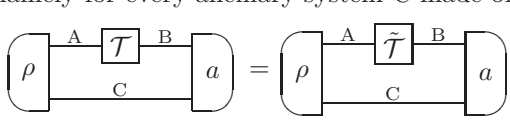
namely it is obtained by dropping all terms that contain an odd number of field operators in any of the LFMs in B_2, while in the remaining terms one erases the field operators in B_2.

3.4. *Derivation of the parity superselection rule*

In the following, we will show that the Wigner parity superselection rule[16,17] can be derived operationally from postulates (iv) and (v).

Theorem 3.1. *Every transformation between N LFMs is operationally equivalent to a map where each Kraus operator is a combinations of products of either odd or even numbers of field operators.*

Proof. Let us take an arbitrary transformation $\mathcal{T} \in \mathrm{Transf}_{\mathbb{R}}(\mathrm{A},\mathrm{B})$ with A, B N-LFM systems. Since by assumption (iii) $\mathcal{T}$ is Hermitian preserving, it can be written as the difference between two CP maps, hence, for an arbitrary ρ, $\mathcal{T}(\rho) = \sum_i s_i K_i \rho {K_i}^{\dagger}$, where K_i are Kraus operators, and $s_i = \pm 1$ for every i. Every K_i can be decomposed as $K_i = E_i + O_i$ with $E_i, O_i \in \mathcal{L}(\mathbb{C}^{2^N})$, and E_i and O_i being the part of K_i containing only superposition of an even and odd number of field operators, respectively. Thus, we have $\mathcal{T} = \sum_i s_i \big(E_i \cdot {E_i}^{\dagger} + O_i \cdot {O_i}^{\dagger} + E_i \cdot {O_i}^{\dagger} + O_i \cdot {E_i}^{\dagger}\big)$. We want to show that $\mathcal{T}$ is equivalent to the map $\tilde{\mathcal{T}} := \sum_i s_i \big(E_i \cdot {E_i}^{\dagger} + O_i \cdot {O_i}^{\dagger}\big)$, namely for every ancillary system C made of M LFMs



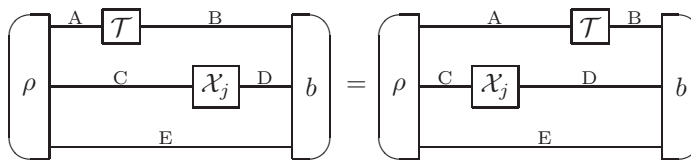
$$, \quad \forall |\rho) \in \mathrm{St}(\mathrm{AC}), \quad (a| \in \mathrm{Eff}(\mathrm{BC}).$$

Using assumption (vii), the previous relation is equivalent to

$$\sum_i s_i \operatorname{Tr}\left[\left(E_i \rho {O_i}^\dagger + O_i \rho {E_i}^\dagger\right) a^\dagger\right] = 0\,, \tag{18}$$

for every $|\rho) \in \mathrm{St}(\mathrm{AC})$, $(a| \in \mathrm{Eff}(\mathrm{BC})$ and every ancillary system C.

In order to prove Eq. (18) we consider the physical map — by assumption (iv) — $\mathrm{Transf}(\mathrm{C},\mathrm{D}) \ni \mathcal{X}_j$, j denoting a Fermionic subsystem belonging to C, and D a Fermionic system made of M LFMs too. Being $\mathcal{X}_j$ and $\mathcal{T}$ two transformations acting on different subsystems, by Eq. (1) their sequential composition shall commute; i.e. for every ancillary system E, for every state $|\sigma) \in \mathrm{St}(\mathrm{ACE})$, and for every effect $(b| \in \mathrm{Eff}(\mathrm{BDE})$



Consider the case where D is the system made of 0 LFMs, i.e. the ancillary system is the trivial system I. Then we have by assumption (vii) that a necessary condition for the commutation of the maps $\mathcal{T}$ and $\mathcal{X}_j$ is given by

$$\begin{aligned}
&\sum_i s_i \operatorname{Tr}\Big[\Big(X_j E_i \rho {E_i}^\dagger X_j^\dagger + X_j O_i \rho {O_i}^\dagger X_j^\dagger + X_j E_i \rho {O_i}^\dagger X_j^\dagger + X_j O_i \rho {E_i}^\dagger X_j^\dagger\Big) b\Big] \\
&\quad = \sum_i s_i \operatorname{Tr}\Big[\Big(E_i X_j \rho X_j^\dagger {E_i}^\dagger + O_i X_j \rho X_j^\dagger {O_i}^\dagger + E_i X_j \rho X_j^\dagger {O_i}^\dagger + O_i X_j \rho X_j^\dagger {E_i}^\dagger\Big) b\Big] \\
&\qquad \forall |\rho) \in \mathrm{St}(\mathrm{ACE})\,, \quad \forall (b| \in \mathrm{Eff}(\mathrm{BDE})\,.
\end{aligned}$$

Since the Kraus operators E_i, O_i contain respectively an even and an odd number of field operators, the anticommutation relations for the fields and the invariance of the trace under cyclic permutation give us

$$\sum_i s_i \operatorname{Tr}\Big[\Big(E_i \rho {O_i}^\dagger + O_i \rho {E_i}^\dagger\Big) X_j^\dagger b X_j\Big] = 0\,.$$

If we now choose $b = X_j a X_j^\dagger$ for an arbitrary $a \in \mathrm{Eff}(\mathrm{AC})$, by Proposition 3.1, we obtain

$$\sum_i s_i \operatorname{Tr}\Big[\Big(E_i \rho {O_i}^\dagger + O_i \rho {E_i}^\dagger\Big) a\Big] = 0 \quad \forall |\rho) \in \mathrm{St}(\mathrm{AC})\,, \quad (a| \in \mathrm{Eff}(\mathrm{BC})\,, \tag{19}$$

namely Eq. (18). We conclude therefore that the compatibility condition of commutation between local transformation implies the thesis. □

The previous theorem allows us to consider the transformations with each Kraus operator involving only an even or an odd number of field operators as the representatives of the equivalence class they belong to. This fact allows us to prove the following corollary.

Corollary 3.1. *Effects of the* FQT *are positive operators made of products of an even number of field operators.*

Proof. Since in a causal theory we have  for some transformation $\mathcal{T}$, every effect can be written as $a = \sum_i s_i {E_i}^\dagger E_i + s_i {O_i}^\dagger O_i$, namely an operator involving only products of even number of field operators. □

Lemma 3.3. *The even part of a state ρ is a density matrix.*

Proof. By assumption (ii) a state of N LFMs is a positive operator ρ on $\mathscr{F}_N$, then it can be expressed as $\rho = X^\dagger X$. Writing $X = E + O$, with E combination of even products of field operators and O combination of odd products, we have $\rho = E^\dagger E + O^\dagger O + E^\dagger O + O^\dagger E$. Finally, the even part of ρ is given by $E^\dagger E + O^\dagger O \geq 0$, which is positive. □

Proposition 3.2. *States of* FQT *satisfy the parity superselection rule.*

Proof. Consider the state $|\rho) = \sum_j E_j + O_j$, and its even part $|\rho_E) \coloneqq \sum_j E_j$ with E_i and O_i made of linear combinations of an even and an odd number of field operators, respectively. Since $\mathrm{Tr}[O_j a] = 0$, due to a being made of products of an even number of field operators (see Corollary 3.1), we have that $|\rho)$ is operationally equivalent to $|\rho_E)$, that is for every effect $(a|$, $(a|\rho) = (a|\rho_E)$. Hence it is not restrictive to consider only the states represented by density matrices that are linear combinations of products of even number of field operators, as representatives of the resulting equivalence classes of states. One can now decompose the Fock space $\mathscr{F}_N$ in the direct sum

$$\mathscr{F}_N = \mathscr{F}_N^0 \oplus \mathscr{F}_N^1 \,, \tag{20}$$

where $\mathscr{F}_N^0$ and $\mathscr{F}_N^1$ are the eigenspaces of the parity operator

$$P = \frac{1}{2}\left(I + \prod_{i=1}^{n} \left(\varphi_i \varphi_i^\dagger - \varphi_i^\dagger \varphi_i \right) \right) \tag{21}$$

corresponding to the eigenvalues $p = 0, 1$ — i.e. corresponding to an even/odd total occupation number. We conclude that every state — being represented by a combination of products of an even number of fields — commutes with P, thus it has a well defined parity, i.e. states satisfy the parity superselection rule. □

Corollary 3.2. *The vacuum state $|\Omega\rangle\langle\Omega|$ is physical.*

Proof. Being $|\Omega\rangle\langle\Omega| = \prod_i \varphi_i \varphi_i^\dagger$ a state with an even number of field operators, it is valid state of the FQT. □

Finally, since effects of a system of N LFMs are linear combinations of even products of field operators, they commute with the parity operator, too. This allows us to derive the parity superselection rule also for effects.

Corollary 3.3. *Effects of* FQT *satisfy the parity superselection rule.*

3.5. *Set of states, effects and transformations*

In the following, we will analyze the consequences of the parity prescription on the states, effects, and transformations of FQT.

For this purpose, we remind that if A is a N qubits system, then the linear spaces of states and effects correspond to the set of $2^N \times 2^N$ Hermitian matrices

$$\mathrm{St}_{\mathbb{R}}(\mathrm{A}) = \mathrm{Eff}_{\mathbb{R}}(\mathrm{A}) = \mathrm{Herm}\big((\mathbb{C}^2)^{\otimes N}\big) \tag{22}$$

and the dimension of the set of states and effects is

$$D_{\mathrm{A}} = d_{\mathrm{A}}^2 = 2^{2N}\,, \tag{23}$$

with $d_{\mathrm{A}} = 2^N$ the Hilbert dimension of N qubits. On the other hand, a system of N LFMs must obey the parity superselection rule, which forbids any pure state corresponding to a superposition of vectors belonging to $\mathscr{F}_N^0$ and $\mathscr{F}_N^1$, i.e. pure states are given by projections on superpositions of Fock vectors with total occupation numbers equal modulo 2. Hence the elementary system — the one-LFM — has only the pure states $|0\rangle\langle 0|, |1\rangle\langle 1|$, thus corresponding to the classical *bit* (indeed the Fock vectors $|0\rangle$ and $|1\rangle$ belong to $\mathscr{F}_1^0$ and $\mathscr{F}_1^1$, respectively, and then one cannot consider their superpositions). In general, for a system A of N LFMs we can identify two disjoint sectors with different parity in the linear sets of states and effects: $\mathrm{St}_{\mathbb{R}}(\mathrm{A}) = \mathrm{Eff}_{\mathbb{R}}(\mathrm{A}) = \mathrm{Herm}\left(\mathscr{F}_N^0\right) \oplus \mathrm{Herm}\left(\mathscr{F}_N^1\right)$. Since $\dim \mathscr{F}_N^0 = \dim \mathscr{F}_N^1 = 2^{N-1}$ we have $\dim \mathrm{Herm}\left(\mathscr{F}_N^i\right) = 2^{2(N-1)}$. Being the dimension of the linear space of states of $N-1$ qubits exactly $2^{2(N-1)}$, we have that each parity sector of the linear set of states of N LFMs is isomorphic to that of $N-1$ qubits, making $\mathrm{St}_{\mathbb{R}}(\mathrm{A}) = \mathrm{Eff}_{\mathbb{R}}(\mathrm{A})$ equivalent to the direct sum of two $N-1$ qubit state spaces, with

$$D_{N\,\mathrm{LFMs}} = 2D_{N-1\,\mathrm{QUBITs}} = \frac{1}{2} D_{N\,\mathrm{QUBITs}} = 2^{2N-1}\,. \tag{24}$$

A general element of $\mathrm{St}_{\mathbb{R}}(\mathrm{A}) = \mathrm{Herm}\left(\mathscr{F}_N^0\right) \oplus \mathrm{Herm}\left(\mathscr{F}_N^1\right)$ has a block diagonal form, that characterizes also the actual sets of states and effects: reordering the basis of the Fock space $\mathscr{F}_N$ in such a way that all the even vectors precede all the

odd ones, one has that for every state $\rho \in \mathrm{St}(\mathrm{A})$ and every effect $a \in \mathrm{Eff}(\mathrm{A})$

$$\rho = \left(\begin{array}{c|c} \rho_0 & 0 \\ \hline 0 & \rho_1 \end{array}\right), \quad \rho_i \geq 0\,, \quad \mathrm{Tr}[\rho_0] + \mathrm{Tr}[\rho_0] \leq 1\,,$$

$$a = \left(\begin{array}{c|c} a_0 & 0 \\ \hline 0 & a_1 \end{array}\right), \quad 0 \leq a_i \leq I_{2^{N-1}}\,,$$

corresponding to $\mathrm{St}(\mathrm{A}) = \mathrm{Conv}[(\mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^0)) \cup (\mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^1))]$ and $\mathrm{St}(\mathrm{A}) = \mathrm{Conv}[(\mathrm{Eff}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^0)) \cup (\mathrm{Eff}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^1))]$, with $\mathrm{Conv}(X)$ representing the convex hull of the set X.

Notice that, thanks to the definition of Eq. (12) the Fock space $\mathscr{F}(N)$ is isomorphic to a N-qubit Hilbert space, by the trivial identification of the occupation number basis $|s_1, \ldots, s_N\rangle_F$ with the qubit computational basis $|s_1, \ldots, s_N\rangle_Q$ of eigenvectors of the Pauli matrices σ_i^z with $1 \leq i \leq N$. Hence the two parity sectors $\mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^i)$ are actually isomorphic to the $(N-1)$-qubit states set, with pure states given by the rank one projectors $|\Psi\rangle\langle\Psi|$ with $|\Psi\rangle$ normalized superposition of Fock vectors belonging to $\mathscr{F}_N^i$, while the two sectors $\mathrm{Eff}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^i)$ are isomorphic to the $(N-1)$-qubit effects set, whose atomic elements coincide with pure states.

Proposition 3.3. *Let* A, B *be two* N-LFM *systems. Then, transformations from* A *to* B *are CP maps from* St(A) *to* St(B)*.*

Proof. Since the parity superselection implies the presence of the two parity sectors $\mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^i)$ with $i = 0, 1$, we have that an arbitrary transformation $\mathcal{T} \in \mathrm{Transf}(\mathrm{A}, \mathrm{B})$ can be written as $\mathcal{T} = \mathcal{T}_{00} + \mathcal{T}_{01} + \mathcal{T}_{10} + \mathcal{T}_{11}$, with

$$\mathcal{T}_{xy} : \mathrm{St}(\mathrm{A}) \to \mathrm{St}(\mathrm{B})$$

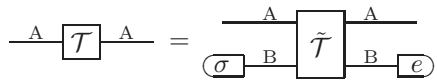
$$|\rho) \mapsto \begin{cases} \mathcal{T}_{xy}|\rho) \in \mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^y) & \text{if } |\rho) \in \mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}\left(\mathscr{F}_N^x\right), \\ 0 & \text{if } |\rho) \in \mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^{\bar{x}})\,. \end{cases}$$

Since $\mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^0) \sim \mathrm{St}(\mathrm{A}) \cap \mathrm{Herm}(\mathscr{F}_N^1) \sim \mathrm{St}(\mathrm{B})$ with B a $N-1$ qubits system, we have that all the $\mathcal{T}_{xy}$ with $x, y = 0, 1$ are actually quantum maps from $N-1$ to $N-1$ qubits, i.e. CP maps. □

Since we are admitting the no-restriction hypothesis, all the transformations with Kraus operators being superpositions of products of either an even number or an odd number of field operators belong to the theory (since they are admissible). Finally, every admissible transformation can be dilated to a single-Kraus one thanks to the following proposition.

Proposition 3.4. *Every multi-Kraus transformation can be dilated to a single-Kraus one.*

Proof. Let $\mathcal{T} = \sum_{i\in\eta_{\rm E}} E_i \cdot E_i^\dagger + \sum_{i\in\eta_{\rm O}} O_i \cdot O_i^\dagger$ be a transformation of the N-LFM system A, with E_i, O_i even and odd Kraus operators respectively. We want to show that we can find $\tilde{\mathcal{T}} = \tilde{T} \cdot \tilde{T}^\dagger$ acting on AB with B a M-LFM system and a state $|\sigma) \in \mathrm{St(B)}$ such that

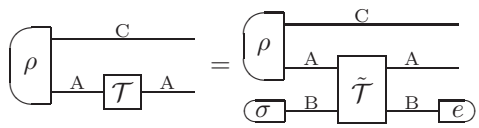


(25)

In a K-LFM system there are 2^{K-1} Fock vectors of Eq. (12) involving an even number of fields (as well as 2^{K-1} involving an odd number of them). An even Fock vector $|e_i\rangle$ can be written as $\tilde{E}_i|\Omega\rangle$, with E_i an operator involving an even number of fields (similarly we have $|o_i\rangle = \tilde{O}_i|\Omega\rangle$ for the odd ones). We set B to be a M-LFM system with $M \coloneqq \max[\mathrm{ceiling}(\log_2 |\eta_{\rm E}|), \mathrm{ceiling}(\log_2 |\eta_{\rm O}|)] + 1$: in this way B is just big enough to allocate a number of even and odd Fock vectors equals respectively to the number of even and odd Kraus operators appearing in $\mathcal{T}$. Moreover, let $|\sigma)$ be the vacuum state of M LFMs, then a dilation of $\mathcal{T}$ is given by the transformation $\tilde{\mathcal{T}}$ with even single-Kraus

$$\tilde{T} = \sum_{i\in\eta_{\rm E}} E_i \tilde{E}_i + \sum_{i\in\eta_{\rm O}} O_i \tilde{O}_i \,, \tag{26}$$

where $\tilde{E}_i$ and $\tilde{O}_i$ are the even and odd field operators defining the even and odd orthonormal Fock vectors for the system B. Let us show the equality of Eq. (25), namely



(27)

for an arbitrary system C of P LFMs and an arbitrary state $|\rho) \in \mathrm{St(CA)}$. The LHS of Eq. (27) is given by

$$\sum_{i\in\eta_{\rm E}} E_i \rho E_i^\dagger + \sum_{i\in\eta_{\rm O}} O_i \rho O_i^\dagger \,.$$

On the other hand, being $\tau \coloneqq |\rho) \otimes |\sigma) = \rho \prod_{i=1}^M \tilde{\varphi}_i \tilde{\varphi}_i{}^\dagger$, with $\tilde{\varphi}_i$, $\tilde{\varphi}_i{}^\dagger$ the field operators on the subsystem B, the RHS of Eq. (27) is

$$\begin{aligned}
\mathrm{Tr_B}[\tilde{T}\tau\tilde{T}^\dagger] = &\sum_{j,k\in\eta_{\rm E}} \mathrm{Tr_B}\Big[E_j \tilde{E}_j \tau \tilde{E}_k^\dagger E_k^\dagger\Big] + \sum_{j,k\in\eta_{\rm O}} \mathrm{Tr_B}\Big[O_j \tilde{O}_j \tau \tilde{O}_k^\dagger O_k^\dagger\Big] \\
&+ \sum_{j\in\eta_{\rm E}} \sum_{k\in\eta_{\rm O}} \Big\{ \mathrm{Tr_B}\Big[E_j \tilde{E}_j \tau \tilde{O}_k^\dagger O_k^\dagger\Big] + \mathrm{Tr_B}\Big[O_k \tilde{O}_k \tau \tilde{E}_j^\dagger E_j^\dagger\Big] \Big\} \\
= &\sum_{j,k\in\eta_{\rm E}} \mathrm{Tr_B}\left[E_j \rho E_k^\dagger \tilde{E}_k^\dagger \prod_{i=1}^M \tilde{\varphi}_i \tilde{\varphi}_i{}^\dagger \tilde{E}_j\right]
\end{aligned}$$

$$+\sum_{j,k\in\eta_{\rm O}}\mathrm{Tr}_{\rm B}\left[O_j\rho O_k^\dagger\tilde{O}_k^\dagger\prod_{i=1}^{M}\tilde{\varphi}_i\tilde{\varphi}_i{}^\dagger\tilde{O}_j\right]$$

$$-\sum_{j\in\eta_{\rm E}}\sum_{k\in\eta_{\rm O}}\left\{\mathrm{Tr}_{\rm B}\left[E_j\rho O_k^\dagger\tilde{O}_k^\dagger\prod_{i=1}^{M}\tilde{\varphi}_i\tilde{\varphi}_i{}^\dagger\tilde{E}_j\right]\right.$$

$$\left.-\mathrm{Tr}_{\rm B}\left[O_k\rho E_j^\dagger\tilde{E}_j^\dagger\prod_{i=1}^{M}\tilde{\varphi}_i\tilde{\varphi}_i{}^\dagger\tilde{O}_k\right]\right\}.$$

Due to the orthogonality relation between the Fock vectors, Eq. (17) shows that the previous equation is equal to Eq. (27). Notice that with a similar procedure we could have dilated $\mathcal{T}$ to an *odd* single-Kraus transformation. □

4. Informational Features

In this section, we derive the consequences of the parity superselection on the structure of FQT. We will explore the tomography of Fermionic states (which results to be nonlocal), the properties of Fermionic entanglement (which exhibits differences with respect to the quantum case), and some issues regarding the computation in the FQT.

First of all we introduce the Jordan–Wigner isomorphism, which will be useful to compare FQT with QT and to address the issue of simulation.

4.1. *The Jordan–Wigner map*

Thanks to Eq. (12), the Fock space $\mathscr{F}_N$ and Hilbert space of N qubits $(\mathbb{C}^2)^{\otimes N}$ are isomorphic. A simple way to map unitarily an orthonormal basis of the former to an orthonormal basis of the latter is

$$|s_1,\ldots,s_N\rangle_{\rm F}\xrightarrow{U}|s_1,\ldots,s_N\rangle_{\rm Q}\,,$$

where $|s_1,\ldots,s_N\rangle_{\rm Q}$ is the joint eigenvector of the qubit operators σ_j^z with $j=1,\ldots,N$. Notice that such an encoding necessarily depends on the chosen ordering for the LFMs in Eq. (12). Indeed, had we chosen a different ordering $\pi\in S_N$ in Eq. (12) we would have got the Fock vectors

$$|s_1,\ldots,s_N\rangle_{\rm F}^{\pi}\coloneqq\left(\varphi_{\pi(1)}^\dagger\right)^{s_{\pi(1)}}\cdots\left(\varphi_{\pi(N)}^\dagger\right)^{s_{\pi(N)}}|\Omega\rangle\equiv(-1)^{\mathrm{sign}(\pi)}|s_1,\ldots,s_N\rangle_{\rm F}\,,$$

and the new unitary map would have been

$$|s_1,\ldots,s_N\rangle_{\rm F}^{\pi}\xrightarrow{U^\pi}|s_1,\ldots,s_N\rangle_{\rm Q}\,.$$

For a given ordering π the map U^π induces a *-algebra isomorphism between the CAR algebra of the fields and the algebra of the Pauli matrices $\{\sigma_j^\alpha\}$ known

as *Jordan–Wigner transform* (JWT). For example, for a given ordering π the JWT gives

$$\varphi_i \to J_\pi(\varphi_i) = \prod_{k=\pi(1)}^{\pi(i-1)} \sigma_k^z \sigma_{\pi(i)}^- , \quad \text{with} \quad \sigma_k^\pm \coloneqq \frac{\sigma_k^x \pm i\sigma_k^y}{2} .$$

From the previous equation, we notice that under JWT a single LFM field operator is in general mapped to a many qubits operator. This is a general property of the JWT regardless the number of LFMs involved. For instance, the two-LFM field operator $\varphi_i^\dagger \varphi_j$ is mapped under a JWT to

$$\varphi_i^\dagger \varphi_j \to J_\pi(\varphi_i^\dagger \varphi_j) = \sigma_{\pi(i)}^+ \prod_{\pi(i)<k<\pi(j)} \sigma_k^z \sigma_{\pi(j)}^- ,$$

namely the corresponding qubit operator involves more than two subsystems, the only exception when the chosen ordering gives $\pi(j) = \pi(i) + 1$.

In the following, we will denote by J the JWT representation corresponding to the trivial ordering permutation. Under the trivial ordering, the Pauli matrices can be expressed in terms of the Fermionic operators $\varphi_1, \ldots, \varphi_N$ as follows

$$\sigma_i^x = \sigma_1^z \cdots \sigma_{i-1}^z J\big(\varphi_i + \varphi_i^\dagger\big) , \tag{28}$$

$$\sigma_i^y = -i\sigma_1^z \cdots \sigma_{i-1}^z J\big(\varphi_i - \varphi_i^\dagger\big) , \tag{29}$$

$$\sigma_i^z = J\big(\varphi_i^\dagger \varphi_i - \varphi_i \varphi_i^\dagger\big) . \tag{30}$$

Notice that, the parity superselection rule in the Fock space $\mathscr{F}_N$ is trivially translated in the qubit space thanks to the JWT; i.e. defining *total occupation number* for the qubit vector $|s_1, \ldots, s_N\rangle_{\mathrm{Q}}$ as the sum $\sum_i s_i$, the Wigner superselection forbids states that are projections on superposition of qubit vectors with total occupation numbers different modulo 2.

4.2. *Bilocal tomography*

In the following, we exploit the JWT to represent the states of the FQT. For the sake of simplicity, we will drop the J symbol when this causes no confusion.

Thanks to Sec. 3, we know that FQT is the parity superselected version of the QT of qubits. Using the generalized theory of superselected OPT developed in Subsec. 2.2, we can see that FQT can be regarded as a minimal superselection of QT:

Proposition 4.1. FQT *is a minimal superselection of* QT *with the following linear constraints on the qubit system* B

$$\mathrm{St}(\sigma(\mathrm{B})) \coloneqq \{\rho \in \mathrm{St}(\mathrm{B});\ \mathrm{Tr}[\sigma^x \rho] = \mathrm{Tr}[\sigma^y \rho] = 0\} . \tag{31}$$

Proof. Let A and B be a one-LFM and a one-qubit system, respectively. We have noticed already that, due to the parity prescription, A has only two pure states

$|0\rangle\langle 0|$, $|1\rangle\langle 1|$. Then the density matrices $\rho \in \mathrm{St}(\mathrm{A})$ shall be diagonal

$$\mathrm{Tr}[\sigma^x \rho] = \mathrm{Tr}[\sigma^y \rho] = 0\,, \tag{32}$$

showing that the superselection on the elementary systems is as in Eq. (31) with

$$\mathrm{A} = \sigma(\mathrm{B})\,, \quad D_{\mathrm{A}} = D_{\mathrm{B}} - V^{\sigma}_{\mathrm{B}}\,, \quad V^{\sigma}_{\mathrm{B}} = 2\,.$$

Now we have to show that the whole FQT is built bottom-up extending in the minimal way the constraint (31) on the composite systems. Let B_N, B_M be two systems made of N and M qubits respectively. According to Definition 2.5, we can simply check that FQT achieves the lower bound of Eq. (4),

$$V^{\sigma}_{\mathrm{B}_N\mathrm{B}_M} = D_{\mathrm{B}_N} V^{\sigma}_{\mathrm{B}_M} + D_{\mathrm{B}_M} V^{\sigma}_{\mathrm{B}_N} - 2 V^{\sigma}_{\mathrm{B}_N} V^{\sigma}_{\mathrm{B}_M}\,. \tag{33}$$

Using Eq. (24), we have $V^{\sigma}_{\mathrm{B}_N} = \frac{1}{2} D_{\mathrm{B}_N}$ and $V^{\sigma}_{\mathrm{B}_M} = \frac{1}{2} D_{\mathrm{B}_M}$, hence Eq. (33) is satisfied. □

Since QT is local-tomographic, thanks to Proposition 2.2 the FQT is maximally bilocal-tomographic. This can also be verified counting the number of independent local and two-local effects for a system of N LFMs and noticing that it is exactly its states space dimension:

$$\sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N}{2k} D_{1\,\mathrm{LFM}}^{N-2k} \tilde{D}_{2\,\mathrm{LFMs}}^{k} = 2^{2N-1} = D_{N\,\mathrm{LFMs}}\,.$$

We emphasize that FQT provides an example of a bilocal-tomographic theory whose systems do not satisfy the dimensional prescription in Ref. 23. Indeed, after showing that the dimension of the nonlocal component of a bipartite system $\tilde{D}_{\mathrm{AB}} = D_{\mathrm{AB}} - D_{\mathrm{A}} D_{\mathrm{B}}$ can be factorized as $\tilde{D}_{\mathrm{AB}} = L_{\mathrm{A}} L_{\mathrm{B}}$, and assuming that the two functions

$$D_{\mathrm{A}} + L_{\mathrm{A}}\,, \quad D_{\mathrm{A}} - L_{\mathrm{A}}\,,$$

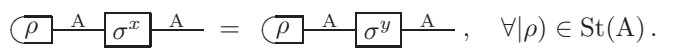
are strictly increasing functions of the number of perfectly discriminable states d_{A}, the authors of Ref. 23 prove that in a bilocal-tomographic theory, the dimension of the system A must be

$$D_{\mathrm{A}} = \frac{1}{2}(d^r_{\mathrm{A}} + d^s_{\mathrm{A}})\,, \tag{34}$$

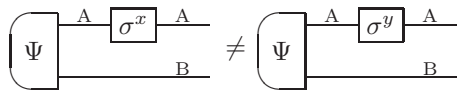
for some integers r, s satisfying $r \geq s > 0$. This is not true for the Fermionic computation where for example $D_{\mathrm{A}} = 8$ cannot be achieved in this way. The strict monotonicity of the function $D_{\mathrm{A}} - L_{\mathrm{A}}$ is too restrictive and excludes the Fermionic case from the set of admissible bilocal-tomographic theories, since we have $D_{\mathrm{A}} - L_{\mathrm{A}} = 0$ for any system A made of an arbitrary number N of LFMs.

Not satisfying local tomography, the FQT does not satisfy the property of Remark 2.1 in Subsec. 2.1. Indeed consider the unitary maps on a single LFM system given by I, σ^x, σ^y, σ^z. Being $|0\rangle$, $|1\rangle$ the only pure normalized states of a single

LFM, the maps σ^x and σ^y (and similarly I and σ^z) are equal when evaluated on a one-LFM system A; pictorially:



We need a n-LFM state with $n \geq 0$ to verify that the two maps σ^x and σ^y are indeed different; e.g. considering $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ we get $\sigma^x \otimes I|\Psi\rangle \neq e^{i\gamma}\sigma^y \otimes I|\Psi\rangle$ for every $\gamma \in \mathbb{R}$; pictorially



4.2.1. *Other superselected quantum theories*

It is worth mentioning that FQT is not the unique minimal superselection of QT. Another example is given by the RQT defined[23] as the restriction of the quantum case to real matrices. The elementary system of RQT, with two perfectly distinguishable states, is denoted *rebit* and its convex set of states is the disk obtained by the equatorial section of the qubit. According to Definition 2.4, the RQT is a superselection of the standard QT, being the requirement of reality of a quantum state ρ given by the linear constraint $\rho - \rho^T = 0$, with T denoting transposition with respect to a fixed basis taken as real. Hence, if A is the multipartite system of N *rebits* having Hilbert dimension $d_{\mathrm{A}} = 2^N$, the dimension of $\mathrm{St}_{\mathbb{R}}(\mathrm{A})$ is given by

$$D_{\mathrm{A}} = d_{\mathrm{A}}(d_{\mathrm{A}} + 1)/2\,.$$

Thus, if B is a system of N qubits, one has $\mathrm{A} = \sigma(\mathrm{B})$ where the number of linear constraints for the system A is given by

$$V^{\sigma}_{\mathrm{A}} = D_{\mathrm{B}} - D_{\mathrm{A}}\,.$$

One can easily check that also RQT is minimally superselected; indeed, for a couple of systems A, C of N and M rebits respectively, the number of constraints for the composite system AC

$$V^{\sigma}_{\mathrm{AB}} = \frac{1}{2} d_{\mathrm{A}} d_{\mathrm{B}} (d_{\mathrm{A}} d_{\mathrm{B}} - 1)$$

saturates the lower bound of Eq. (5). Hence, from the linear constraint of a one-rebit system $\mathrm{Tr}[\sigma^y \rho] = 0$, we build the whole RQT by taking the minimal extension of this constraint to the composite systems. Therefore, according to Proposition 2.2, the RQT is maximally bilocal-tomographic (see also Ref. 23).

In Proposition 2.2, we have considered the extremal cases of minimal and maximal superselection, which lead respectively to bilocal- and local-tomographic theories. On the other hand, there is a full range of possible constraints between these two cases — i.e. V^{σ}_{AB} strictly included in the bounds of Eqs. (4) and (5) — where one can find superselected theories with different degrees of discriminability.

As already pointed out at the end of Subsec. 4.1, the parity superselection of the FQT is trivially translated in the QT representation by allowing only pure qubits states that are projections on superpositions of vectors with total occupation numbers equal modulo 2. A more general scenario is given by considering a *number superselected* QT, namely superselected QT theories of qubits where the admissible pures states are projections on superpositions of vectors with the same total occupation numbers.

Proposition 4.2. *There is no $n \in \mathbb{N}$ such that a number superselected* QT *is n-local-tomographic.*

Proof. For any n, we will present a suitable composite system $\mathrm{B} \coloneqq \mathrm{B}_1 \cdots \mathrm{B}_n \mathrm{B}_{n+1}$ and a couple of state $|\psi_+\rangle, |\psi_-\rangle \in \mathrm{St}(\mathrm{B})$ we cannot distinguish by means of n-local effects (see Definition 2.3). Set n to be an arbitrary integer, then for $1 \leq i \leq n$ each subsystem B_i is the elementary system of the number superselected QT, while B_{n+1} is the parallel composition of n of such elementary systems. $|\psi_+\rangle$, $|\psi_-\rangle$ are the pure states corresponding to the projections on the Hilbert space vectors

$$|\psi_\pm\rangle \coloneqq |0\rangle_{\mathrm{B}_1}|0\rangle_{\mathrm{B}_n}|1,1,\ldots,1\rangle_{\mathrm{B}_{n+1}} \pm |1\rangle_{\mathrm{B}_1}|1\rangle_{\mathrm{B}_n}|0,0,\ldots,0\rangle_{\mathrm{B}_{n+1}} .$$

There is no n-local effect able to discriminate the two states, i.e. no discriminating effect has the form $E^{(n)} \otimes E^{(1)}$, with $E^{(n)}$ an effect for n subsystems (hence either $E^{(n)} \in \mathrm{Herm}((\mathbb{C}^2)^{\otimes n})$ or $E^{(n)} \in \mathrm{Herm}((\mathbb{C}^2)^{\otimes(n-1)} \otimes (\mathbb{C}^2)^{\otimes n})$), and $E^{(1)}$ a one-effect (and so either $E^{(1)} \in \mathrm{Herm}((\mathbb{C}^2)^{\otimes n}$ or $E^{(1)} \in \mathrm{Herm}(\mathbb{C}^2)$. Indeed, since the two states differ only in the sign of the off-diagonal terms, a suitable effect to tell them apart should have a non null component $|\psi_+\rangle\langle\psi_-| = |0\rangle\langle 1|_{\mathrm{B}_1} \otimes \cdots \otimes |0\rangle\langle 1|_{\mathrm{B}_n} \otimes |1\cdots 1\rangle\langle 0\cdots 0|_{\mathrm{B}_{n+1}}$. However $|\psi_+\rangle\langle\psi_-|$ cannot be spanned by the tensor product of the two effects $E^{(1)}$, $E^{(n)}$, since due to the superselection rule each $E^{(1)}$, $E^{(n)}$ have just the matrix elements $|s_1,\ldots,s_k\rangle\langle t_1,\ldots,t_k|$ with $\sum s_i = \sum t_i$. □

The previous result shows that a QT with number superselection has a cumbersome tomographic property: given a n-partite system there is always a couple of states that cannot be discriminated without resorting to a nonlocal effect involving all the subsystems.

4.3. *Fermionic entanglement*

Entanglement is commonly regarded as the peculiar trait of QT and it has been studied extensively also in relation to the other quantum features. A pure state of a pair of quantum systems is called entangled if it cannot be factorized, while a mixed state is entangled if it cannot be written as a mixture of factorized pure states, i.e. it is not *separable*. The main goal in the study of entanglement is to find criteria for testing whether a state is separable or not (see for example the *partial transpose condition* proposed by Peres in Ref. 28), and to provide consistent measures for quantifying entanglement. Among the measures of entanglement

considered in the literature we can cite the *entanglement of formation*,[29–32] the *distillable entanglement*,[33] and the *relative entropy of entanglement*[34] (for a review on the entanglement measures see Ref. 35).

Despite entanglement in QT has been largely investigated, the nature of entanglement in general OPTs is almost an unexplored field. Because of the physical relevance of the Fermionic field some authors[7] have recently wondered how separable states can be defined for Fermionic systems, taking into account the nonlocal action of mode creation and annihilation operators. Here we study the entanglement in FQT and show how the parity superselection derived in Subsec. 3.4 affects the features of the resulting theory.

While the notion of entangled state as a nonseparable state — i.e. a state that cannot be prepared by LOCC — can immediately be generalized to arbitrary OPTs, it is not clear whether this notion is operationally relevant in the absence of local tomography or not. For example, it may be that in order to discriminate an entangled state from a separable one, one needs bipartite effects, and then one cannot use this kind of entanglement to violate Bell-like inequalities. The nontriviality of the operational notion of Fermionic entanglement has been the focus of Ref. 7. There the authors propose four different definitions of entanglement for Fermionic systems and provide a careful analysis of their mutual relations. Fortunately, as we will see in this section, it turns out that in FQT any entangled state can be discriminated from any separable one by local effects, provided that two copies of the state are available, thus establishing nonseparability as the unique notion of entanglement in FQT.

Once an OPT is provided with a notion of "entangled state," the amount of entanglement in a given state of the theory should be quantified in operational terms. Having the notion of *entanglement of formation* a clear operational interpretation, here we will extend this measure of quantum entanglement to the Fermionic case. The entanglement of formation, introduced in Refs. 29 and 30, focuses on the resources needed in order to generate a given amount of entanglement when state manipulation is restricted to LOCC. In QT, all measures of entanglement for bipartite states refer to a standard unit: the *ebit*, which is the amount of entanglement of a bipartite *singlet state*. The entanglement of formation of a quantum state ρ represents the minimum number of ebits needed to achieve a decomposition of ρ into pure states by means of LOCC, where the minimization is over all possible decompositions. The constraint of LOCC plays a fundamental role in order to view entanglement as a *resource*. Indeed, the amount of entanglement does not increase under LOCC transformations, inducing a hierarchy of states based on their "usefulness" under LOCC operations. Accordingly a state is called *maximally entangled* when it can be transformed into any other by means of LOCC. In QT, we can find a single two-qubit state that can be used to achieve all the other two-qubit states by means of LOCC: the singlet state. As soon as we increase the dimension of quantum systems, it is no longer possible to identify a unique state we can use to get all the others.[25] The customary notion of maximally entangled state

has to be superseded by that of MES of n-partite states, namely the set of states maximally useful under LOCC manipulation, i.e. any state outside this set can be obtained via LOCC from one of the states within the set, and no state in the set can be achieved from any other state via LOCC. It is still not clear in QT whether the MES is stable once we study the asymptotic quantification of entanglement. In a general OPT, we cannot expect that the MES for bipartite states reduces to a unique maximally entangled state, as in QT for bipartite qubit entanglement.

Allowing for classical communication in LOCC implies that LOCC protocols are not completely local, introducing a complicate structure whose complete characterization is still an open problem in QT. A full theory of Fermionic entanglement would require the introduction of similar notions, involving a complete analysis of the transformations of states under LOCC, which goes beyond the scope of this paper. Nevertheless we can find some relevant features of entanglement in FQT, and show that FQT and QT are very different from the entanglement point of view.

Here is a brief summary of the results presented in this section:

(a) nonseparability is the unique notion of entanglement in FQT;
(b) there is a simple linear criterion for testing the full separability of states;
(c) Fermionic LOCC correspond to quantum LOCC with a polynomial overhead of classical communication;
(d) MES are needed also for bipartite states;
(e) there are mixed states that are not separable and with maximal entanglement of formation;
(f) there are states with maximal entanglement of formation that do not belong to a MES;
(g) the monogamy of entanglement is violated (taking as measure the Fermionic concurrence in relation with the Fermionic entanglement of formation).

Some of these results can also be found in Ref. 24.

Again, in the following we exploit the JWT to represent the states of the FQT and we will drop the J symbol for the sake of clarity.

4.3.1. *Nonseparability as the unique notion of Fermionic entanglement*

We show that in FQT any entangled state can be discriminated from any separable one by local effects, provided that two copies of the state are available. This feature stems from Theorem 2.1 and indicates nonseparability as the unique notion of entanglement in FQT.

Suppose that two states ρ and σ in $\mathrm{St}(\mathrm{AB})$ are different. This implies that there exists an effect $a \in \mathrm{Eff}(\mathrm{AB})$ such that $(a|\rho) \neq (a|\sigma)$. Either a is in $\mathrm{Eff}_{\mathbb{R}}(\mathrm{A}) \otimes \mathrm{Eff}_{\mathbb{R}}(\mathrm{B})$, in which case local measurements are sufficient to discriminate between ρ and σ, or a has a genuinely bipartite term in $\widetilde{\mathrm{Eff}}_{\mathbb{R}}(\mathrm{AB}) \coloneqq \mathrm{Eff}_{\mathbb{R}}(\mathrm{AB})/\mathrm{Eff}_{\mathbb{R}}(\mathrm{A}) \otimes \mathrm{Eff}_{\mathbb{R}}(\mathrm{B})$, where the quotient is modulo the equivalence relation $a \sim b$ iff $a - b \in \mathrm{Eff}_{\mathbb{R}}(\mathrm{A}) \otimes \mathrm{Eff}_{\mathbb{R}}(\mathrm{B})$. This implies that if we have to discriminate between $\rho \otimes \rho$ and

$\sigma\otimes\sigma$, we need an effect in $\widetilde{\mathrm{Eff}}_{\mathbb{R}}(\mathrm{AB})\widetilde{\otimes}\widetilde{\mathrm{Eff}}_{\mathbb{R}}(\mathrm{A'B'})$. Now, by Theorem 2.1, this space is also spanned by functionals in $\widetilde{\mathrm{Eff}}_{\mathbb{R}}(\mathrm{AA'})\otimes\widetilde{\mathrm{Eff}}_{\mathbb{R}}(\mathrm{BB'})$. Finally, this means that a factorized effect $c\otimes d$ with $c\in\mathrm{Eff}(\mathrm{AA'})$ and $d\in\mathrm{Eff}(\mathrm{BB'})$ is sufficient to detect entanglement between Alice's systems AA′ and Bob's BB′. Any state that is not separable is then actually entangled in any operational sense, namely its statistics on LOCC effects is different from that of any separable state. Notice also that two copies of the state are sufficient to detect entanglement.

4.3.2. *Full separability criterion for multi-*LFM *states*

Unlike QT, FQT admits a linear criterion for establishing whether a state of many LFMs is fully separable. By definition, a state of N LFMs is fully separable if it can be written as a convex combination of product states, namely

$$\rho=\sum_{i=1}p_i\rho_1^{(i)}\otimes\rho_2^{(i)}\otimes\cdots\otimes\rho_N^{(i)}\,,\quad\text{with}\quad\sum_i p_i=1\,,\quad p_i\geq 0\,. \tag{35}$$

Since the local states of the ith LFM are convex combination of $|0\rangle\langle 0|_i$ and $|1\rangle\langle 1|_i$, an arbitrary N LFMs state is fully separable if and only if it is diagonal in the Fock basis of vectors $|s_1,\ldots,s_N\rangle$. If we consider now a state of a system A made of N composite systems A_1, $\mathrm{A}_2,\ldots,\mathrm{A}_N$, by definition a state of A is separable if it can be expressed as in Eq. (35), with $\rho_j^{(i)}\in\mathrm{St}(\mathrm{A}_j)$. Then it is clear that a necessary condition for separability is that the full state ρ commutes with all local parity operators. Moreover, a state $\rho\in\mathrm{St}(\mathrm{A})$ that commutes with local parity operators is separable if and only if the projections of ρ in every parity sector correspond to density matrices of quantum separable states.

4.3.3. *Fermionic* LOCC

We will now show that every LOCC protocol in the FQT is simulated by a LOCC protocol in QT. Notice that we can find three classes of FQT transformations: (i) transformations whose Kraus operators are even (i.e. superpositions of products of even number of field operators), (ii) transformations whose Kraus operators are odd, and (iii) transformations with both even and odd Kraus operators. We can then refine every transformation $\mathcal{T}$ to a test $\{\mathcal{T}_e,\mathcal{T}_o\}$ where $\mathcal{T}_e$ has only even Kraus operators, while $\mathcal{T}_o$ has only odd ones. Thanks to this decomposition, we can prove the following lemma.

Proposition 4.3. *Every Fermionic* LOCC *corresponds to a quantum* LOCC *on qubits under* JWT.

Proof. Let $\mathrm{C}=\mathrm{C}_1\cdots\mathrm{C}_N$ be the Fermionic system made of N LFMs, and let A be one subsystems $\mathrm{A}\coloneqq\mathrm{C}_{i_1}\cdots\mathrm{C}_{i_M}$ with $i_j\in\chi_{\mathrm{A}}\subseteq\Gamma_N\coloneqq\{1,\ldots,N\}$ made of M

LFMs. Consider now the most general bipartite LOCC on C between Alice controlling the subsystem A and Bob controlling the subsystem B complementary to A (i.e. C = AB). One can always sort the LFMs in the Jordan–Wigner representation so that the first $N - M$ LFMs correspond to Bob's subsystem B. Denoting with $E_{\rm X}$ and $O_{\rm X}$ an even and an odd Kraus operator for the subsystem X, the JWT maps single-Kraus transformations local on A and B in the following way

$$J(O_A) = \bigotimes_{i\in\Gamma_N\setminus\chi_{\rm A}} \sigma_i^z \otimes O'_A\,, \quad J(E_A) = I_B \otimes E'_A\,, \tag{36}$$

$$J(O_B) = O'_B \otimes I_A\,, \qquad J(E_B) = E'_B \otimes I_A\,, \tag{37}$$

where $O'_{\rm X}$, $E'_{\rm X}$ correspond to Kraus operators of quantum maps on the subsystem X. Equations (36) and (37) show that if Alice and Bob perform a Fermionic LOCC protocol, this is equivalent to a quantum LOCC protocol in the JWT representation. Indeed, whenever Alice needs to apply a Fermionic transformation $\mathcal{T}$, she can achieve it in the qubit case by performing the test $\{\mathcal{T}_e, \mathcal{T}_o\}$, and then she just needs to tell Bob whether the event o or e occurred: in this way Bob knows if he has to apply a string of σ_z operators locally on his subsystem or not. On the other hand, Bob's Fermionic transformations are local also in the qubit case. We conclude that the JWT mapping preserves the LOCC nature of bipartite transformations, with an overhead of one classical bit at each round in order to communicate the parity of the Kraus operators.

In general, one can consider an n-partite LOCC. In this case let $\mathrm{A}_1, \ldots, \mathrm{A}_n$ be the n subsystems partitioning C, and let us sort the LFMs such that the ones belonging to the system A_i precede the ones of A_j if $i < j$. The ith party needs a bit of classical information to communicate to the $(i-1)$th party the total parity of all the Kraus operators occurred up to the $(n-i+1)$th round, in this way the $(i-1)$th party knows whether he needs to apply the string of σ_z's on his subsystems or not. Iterating this process, we find that a Fermionic n-partite LOCC corresponds under JWT to an n-partite qubit LOCC with an overhead of $n-1$ bits of classical information. □

4.3.4. *Maximally entangled sets for two* LFMs

As already stated, the concept of "maximally entangled state" has to be superseded by that of MES[25] even for two LFMs. In FQT, a single LFM ρ is operationally equivalent to a bit, so we can perform locally only the unitary gates with Kraus

$$\sigma^x\,,\ \sigma^y\,, \quad \sin\vartheta I + i\sin\vartheta\sigma^z\,, \quad \vartheta \in [0, 2\pi)\,,$$

which do not allow to transform the vectors $|0\rangle$, $|1\rangle$ into any superposition. Thus, given a state $|\Psi\rangle\langle\Psi|$ with Schmidth decomposition $|\Psi\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$, one cannot change the magnitude of the coefficients α and β by local unitary operations. By acting locally, one can simply change the parity sector by means of the Kraus σ^x,

σ^y (which locally are the same), and apply an arbitrary relative phase $\exp(2i\vartheta)$ via the Kraus $\sin\vartheta\ I + i\sin\vartheta\ \sigma^z$.

We can moreover get any arbitrary factorized state of the FQT — i.e. projections on $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ — from any state in the MES by means of LOCC operations: Alice measures her LFM in the computational basis by the Kraus operators $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and conditionally on the outcome she tells Bob the local operation he has to apply on his LFM — i.e. the identity with Kraus I or the bit flip with Kraus σ^x. Clearly, one cannot do the opposite. Hence, examples of MES's for two LFMs are given by MES_0, MES_1, which are defined as

$$\mathrm{MES}_0 \coloneqq \{|\Psi_{\alpha,\beta}\rangle\langle\Psi_{\alpha,\beta}| \mid |\Psi_{\alpha,\beta}\rangle := \alpha\,|00\rangle + \beta\,|11\rangle\,,\ \alpha,\beta > 0\}\,,$$

$$\mathrm{MES}_1 \coloneqq \{|\Psi_{\alpha,\beta}\rangle\langle\Psi_{\alpha,\beta}| \mid |\Psi_{\alpha,\beta}\rangle := \alpha\,|01\rangle + \beta\,|10\rangle\,,\ \alpha,\beta > 0\}\,.$$

4.3.5. *The Fermionic entanglement of formation*

In the usual QT scenario, the *entanglement cost* of a given, generally entangled, state $\rho \in \mathrm{St(AB)}$ shared by distant observers Alice and Bob quantifies the amount of resources needed by the two parties in order to create the state ρ. Consider then the protocol

$$|\Sigma\rangle\langle\Sigma|^{\otimes m} \xrightarrow{\text{LOCC}} \rho^{\otimes n}\,,$$

where m singlet states $|\Sigma\rangle\langle\Sigma|$ are converted into n copies of the target state ρ by means of LOCC. Perfect transformation by LOCC is usually impossible and one requires it only asymptotically, say in the limit where the number of created copies of ρ approaches infinity. The entanglement cost E_c is thus defined as the optimal asymptotic ratio $r = m/n$. The last one is very difficult to compute, while the *entanglement of formation*, which also has an operational interpretation, can be more easily computed in terms of the density matrix ρ.

The definition of entanglement of formation is based on the result of Ref. 36 for the entanglement cost of pure states. In the paper, the authors show that the entanglement cost of a pure state $\rho = |\Psi\rangle\langle\Psi|$ coincides with the von Neumann entropy of either of its marginal states, say $E_c(|\Psi\rangle\langle\Psi|) = S(\mathrm{Tr_A}\,|\Psi\rangle\langle\Psi|)$. Therefore to produce $|\Psi\rangle\langle\Psi|^{\otimes n}$ one needs $m \approx nS(\mathrm{Tr_A}\,|\Psi\rangle\langle\Psi|)$ singlets with the equality achieved in the asymptotic limit. The entanglement of formation of a mixed state $\rho \in \mathrm{St(AB)}$ is then defined as

$$E(\rho) \coloneqq \min_{\mathcal{D}_\rho} \sum_i p_i S(\mathrm{Tr_A}\,|\Psi_i\rangle\langle\Psi_i|)\,, \tag{38}$$

where

$$\mathcal{D}_\rho \coloneqq \left\{ \{p_i, |\Psi_i\rangle\} \,\middle|\, \rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \right\}$$

is the set of all the pure decompositions of the mixed state ρ. The operational interpretation[b] of the entanglement of formation has been pointed out by Wootters in Ref. 32, where it is noticed that

$$E(\rho) \equiv \lim_{n\to\infty} m_n(\rho)/n\,, \tag{39}$$

with $m_n(\rho)$ the minimum number of singlet states needed by two parties to prepare via LOCC random tensor products $\bigotimes_{l=1}^{n} |\Psi_{i_l}\rangle\langle\Psi_{i_l}|$ of states in a decomposition $\{p_i, |\Psi_i\rangle\langle\Psi_i|\}$ of ρ, sampled by the distribution $p(i_1,\ldots,i_n) = p(i_1)\cdots p(i_n)$, minimized over all possible decompositions:

$$|\Sigma\rangle\langle\Sigma|^{\otimes m} \xrightarrow{\text{LOCC},\,\mathcal{D}_\rho} \rho^{\otimes n}\,.$$

In Refs. 30 and 31, a formula is also provided for evaluating the entanglement of formation (38) of a state ρ just in terms of its density matrix. For a mixed state ρ of two qubits, one has

$$E(\rho) = \mathcal{E}(C(\rho))\,, \tag{40}$$

with $\mathcal{E}(x) := h\Big(\frac{1+\sqrt{1-x^2}}{2}\Big)$, h the binary Shannon entropy, and the expression of the *concurrence* $C(\rho)$ depending only on the density matrix ρ (see Refs. 30 and 31 for the explicit formula of the concurrence). As for the entanglement of formation, also the concurrence of a generally mixed state ρ is given by

$$C(\rho) := \min_{\mathcal{D}_\rho} \sum_i p_i C(|\Psi_i\rangle\langle\Psi_i|)\,. \tag{41}$$

Both the entanglement of formation and the concurrence are zero if and only if the state ρ is separable, and for two qubits they reach the maximum value 1, if and only if ρ is a maximally entangled state.

In analogy to the quantum case we can define the operational Fermionic entanglement of formation. Given a Fermionic state $\rho = p_0\rho_0 + p_1\rho_1$ its entanglement of formation is defined as

$$E_{\mathrm{F}}(\rho) = \lim_{n\to\infty} m_n(\rho)/n\,, \tag{42}$$

with $m_n(\rho)$ the minimum number of states in a Fermionic MES needed by two parties to prepare via Fermionic LOCC random tensor products $\bigotimes_{l=1}^{n} |\Psi_{i_l}\rangle\langle\Psi_{i_l}|$ of states in

[b]Notice that the entanglement of formation of a mixed state ρ is not proven to correspond to its entanglement cost, and in general it is $E(\rho) \geq E_c(\rho)$. However, in Ref. 37 it has been shown that

$$E_c(\rho) = \lim_{n\to\infty} E(\rho^{\otimes n})/n\,,$$

where the right hand side of the equality is the so called *regularized entanglement of formation*. If the entanglement of formation turns out to be additive, the entanglement cost will be equal to the entanglement of formation.

a decomposition $\{p_i, |\Psi_i\rangle\langle\Psi_i|\}$ of ρ, sampled by the distribution $p(i_1, \ldots, i_n) = p_{i_1} \cdots p_{i_n}$, minimized over all possible decompositions:

$$\bigotimes_{k=1}^{m} |\Sigma_k\rangle\langle\Sigma_k| \quad (|\Sigma_k\rangle\langle\Sigma_k| \in \text{MES}) \xrightarrow{\text{LOCC}_\text{F},\, \mathcal{D}_\rho^\text{F}} \rho^{\otimes n}\,. \tag{43}$$

It is important to notice that the Fermionic entanglement of formation of a mixed state corresponds to the convex-roof extension of the Fermionic entanglement of formation of pure states, as follows

$$E_\text{F}(\rho) = \min_{\mathcal{D}_\rho^\text{F}} \sum_i p_i E_F(|\Psi_i\rangle\langle\Psi_i|)\,, \tag{44}$$

where $\mathcal{D}_\rho^\text{F}$ is the set of all the pure decompositions of ρ satisfying the parity superselection rule.[c] For pure states, we have the following result.

Proposition 4.4. *For pure states* $|\Psi\rangle\langle\Psi|$*, the function*

$$\tilde{E}_F(|\Psi\rangle\langle\Psi|) := S(\text{Tr}_A\, |\Psi\rangle\langle\Psi|) \tag{45}$$

is a lower bound for the Fermionic entanglement of formation (42)*.*

Proof. First notice that $E_\text{F}(|\Psi\rangle\langle\Psi|)$ in Eq. (42) corresponds to the maximal rate of conversion of states in the Fermionic MES to the state $|\Psi\rangle\langle\Psi|$ via fermionic LOCCs, as in Eq. (43). Now consider the following protocol for qubit states

$$|\Sigma\rangle\langle\Sigma|^{\otimes m'} \xrightarrow{\text{LOCC}} \bigotimes_{k=1}^{m} |\Sigma_k\rangle\langle\Sigma_k| \quad (|\Sigma_k\rangle\langle\Sigma_k| \in \text{MES}) \xrightarrow{\text{LOCC}_\text{F}} |\Psi\rangle\langle\Psi|^{\otimes n}\,, \tag{46}$$

where m' quantum singlets are converted via quantum LOCC into m states in the Fermionic MES that are then converted at a rate $E_\text{F}^\text{op}(|\Psi\rangle\langle\Psi|)$ to n copies of the target state $|\Psi\rangle\langle\Psi|$ via Fermionic LOCC. Since any Fermionic state in the MES has a quantum entanglement of formation smaller than (or equal to) 1, the protocol (46) allows for a conversion rate

$$\frac{m'}{m} E_\text{F}(|\Psi\rangle\langle\Psi|) \leq E_\text{F}(|\Psi\rangle\langle\Psi|)\,. \tag{47}$$

Moreover, since any Fermionic LOCC is also a quantum LOCC (see Proposition 4.3), the protocol (46) is a particular instance of the general protocol for LOCC conversion of m' singlet states to n copies of the target state $|\Psi\rangle\langle\Psi|$, and then we have

$$\tilde{E}_\text{F}(|\Psi\rangle\langle\Psi|) = E(|\Psi\rangle\langle\Psi|) \leq \frac{m'}{m} E_\text{F}(|\Psi\rangle\langle\Psi|) \leq E_\text{F}(|\Psi\rangle\langle\Psi|)\,. \tag{48}$$

This proves the thesis. □

[c]In Ref. 38, the authors do the same for RQT considering the decompositions $\mathcal{D}_\rho^\text{R}$ on real states.

Now, if we extend the definition of $\tilde{E}_{\rm F}(\rho)$ to mixed states by convex-roof extension, we have

$$\tilde{E}_{\rm F}(\rho) \coloneqq \min_{\mathcal{D}^{\rm F}_\rho} \sum_i p_i \tilde{E}_{\rm F}(|\Psi_i\rangle\langle\Psi_i|) = p_0 E(\rho_0) + p_1 E(\rho_1)\,, \tag{49}$$

$$C_{\rm F}(\rho) \coloneqq \min_{\mathcal{D}^{\rm F}_\rho} \sum_i p_i C(|\Psi_i\rangle\langle\Psi_i|) = p_0 C(\rho_0) + p_1 C(\rho_1)\,, \tag{50}$$

where we introduced the quantity $C_{\rm F}(\rho)$ that extends the notion of *concurrence* to the Fermionic case.[d] The last equalities in Eqs. (49) and (50) are obtained upon noticing that the state ρ admits the unique parity decomposition $\rho = p_0\rho_0 + p_1\rho_1$, with $p_0 + p_1 = 1$ and ρ_0, ρ_1 states in the even and odd parity sector respectively, and that all decompositions in $\mathcal{D}^{\rm F}_\rho$ shall preserve the probabilities p_0 and p_1. Moreover, since $\mathcal{D}^{\rm F}_{\rho_i} \equiv \mathcal{D}_{\rho_i}$, we have $\tilde{E}_{\rm F}(\rho_i) = E(\rho_i)$ and $C_{\rm F}(\rho_i) = C(\rho_i)$. Notice that for a pure state $|\Psi\rangle\langle\Psi|$, we have $C_F(|\Psi\rangle\langle\Psi|) = C(|\Psi\rangle\langle\Psi|)$.

Now, thanks to Proposition 4.4, we clearly have

$$\sum_i p_i \tilde{E}_{\rm F}(|\Psi_i\rangle\langle\Psi_i|) \leq \sum_i p_i E_{\rm F}(|\Psi_i\rangle\langle\Psi_i|)\,, \tag{51}$$

for every Fermionic pure state decomposition $\{p_i, |\Psi_i\rangle\}$ of ρ, and then by Eq. (44)

$$\tilde{E}_{\rm F}(\rho) \leq E_{\rm F}(\rho)\,. \tag{52}$$

Notice that, unlike in QT[30] and in RQT,[38] the quantities $E_{\rm F}$ and $C_{\rm F}$ do not satisfy the relation $E_{\rm F}(\rho) = \mathcal{E}(C_{\rm F}(\rho))$ (see Eq. (40)). On the other hand it is $\tilde{E}_{\rm F}(\rho) \geq \mathcal{E}(C_{\rm F}(\rho))$, and for a state Φ with $C_{\rm F}(\Phi) = 1$ we have $\tilde{E}_{\rm F}(\Phi) = \mathcal{E}(C_{\rm F}(\Phi)) = 1$. Therefore, when $C_{\rm F}(\rho) = 1$, the quantity $\tilde{E}_{\rm F}$ coincides with the operational Fermionic entanglement of formation $E_{\rm F}$.

4.3.6. *Mixed states with maximal entanglement of formation*

Using the quantities $E_{\rm F}$ and $C_{\rm F}$, and the separability criterion, we can show that in FQT there are mixed states with maximal entanglement of formation. Consider the state

$$\Phi \coloneqq \frac{1}{4}(I \otimes I + \sigma^x \otimes \sigma^x)\,, \tag{53}$$

corresponding to the mixture with $p = 1/2$ of the Fermionic pure states $|\Psi_0\rangle\langle\Psi_0|$ and $|\Psi_1\rangle\langle\Psi_1|$ with

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\,, \quad |\Psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\,.$$

[d]The expressions (49) and (50) were already proposed in Ref. 7. Here, we show that Eq. (49) provides a lower bound for the Fermionic entanglement of formation.

Despite being mixed, Φ has maximal entanglement of formation and concurrence

$$E_{\rm F}(\Phi) = \frac{1}{2}E(|\Psi_0\rangle\langle\Psi_0|) + \frac{1}{2}E(|\Psi_1\rangle\langle\Psi_1|) = 1\,,$$

$$C_{\rm F}(\Phi) = \frac{1}{2}C(|\Psi_0\rangle\langle\Psi_0|) + \frac{1}{2}C(|\Psi_1\rangle\langle\Psi_1|) = 1\,.$$

It is easy to verify that Φ is not separable; indeed Φ does not satisfy the separability criterion of Subsec. 4.3.2

$$\mathrm{Tr}\big[\rho_{\rm sep}(\sigma^i \otimes \sigma^j)\big] = 0\,, \quad i,j = x,y\,.$$

Other mixed maximally entangled states can be found by replacing every occurrence of σ_i^x with an arbitrary linear combination of σ_i^x and σ_i^y in Eq. (53). Notice that all these states, which have maximal entanglement of formation, do not belong to a MES (see Subsec. 4.3.4).

Also RQT has mixed maximally entangled states. Being the rebit defined by the linear constraint $\mathrm{Tr}[\sigma_y\rho] = 0$, in RQT a mixed maximally entangled state is achieved by replacing σ_x with σ_y in the state of Eq. (53).[38]

Notice that the state (53) is separable in QT, since it is the mixture with $p = 1/2$ of the pure product states $|\Pi_+\rangle\langle\Pi_+|$ and $|\Pi_-\rangle\langle\Pi_-|$ with

$$|\Pi_+\rangle := |+\rangle|+\rangle\,, \quad |\Pi_-\rangle := |-\rangle|-\rangle \quad \text{and} \quad |\pm\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\,. \tag{54}$$

Such a decomposition is not allowed neither in FQT nor in RQT, because of the violation of their respective superselection rules by the vectors $|\pm\rangle$.

4.3.7. *Violation of entanglement monogamy*

The shareability of correlations between many parties is one of the main differences between quantum and classical correlations. While in the classical information theory correlations can be shared among arbitrary many parties, in QT a system maximally entangled with a second system cannot share quantum correlations with a third one. This has been dubbed the "monogamy of entanglement" and a big effort has been devoted to its quantification: see Refs. 39–46, or Refs. 47, 48 for a recent review on the subject.

Entanglement monogamy is usually stated by means of inequalities involving some entanglement measures, i.e.

$$M(\rho_{\rm AB}) + M(\rho_{\rm AC}) \leq M(\rho_{\rm A(BC)})\,, \tag{55}$$

where $M(\rho_{\rm AB})$ is a measure of the entanglement between systems A and B. It is worth mentioning that not every entanglement measure satisfies the inequality of Eq. (55), so not all the entanglement measures are good indicators for monogamy. A measure satisfying the inequality (55) is called *monogamous*. In Ref. 43, it has been shown that in QT, the concurrence is monogamous and satisfies

$$C^2(\rho_{\rm AB}) + C^2(\rho_{\rm AC}) \leq 1\,. \tag{56}$$

Notice that, if ρ_{AB} has maximal concurrence — $C(\rho_{\mathrm{AB}}) = 1$ — then ρ_{AC} must have concurrence equal to 0.

The Fermionic entanglement (say the Fermionic concurrence) is not monogamous. For instance, consider the pure state of three LFMs $|\Phi'\rangle\langle\Phi'|$ with

$$|\Phi'\rangle := \tfrac{1}{2}(|000\rangle + |110\rangle + |011\rangle + |101\rangle)\,. \tag{57}$$

Tracing the state $|\Phi'\rangle\langle\Phi'|$ over any one of the three LFMs, we find that the reduced bipartite state is the mixed state Φ of Eq. (53), having maximal entanglement of formation and concurrence. Therefore, in the FQT as well as in RQT[26] each pair of subsystems can share any amount of entanglement of formation.

4.4. *Fermionic computation*

Recently some authors have been wondering whether models of Fermionic quantum computation might support universal computation and/or exhibit different computational power with respect to the standard quantum computational model. As already stressed, one can build different computational models based on LFMs, according to: (i) the degree of superselection on the states (e.g. conservation of the parity number instead of the total excitation number), and (ii) the admitted transformations of the theory. In Ref. 1 Bravyi and Kitaev considered an LFM computational model with a parity superselection where the unitary transformations are the parity-preserving ones (i.e. the CP maps with a single-Kraus operator which is a linear combination of products of an even number of field operators). They showed that such a computational model supports universal computation and that it can be simulated by regular unitary gates of qubits with a computational overhead that goes as the logarithm of the number of the LFMs, thus proving the computational equivalence of the two models. The same result can be extended to the FQT presented in this paper which is the largest computational model based on LFMs satisfying the assumptions (i)–(viii) in Subsec. 3.2.

In extending the results of Ref. 1 to the FQT, we also review the original proofs for the sake of completeness. The proofs for the FQT relies on the following observation: unitary transformations of Fermionic quantum computation of Ref. 1 are parity-preserving, while the FQT allows also parity-changing transformations, i.e. the sets of transformations of the FQT are strictly larger than the ones considered in Ref. 1. However, a parity-nonpreserving map $\mathcal{T}$ on N LFMs — i.e. $\mathcal{T}$ has Kraus operators that are linear combinations of products of an odd number of field operators — can always be seen as the sequential composition $\mathcal{X}_i \circ \mathcal{X}_i{}^{-1} \circ \mathcal{T}$ acting on N LFMs, where $\mathcal{X}_i$ is the unitary map that flips the ith LFM from occupied to unoccupied, and vice versa. Notice that $\mathcal{X}_i{}^{-1} \circ \mathcal{T}$ is now parity-preserving. We conclude therefore that in the FQT a nonparity-preserving map can be seen as the sequential composition $\mathcal{X}_i \circ \mathcal{R}$ of a parity-preserving map $\mathcal{R}$ and a local flip $\mathcal{X}_i$.

4.4.1. *Universality of computation*

We want to prove that in the FQT, there is a finite set of Fermionic gates that allows us to build every FQT circuit. Given a system of N LFMs, there are the parity-preserving transformations and the parity changing ones, which can be written as the sequential composition $\mathcal{X}_i \circ \mathcal{R}$, with $\mathcal{R}$ parity-preserving. In Ref. 1 a universal set Υ of LFM gates for the parity-preserving transformations is given; then it follows that a universal set for the FQT is given by $\Upsilon \cup \{\mathcal{X}_i\}$ for some $i \in \{1, \ldots, N\}$.

Let us now review the derivation of the universal set for parity-preserving transformations. The proof[1] relies on the universality of computation in QT, and on the possibility of expressing every parity-preserving LFM gate by means of qubit gates. It is important to notice some differences between the qubit computation and the Fermionic one. Consider a gate $\mathcal{G}$ acting on M qubits. Such a gate is represented by a unitary operator G acting on $(\mathbb{C}^2)^{\otimes M}$. When such a gate is used in a quantum circuit of $N > M$ qubits, its operator representative is always given by the unitary operator $G \otimes I$, modulo a relabeling of the subsystems; more precisely since the Hilbert space of N qubits $(\mathbb{C}^2)^{\otimes N}$ can be identified with $(\mathbb{C}^2)^{\otimes M} \otimes (\mathbb{C}^2)^{\otimes (N-M)}$ by the qubit permutation $P : |s_1, \ldots, s_N\rangle_{\mathrm{Q}} \mapsto |s_{j_1}, \ldots, s_{j_M}\rangle_{\mathrm{Q}} \otimes |s_{j_{M+1}}, \ldots, s_{j_N}\rangle_{\mathrm{Q}}$, the action of $\mathcal{G}$ on the qubits $s_{j_1}, \ldots, s_{j_M}$ is given by the operator

$$\tilde{G}(j_1, \ldots, j_M) := P^{-1} G \otimes I P \,. \tag{58}$$

Clearly such a property is of paramount importance for the universality of computation, since the gate $\mathcal{G}$ is "always" represented by the operator G irrespective of the number of the qubits of the whole circuit, and irrespective of the specific choice of the qubits the gate acts on.

In the LFM scenario the situation is very different due to the CAR. For instance, a two-LFM gate $\mathcal{F}$ behaves differently depending on the LFM subsystems $\mathcal{F}$ it acts on. For example, let $\varphi_1^\dagger \varphi_2$ be a parity-preserving Fermionic operator; when it is applied to the LFMs j_1, j_2 of a multipartite system of N LFMs it behaves differently depending on the chosen ordering for the N subsystems, since

$$\begin{aligned} &\varphi_{j_1}^\dagger \varphi_{j_2} |\ldots, s_{j_1}, \ldots, s_{j_2}, \ldots\rangle_{\mathrm{F}} \\ &\qquad = \delta_{s_{j_1},0} \times \delta_{s_{j_2},1} \times (-1)^{\sum_{k=j_1+1}^{j_2-1} s_k} \times |\ldots, 1, s_{j_1+1}, \ldots, s_{j_2-1}, 0, \ldots\rangle_{\mathrm{F}} \,. \end{aligned}$$

When we represent the LFM gate $\varphi_{j_1}^\dagger \varphi_{j_2}$ by means of qubits, such a difference in behavior is taken into account by the JWT thanks to the σ_k^z operators at the qubit subsystems ranging from $k = j_1 + 1$ to $k = j_2 - 1$. This fact has the following consequence: a LFM operator has many qubit representations according to the total number of LFMs involved. However, whenever j_1 and j_2 are nearest neighbors there is no contribution from the coefficient $(-1)^{\sum_{k=j_1+1}^{j_2-1} s_k}$ and every two-LFM parity-preserving gate acting on nearest neighbor LFMs admits an unambiguous qubit representation made of parity-preserving qubit gates (by means of the JWT). The same result holds for one-LFM parity-preserving transformations — due to the fact

that parity-preserving transformations are linear combinations of products of an even number of field operators.

This allows us to represent an arbitrary two-LFM gate $\mathcal{T}(j,k)$ acting on the LFMs j, k (w.l.o.g. $j<k$) by means of qubits in an unambiguous way. Indeed, let us call by $S_{\rm F}(j,j+1)$ the Kraus operator of the unitary transformation performing the swap between the jth and the $(j+1)$th LFM, i.e. $S_{\rm F}(j,j+1)\varphi_j S_{\rm F}(j,j+1)^\dagger=\varphi_{j+1}$, and $S_{\rm F}(j,j+1)\varphi_{j+1}S_{\rm F}(j,j+1)^\dagger=\varphi_j$, namely $S_{\rm F}(j,j+1)=I-\varphi_j^\dagger\varphi_j-\varphi_{j+1}^\dagger\varphi_{j+1}+\varphi_{j+1}^\dagger\varphi_j+\varphi_j^\dagger\varphi_{j+1}$. Such an operator acts in the following way: $S_{\rm F}(j,j+1)|\ldots,s_j,s_{j+1},\ldots\rangle_{\rm F}=(-1)^{j(j+1)}|\ldots,s_{j+1},s_j,\ldots\rangle_{\rm F}$. Since the swap between the jth and the $(j+1)$th qubit of a circuit is given by the swap operator $S_{\rm Q}(j,j+1):|\ldots,s_j,s_{j+1},\ldots\rangle_{\rm Q}\mapsto|\ldots,s_{j+1},s_j,\ldots\rangle_{\rm Q}$, we have that

$$J(S_{\rm F}(j,j+1))=S_{\rm Q}(j,j+1)D(j,j+1)\,,$$

where $D(j,j+1):|\ldots,s_j,s_{j+1},\ldots\rangle_{\rm Q}\mapsto(-1)^{j(j+1)}|\ldots,s_j,s_{j+1},\ldots\rangle_{\rm Q}$ is the so-called *swap defect operator*. Notice that also the swap defect operator D is parity-preserving and nearest-neighbor. Since

$$\mathcal{T}(j,k)\equiv S_{\rm F}(k-1,k)\cdots S_{\rm F}(j+1,j+2)\mathcal{T}(j,j+1)S_{\rm F}(j+1,j+2)\cdots S_{\rm F}(k-1,k)\,, \tag{59}$$

we have that

$$\begin{aligned}J(\mathcal{T}(j,k))=&D(k-1,k)\cdots D(j+1,k)S_{\rm Q}(k-1,k)\cdots\\&\times S_{\rm Q}(j+1,j+2)J(\mathcal{T}(j,j+1))S_{\rm Q}(j+1,j+2)\cdots\\&\times S_{\rm Q}(k-1,k)D(j+1,k)\cdots D(k-1,k)\,.\end{aligned} \tag{60}$$

Hence we have found that an arbitrary two-LFM parity-preserving operator is equivalent to a Fermionic circuit involving only gates on nearest neighbor LFMs (Eq. (59)), which can therefore be represented unambiguously by the parity-preserving qubit circuit of Eq. (60). This method works also for operators which act on more than two LFMs. Notice that the term $S_{\rm Q}(j+1,j+2)\cdots S_{\rm Q}(k-1,k)$ in Eq. (60) is just the permutation P of Eq. (58).

Due to the equivalence between parity-preserving Fermionic gates and parity-preserving qubit gates we only need a universal set of parity-preserving qubit gates in order to get a universal set of Fermionic parity-preserving unitary transformations. A universal set for the qubits is given by[1]

$$\Lambda(e^{i\pi/4})\,,\quad\Lambda(\sigma^z)\equiv D\,,\quad\tilde{H}:|a,b\rangle_{\rm Q}\mapsto\frac{1}{\sqrt{2}}\sum_c(-1)^{bc}|a\oplus b\oplus c,c\rangle_{\rm Q}\,, \tag{61}$$

where $\Lambda(U)$ denotes the controlled U with the control system corresponding to the first qubit.

The proof of universality of gates in Eq. (61) proceeds as follows: (i) it is observed that any parity-preserving qubit gate U can be considered as a block-diagonal operators

$$U = \left(\begin{array}{c|c} W_0 & 0 \\ \hline 0 & W_1 \end{array}\right), \tag{62}$$

where W_i acts on the parity sector $\mathscr{H}_i = (\mathbb{C}^2)^{\otimes N-1}$ of the Hilbert space $(\mathbb{C}^2)^{\otimes N}$; (ii) it is shown how to get any parity-preserving operator having $W_0 = W_1$; (iii) the operators having $W_0 = I$ and $W_1 = Y$, which transform the operators having $W_0 = W_1$ to the general form of Eq. (62), are constructed.

Notice that any operator G on $M-1$ qubits can be turned into a parity-preserving one $\tilde{G}$ on M qubits by using an ancillary qubit:

$$\begin{gathered} \tilde{G} = V_M(I \otimes G)V_M\,, \\ V_M^{-1} = V_M : |s_1, \ldots, s_M\rangle_{\mathrm{Q}} \mapsto |s_1 \oplus \cdots \oplus s_M,\, s_2, \ldots, s_M\rangle_{\mathrm{Q}}\,. \end{gathered} \tag{63}$$

Indeed, the unitary operator V_M maps the parity sector $|j\rangle \otimes \mathscr{H}_i$ of M qubits onto the subspace $|i \oplus j\rangle \otimes \mathscr{H}_i$, and then $\tilde{G}$ is parity-preserving, even if G is not. Notice that if G already preserves the parity then $\tilde{G} = I \otimes G$. This is the case of the first two operators $\Lambda(e^{i\pi/4})$ and $\Lambda(\sigma^z)$ in Eq. (61), while the last universal gate $\tilde{H}$ is the parity-preserving extension of the usual Hadamard gate H. Since for every unitary G on $N-1$ qubits the unitary $\tilde{G}$ of Eq. (63) is parity-preserving, we have that $\tilde{G}$ is of the form of Eq. (62). On the other hand, one can easily check that $V_M(\sigma^x \otimes I_{M-1})V_M = \sigma^x \otimes I_{M-1}$, hence $[\sigma^x \otimes I_{M-1}, \tilde{G}] = 0$. Moreover, since $(\sigma^x \otimes I_{M-1})\,|i\rangle \otimes |\xi_j\rangle = |i \oplus 1\rangle \otimes |\xi_j\rangle$, if we identify the bases $|0\rangle \otimes |s_1, \ldots, s_{M-1}\rangle$ and $|1\rangle \otimes |s_1, \ldots, s_{M-1}\rangle$ in the subspaces $|i\rangle \otimes \mathscr{H}_i$, we have

$$\sigma^x \otimes I_{M-1} = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right),$$

which implies that for any $M-1$-qubits gate G the parity-preserving extension $\widetilde{G}$ has $W_0 = W_1$.

Since Eq. (63) defines a $*$-algebra homomorphism, any universal set of gates is mapped to a set of parity-preserving gates that is universal on the even sector. The set of gates $\{\Lambda(e^{i\pi/4}),\, \Lambda(\sigma^z),\, H\}$ is known to be universal, then the corresponding parity-preserving set given by Eq. (61) must be universal. Notice that the homomorphism (63) satisfies the property: $\widetilde{\Lambda(X)} = S^{c,p}\Lambda(\tilde{X})S^{c,p}$ where $S^{c,p}$ is a swap between the control and the parity qubits.

We conclude that the set of Eq. (61) is universal for parity-preserving unitary gates having $W_0 = W_1$. We can use the same set to build parity-preserving unitary operators K with $W_0 = I$ and $W_1 = Y$ to correct the first step. We add one ancillary qubit at the end of our M qubits. Let us define the operator

$$Z : |s_1, \ldots, s_M, s_{M+1}\rangle_{\mathrm{Q}} \mapsto |s_1 \oplus \cdots \oplus s_M, s_2, \ldots, s_M, s_2 \oplus \cdots \oplus s_{M+1}\rangle_{\mathrm{Q}}\,.$$

Let K be a parity-preserving unitary operator with $W_0 = I$ and $W_1 = Y$, and let $\tilde{H}$ have diagonal blocks $W_0 = W_1 = Y$. Denoting by P the permutation $|s_1, s_2, \ldots, s_M, s_{M+1}\rangle_{\rm Q} \mapsto |s_1, s_{M+1}, s_2, \ldots, s_M\rangle_{\rm Q}$, we have

$$Z^{-1}\, P^{-1}(\Lambda(\tilde{H}) \otimes I)PZ = (V^{-1}\Lambda(H)V) \otimes I = K \otimes I\,. \tag{64}$$

We just need to represent the operator Z by means of the operators in the universal set of Eq. (61). This task can be easily accomplished upon noticing that $Z \equiv \Lambda(\widetilde{\sigma^x})(m-1,0,m)\Lambda(\widetilde{\sigma^x})(m-2,0,m)\cdots\Lambda(\widetilde{\sigma^x})(1,0,m)$, where $\Lambda(\widetilde{\sigma^x})(j_1,j_2,j_3)$: $|\ldots, s_{j_1}, \ldots, s_{j_2}, \ldots, s_{j_3}, \ldots\rangle_{\rm Q} \mapsto |\ldots, s_{j_1}, \ldots, s_{j_2} \oplus s_{j_1}, \ldots, s_{j_3} \oplus s_{j_1}, \ldots\rangle_{\rm Q}$. Now, the operator $\Lambda(\widetilde{\sigma^x})$ acting on the qubits A, B, and C can be expressed in terms of the universal set as $\Lambda(\widetilde{\sigma^x})_{\rm ABC} = (\tilde{H}_{\rm AC} \otimes I_{\rm B})\ (\Lambda(\sigma^z)_{\rm BC} \otimes I_{\rm A})\ (\tilde{H}_{\rm AC} \otimes I_{\rm B})$.

Now we just need to represent the gates of Eq. (61) in terms of the creation and the annihilation operators. The first two operators are

$$\Lambda(e^{i\pi/4}) = \exp\left(i\frac{\pi}{4}\varphi_0^\dagger\varphi_0\right), \quad \Lambda(\sigma^z) = \exp\Big(i\pi\varphi_0^\dagger\varphi_0\varphi_1^\dagger\varphi_1\Big)\,. \tag{65}$$

The gate $\tilde{H}$ can be represented in the LFM case by means of the decomposition

$$\tilde{H} = [I \otimes \Lambda(-i)] \cdot \tilde{G} \cdot [I \otimes \Lambda(-i)]\,, \quad G = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Hence a universal set for the parity-preserving gates of the FQT is given by the gates of Eq. (65) together with

$$\begin{aligned}\tilde{G} &= \exp\left[-i\frac{\pi}{4}\Big(\varphi_0 - \varphi_0^\dagger\Big)\Big(\varphi_1 + \varphi_1^\dagger\Big)\right] \\ &= \exp\left[i\frac{\pi}{4}\Big(\varphi_0^\dagger\varphi_1 + \varphi_1^\dagger\varphi_0\Big)\right]\exp\left[i\frac{\pi}{4}\Big(\varphi_1\varphi_0 + \varphi_0^\dagger\varphi_1^\dagger\Big)\right].\end{aligned}$$

4.4.2. *Simulation*

We now address the issue of simulating a qubit circuit by means of a FQT circuit, and vice versa. The proof of the universality given in the previous section gives already a way to simulate a LFM circuit by means of qubits, relying on the Jordan–Wigner isomorphism between the qubit algebra and the Fermionic one. Moreover, thanks to Subsec. 4.3.3, we know that a Fermionic LOCC can be simulated by a LOCC on qubits. In order to address the simulation in the other way round, and to tight the simulation cost of the previous section, we will present the scheme of Ref. 1 which does not rely on the identification $|s_1, \ldots, s_N\rangle_{\rm F} \leftrightarrow |s_1, \ldots, s_N\rangle_{\rm Q}$ — i.e. the JWT. As we will see, this time the scheme of Ref. 1 works out of the box even for the FQT. In the following, for the sake of convenience, we will label the LFM and the qubit systems starting from "zero," and not from "one" as we did in the rest of the paper.

Given a circuit of the FQT, a procedure to simulate a K-LFM gate $\mathcal{T}$ can be summarized as: (i) we embed the M LFMs system in a M qubits system, (ii) we add K ancillary qubits initialized in the state $|0,\ldots,0\rangle_{\mathrm{Q}}$, (iii) we exchange the qubits corresponding to the LFMs involved in the computation with the ancillas, taking into account possible global phases due to the anticommutation relation of the original LFM systems, (iv) we perform the computation on the ancilla by means of the corresponding qubit gate, (v) we revert the extracted qubits in their original position, (vi) we re-encode the resulting qubits — excluding the ancillary qubits — in the original M LFMs.

Clearly, one possible way of encoding is given by the JWT, namely $|s_0,\ldots,s_{M-1}\rangle_{\mathrm{F}} \leftrightarrow |s_0,\ldots,s_{M-1}\rangle_{\mathrm{Q}}$. This is actually the same encoding used in the previous section in order to derive the universal set for the Fermionic computation. In such a case, the process of embedding and of extraction (of the jth LFM) is synthetically given by

$$\begin{aligned}
&|s_0,\ldots,s_j,\ldots,s_{M-1}\rangle_{\mathrm{F}}\\
&\quad\to |s_0,\ldots,s_j,\ldots,s_{M-1}\rangle_{\mathrm{Q}} \to |0,s_0,\ldots,s_j,\ldots,s_{M-1}\rangle_{\mathrm{Q}}\\
&\quad\to (-1)^{s_j\oplus_{i=0}^{j-1}s_i}|s_j,s_0,\ldots,0,\ldots,s_{M-1}\rangle_{\mathrm{Q}}\,. \qquad (66)
\end{aligned}$$

A simulation scheme resorting to the above JWT encoding is not very efficient, since every time we perform the extraction of one qubit we shall evaluate a phase given by the coefficient $(-1)^{s_j\oplus_{i=0}^{j-1}s_i}$ of Eq. (66). Therefore, in the worst case scenario, for every one-LFM gate we have to use $O(M)$ qubit gates. We can do better using a different embedding.

Let us introduce a partial ordering in the space of the binary strings: we say that the binary string $\alpha := \alpha_{T-1}\cdots\alpha_0$ precedes $\beta := \beta_{T-1}\cdots\beta_0$, and we write $\alpha \preceq \beta$, whenever for some $0 \le l_0 \le T-1$ we have $\alpha_l = \beta_l$ for $l \ge l_0$, and $\beta_l = 1$ for $l < l_0$. If we denote with j_{bin} the binary string corresponding to the decimal number j, we have $j_{\mathrm{bin}} \prec k_{\mathrm{bin}} \implies j < k$, where clearly $j_{\mathrm{bin}} \prec k_{\mathrm{bin}} \Leftrightarrow (j_{\mathrm{bin}} \preceq k_{\mathrm{bin}}) \wedge (j_{\mathrm{bin}} \neq k_{\mathrm{bin}})$. Notice that given a binary string $j_{\mathrm{bin}} = \alpha$ of length T, there are at most T binary strings k_{bin} of the same length satisfying the relation $j_{\mathrm{bin}} \preceq k_{\mathrm{bin}}$. Indeed, since for every $0 \le l_0 \le T-1$ the strings greater than or equal to j_{bin} are precisely those of the form $\alpha_{T-1}\cdots\alpha_{l_0}1\cdots1$, and since there are at most T different strings of this kind, an upper bound to the number of binary strings greater than a given one is given by the string length T.

We can now consider the following encoding scheme:

$$\begin{aligned}
&|s_0,\ldots,s_{M-1}\rangle_{\mathrm{F}} \mapsto |x_0,\ldots,x_{M-1}\rangle_{\mathrm{Q}}\,,\\
&\quad \text{where } x_j = \bigoplus_{i\in S(j)} s_i \text{ and } S(j) := \{k\in\{0,\ldots,M-1\} \mid k_{\mathrm{bin}} \preceq j_{\mathrm{bin}}\}\,. \qquad (67)
\end{aligned}$$

It is very important to notice that since a s_i appears in every x_j satisfying $i_{\mathrm{bin}} \preceq j_{\mathrm{bin}}$, we have that a s_j appears at most in T of the x_j, with $T = \mathrm{ceiling}(\log_2 M)$

being the number of bits required to binary encode the labels of the LFM systems, ranging from 0 to $M-1$.

Let $\chi^{[l_0]}$ be the string having $\chi^{[l_0]}_l = 1$ for $l \leq l_0$ and $\chi^{[l_0]}_l = 0$ otherwise, then the following two properties hold:

(i) the inversion of the relation $x_j = \bigoplus_{i\in S(j)} s_i$ leads to $s_j = x_j - \bigoplus_{i\in K(j)} x_i$, where for $j_{\rm bin} = \beta_{T-1}\cdots\beta_0$

$$K(j) = \left\{\alpha \,\middle|\, \exists 0 \leq l_0 \leq T-1 \text{ s.t. } \chi^{[l_0]}_l \beta_l = \chi^{[l_0]}_l,\ \alpha_l = \beta_l \oplus \delta_{l l_0}\right\};$$

(ii) the quantity $\bigoplus_{i=0}^{j-1} s_i$ can be written in terms of the encoded numbers x_l as $\bigoplus_{i=0}^{j-1} s_i = \bigoplus_{i\in L(j)} x_i$ where $j_{\rm bin} = \beta_{T-1}\cdots\beta_0$

$$L(j) = \left\{\alpha \,\middle|\, \exists 0 \leq l_0 \leq T-1 \text{ s.t. } \alpha_l = \beta_l \oplus \chi^{[l_0]}_l(\beta_l \oplus 1),\ \beta_{l_0}(\alpha_{l_0} \oplus 1) = 1\right\}.$$

Observe that also the sums appearing in the two above expressions contain at most $\log_2 M$ elements.

While the extraction procedure of the qubits with the standard encoding given by the JWT requires a number of computational steps linear in the number of the LFMs of the circuit, with this last encoding the situation is improved: suppose to extract the jth qubit starting from the initial state $|s_0,\ldots,s_{M-1}\rangle_{\rm F}$ encoded in the qubit state $|x_0,\ldots,x_{M-1}\rangle_{\rm Q}$. First of all we add the ancillary qubit $|0\rangle$ at the beginning of the string (let us call it "the qubit at the position -1"), then the extraction goes as follows

$$\begin{aligned} |0,x_0,\ldots,x_{M-1}\rangle_{\rm Q} &\xrightarrow{A} |s_j,x_0,\ldots,x_{M-1}\rangle_{\rm Q} \\ &\xrightarrow{B} |s_j,x'_0,\ldots,x'_{M-1}\rangle_{\rm Q} \\ &\xrightarrow{C} (-1)^{s_j\oplus\bigoplus_{i=0}^{j-1}s_i}|s_j,x'_0,\ldots,x'_{M-1}\rangle_{\rm Q}\,, \end{aligned}$$

where

(A) is a unitary evolution that evaluates the value of s_j from the encoded string $x_{M-1}\cdots x_0$ and writes it into the ancillary qubit. Such an operation can be achieved by means of the unitary map

$$A = \prod_{i\in K(j)\cup\{j\}} \Lambda(\sigma^x)(i,-1)\,, \tag{68}$$

where we remember that $\Lambda(U)(i_0,\ldots,i_p)$ represents the controlled unitary U with control system i_0 and target systems $i_1,\ldots,i_p$. Since the cardinality of $K(j)$ is $O(\log M)$, we will need $O(\log M)$ gates to perform A;

(B) turns the original s_j (not the copy in the ancillary qubit) to zero. The transformation B then must change the encoded string in such a way that the following diagram commutes:

$$\begin{array}{ccc} |s_0,\ldots,s_j,\ldots,s_{M-1}\rangle_{\rm Q} & \longrightarrow & |s_0,\ldots,s_j\oplus s_0,\ldots,s_{M-1}\rangle_{\rm Q}\,, \\ \downarrow E & & \downarrow E \\ |x_0,\ldots,x_j,\ldots,x_{M-1}\rangle_{\rm Q} & \xrightarrow{B} & |x'_0,\ldots,x'_j,\ldots,x'_{M-1}\rangle_{\rm Q}\,. \end{array}$$

This operation is achieved by the following unitary

$$B=\prod_{i_{\rm bin}\succeq j_{\rm bin}}\Lambda(\sigma^x)(-1,i)\,,$$

where again the number of gates required is $O(\log M)$;

(C) is an unitary evolution that evaluates the phase due the exchange of the Fermionic wires:

$$C=\prod_{j\in L(j)}\Lambda(\sigma^z)(-1,i)\,.$$

Again, since $|L(j)|\approx O(\log M)$, the number of the required gates amounts to $O(\log M)$.

In conclusion to simulate a one-LFM gate in a circuit of M LFMs by means of qubit gates we need $O(\log M)$ qubit gates (instead of $O(M)$ gates needed using the JWT encoding). This result holds for every LFM operator — i.e. for a K-LFM gate (with $K\leq M$), one needs to extract K qubits by means of the above procedure. Moreover notice that the proof does not require the gates to be parity-preserving. Indeed the reviewed procedure of Ref. 1 provides an efficient way to perform the qubit extraction (or equivalently to take into account the phase factor given by the σ^z of the JWT) irrespective of the parity features of the gate we want to simulate. Hence, the one-LFM transformation $\mathcal{X}(\rho)=\big(\varphi_i^\dagger+\varphi_i\big)\rho\big(\varphi_i^\dagger+\varphi_i\big)$, which is parity changing, can also be achieved by means of $\log M$ qubit gates: we just need to perform a σ^x on an extracted qubit.

As shown in Ref. 1, an efficient simulation of a N-qubit circuit by means of a LFM circuit is easier. First of all one performs the encoding of the N qubits into $2N$ qubits through the isometric embedding $V:|s_0,\ldots,s_{N-1}\rangle_{\rm Q}\mapsto|s_0,s_0,\ldots,s_{N-1},s_{N-1}\rangle_{\rm Q}$. A quantum gate $\mathcal{G}$ acting on the jth and the kth qubits — thus represented by the unitary operator $\tilde{G}(j,k)$ of Eq. (58) — is represented on the $2N$ qubits by the gate $\tilde{G}'(2j,2j+1,2k,2k+1)=V\tilde{G}(j,k)V^\dagger$. If we embed the resulting $2N$-qubit circuit into $2N$ LFMs by means of the JWT, the resulting $J(\tilde{G}'(j,j+1,k,k+1))$, besides being parity-preserving, is also made of field operators acting only on the LFMs $2j$, $2j+1$, $2k$, $2k+1$, namely no field operators on the rest of the circuit are needed. In conclusion, every qubit gate acting on two qubits can be simulated by means of a four-LFM gate. The same result clearly generalizes for gates with an arbitrary number of qubits.

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