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Isotropic phase squeezing and the arrow of time

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Abstract

We prove that isotropic squeezing of the phase is equivalent to reversing the arrow of time. © 2000 Published by Elsevier
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The concept of 'squeezing' appeared in the literature in the early 70's [1,2] and was extensively studied in order to improve the capacity of quantum information channels and the sensitivity in interferometric measurements [3]. Since then squeezing has been a very popular word in quantum optics. By 'squeezing' one refers to a physical process where the uncertainty of an observable is reduced at the expense of increasing the uncertainty of the conjugated observable, according to the Heisenberg inequalities (for extensive reviews see for example Ref. [4]). In quantum optics quadrature squeezing, namely the squeezing of the probability distribution of the observable $a_\phi = (a^\dagger e^{i\phi} + a e^{-i\phi})/2 - a$ and $a^\dagger$ denoting the annihilation and creation operators of

a given radiation mode – has been achieved experimentally, giving rise to a number of interesting properties, such as phase-sensitive amplification and antibunching [2,4–6]. More recently, the density operator of squeezed states has been measured by optical homodyne tomography [7].

Conjugated quadratures, i.e. quadratures relative to phases ϕ and $\phi + \pi/2$, are generalizations of the couple of observables position-momentum. Thus, we can view quadrature squeezing of radiation states as a narrowing process of the probability distribution in the phase space which occurs in a definite direction, corresponding to the phase of the squeezed quadrature [see Fig. 1(a)]. Classically, one can imagine a similar process in polar coordinates [Fig. 1(b)], where the radial probability distribution is squeezed, while the phase is spread, or viceversa. In the phase space the squared radius corresponds to the total energy of the harmonic oscillator, which is proportional to the photon number operator $N = a^\dagger a$ for a single-mode

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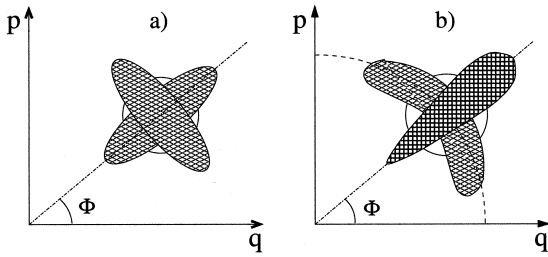*E-mail address:* msacchi@pv.infn.it (M.F. Sacchi).

Fig. 1. Phase-space representation of squeezing: (a) conventional squeezing of two conjugated quadratures; (b) squeezing in phase and photon number.

radiation field. Number squeezing narrows the photon number distribution, with the possibility of achieving sub-Poissonian statistics $\langle \Delta N^2 \rangle < \langle N \rangle$ in photon counting [8]. This process has been investigated extensively and can be experimentally achieved by means of self-phase modulation in Kerr media [9].

The inverse process, namely phase squeezing, is the subject of the present Letter. We will consider isotropic phase squeezing, namely squeezing of the phase probability distribution independently of the mean value of the phase. Such a process corresponds to noise reduction in the measurement of phase, and it would lead to important results for communications and measurements, such as improved sensitivity of interferometric schemes and the achievement of the capacity of quantum communications based on phase coding.

In the following we will prove that isotropic phase squeezing cannot be realized because it would correspond to reversing the arrow of time. The arrow of time is statistically defined by the direction of the irreversible dynamics of open systems [10]. In quantum-mechanical terms, it is associated to a loss of coherence of the quantum state, e.g. dephasing mechanism of the laser light, which corresponds to a random walk on the phase space [11]. We will prove that any dynamical process that isotropically reduces the phase uncertainty can be described only in terms of a 'time-reversed dissipative equation'.

In the literature the Heisenberg-like heuristic inequality $\Delta N \Delta \phi \geq 1$ for the couple number-phase is often reported. However, its meaning is only semiclassical, since the quantum phase does not correspond to any self-adjoint operator [12–14]. There-

fore, in order to investigate isotropic phase squeezing, we have first to introduce the concepts of phase measurement and phase probability distribution in a rigorous way.

The quantum-mechanical definition of the phase is well assessed in the framework of quantum estimation theory [15,16]. In this context the phase of a quantum state is defined by the shift ϕ generated by any operator F with discrete spectrum. For example $F = a^\dagger a$ for the harmonic oscillator, and $F = \sigma_z/2$ for a two-level system, σ_z being the customary Pauli operator.

Quantum estimation theory provides a general description of quantum statistics in terms of POVM's (positive operator-valued measures) and seeks the optimal POVM to estimate one or more parameters of a quantum system on the basis of a cost function which assesses the cost of errors in the estimates. For phase estimation, the optimal POVM for pure states $|\psi\rangle$ with coefficients $\psi_n = |\psi_n| e^{i\chi_n} \neq 0$ on the basis $|n\rangle$ of F eigenvectors is given by

$$d\mu(\phi) = \frac{d\phi}{2\pi} |e(\phi)\rangle \langle e(\phi)| \qquad (1)$$

for the class of Holevo's cost functions – a large class including the maximum likelihood criterion, the 2π-periodicized variance, and the fidelity optimization. In Eq. (1) $|e(\phi)\rangle$ denotes the (Dirac) normalizable vector

$$|e(\phi)\rangle = \sum_{n \in S} e^{i(n\phi - \chi_n)} |n\rangle, \qquad (2)$$

where S is the spectrum of F. In Ref. [17] the solution given in Eqs. (1), (2) has also been proved for *phase-pure* states, namely for states described by a density operator ρ satisfying the condition

$$\rho_{nm} \equiv \langle n| \hat{\rho} |m\rangle = |\rho_{nm}| e^{i(\chi_n - \chi_m)}, \qquad (3)$$

and for a nondegenerate phase-shift generator F[1]. For states that are not of this kind, there is no

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[1] The set of phase pure states indeed has to be restricted excluding the states $\hat{\varrho}$ with $\varrho_{ij} \neq 0$ only for $i - j = nk$, with $n \in \mathbb{N}$ and k denoting an integer constant ≥ 2. In fact, those states have phase properties that are periodic of $2\pi/k$. A simple example is given by the superposition of two coherent states with amplitude $\pm \alpha$ (Schrödinger-cat like states), for which $k = 2$.

available method in the literature to obtain the optimal POVM, and thus the concept itself of phase does not have a well defined meaning.

The phase probability distribution $dp(\phi)$ of a quantum state is evaluated by means of the optimal POVM in Eq. (1) through the Born's rule $dp(\phi) = \text{Tr}[\rho \, d\mu(\phi)]$. The phase uncertainty $\Delta\phi^2$ of the state can then be calculated. However, notice that for periodic distributions the customary r.m.s. deviation depends on the chosen window of integration. Definitions of phase uncertainty that do not depend on the interval of integration given in the literature are monotonic increasing functions f of some average cost of the Holevo's class, namely

$$\delta\phi = f(\langle C \rangle), \qquad (4)$$

where C represents the cost operator

$$C = c_0 - \sum_{n=1}^{\infty} c_n \left(e_+^n + e_-^n \right), \quad c_n \geq 0, \quad \forall n \geq 1, \qquad (5)$$

and $\langle \dots \rangle$ represents the quantum ensemble average. In Eq. (5) we introduced the following notation:

$$e_+ = \sum_{n \in S} e^{i(\chi_{n+1} - \chi_n)} |n+1\rangle\langle n| \quad \text{and} \quad e_- = (e_+)^\dagger. \qquad (6)$$

Typical examples of functions of this kind are the reciprocal peak likelihood and the phase deviation $2(1 - |\langle e_+ \rangle|^2)$. The former corresponds to $f(x) = -1/x$ for the cost operator C with all $c_n = 1$; the latter corresponds to $f = 2\left[1 - (1/4)x^2\right]$ with $c_1 = 1$ and $c_n = 0 \; \forall n \neq 1$ (for phase-pure states $\langle e_+ \rangle$ is a real positive quantity, so $\langle e_- \rangle = \langle e_+ \rangle$ and $-\langle C \rangle^2/4 = -|\langle e_+ \rangle|^2$, with $\langle C \rangle \leq 0$).

Now we introduce the concept of isotropic phase squeezing. For the e.m. field, we remind that ordinary quadrature squeezing is effective in reducing the phase uncertainty of a quantum state provided that the average value of the phase is known a priori. As mentioned above, isotropic phase squeezing should reduce the phase uncertainty of the state ρ independently of the initial mean phase. In mathematical terms, this condition corresponds to a linear

map Γ that is covariant for the rotation group generated by the operator F, namely

$$\Gamma\left(e^{iF\phi} \rho \, e^{-iF\phi}\right) = e^{iF\phi} \, \Gamma(\rho) \, e^{-iF\phi}. \qquad (7)$$

A physically realizable linear map Γ corresponds to a completely positive (CP) map for density operators that can be written in the Lindblad form

$$\frac{\partial \rho}{\partial t} = \sum_n L[V_n] \rho, \qquad (8)$$

where $L[O]\rho \equiv O\rho O^\dagger - \frac{1}{2}(O^\dagger O \rho + \rho O^\dagger O)$ denotes the Lindblad superoperator [18]. We do not take into account the customary Hamiltonian term $-i[H, \rho]$ in the master Eq. (8), because for the covariance condition one has $[H, F] = 0$, and hence the optimal POVM is simply rotated, with the result that the phase uncertainty is not affected by the corresponding unitary evolution (such Hamiltonian term can be equivalently applied in one step before the evolution (8), and preserves the phase purity of the state).

The covariance condition restricts the general form of Eq. (8) to the expression [19]

$$\frac{\partial \rho}{\partial t} = \sum_{m=-\infty}^{+\infty} \sum_j L\left[B_{m,j}\right] \rho, \qquad (9)$$

where

$$B_{mj} = g_{m,j}(F) \, e_+^m, \quad m \geq 0$$

$$B_{mj} = h_{|m|,j}(F) \, e_-^{|m|}, \quad m < 0. \qquad (10)$$

In the following we will focus our attention on the case of a single-mode radiation field, hence we take $F = a^\dagger a$ and the spectrum $S = \mathbb{N}$. We postpone the discussion of the generality of our result at the end of the Letter.

We consider the class of phase-pure states as initial states for the master Eq. (9), since for other kinds of states the phase measurement is not well defined, as mentioned before.

We are now in position to prove the main result of this Letter, namely that isotropic phase squeezing is equivalent to reversing the arrow of time. According to Eq. (4) the time derivative of the phase uncertainty $\delta\phi$ has the same sign as the derivative of

the average cost $\langle C \rangle$, which is obtained from Eq. (9) as follows:

$$\frac{\partial \langle C \rangle}{\partial t} = -2\text{Re}\frac{\partial}{\partial t} \sum_{k=1}^{\infty} c_k \langle e_+^k \rangle . \qquad (11)$$

A straightforward calculation gives the following contribution for the k-th term in the sum of Eq. (11):

$$-2\text{Re}\frac{\partial}{\partial t}\langle e_+^k \rangle$$

$$= \sum_{m=0}^{\infty} \sum_{j} \sum_{l=0}^{\infty} |\rho_{l,l+k}||g_{m,j}(m+l)$$

$$- g_{m,j}(m+l+k)|^2$$

$$+ \sum_{m=1}^{\infty} \sum_{j} \sum_{l=m}^{\infty} |\rho_{l,l+k}||h_{m,j}(l-m)$$

$$- h_{m,j}(l-m+k)|^2$$

$$+ \sum_{m=1}^{\infty} \sum_{j} \sum_{l=\max(0,m-k)}^{m-1} |\rho_{l,l+k}||$$

$$\times h_{m,j}(l-m+k)|^2 \geq 0, \qquad (12)$$

which is manifestly non negative for all values of k. Hence, according to Eqs. (11) and (12), the average cost as well as the phase uncertainty increase versus time. The only possibility to achieve isotropic phase squeezing is then to have a minus sign in front of the master Eq. (8), which means to reverse the arrow of time.

Our proof rules out also the possibility of isotropic squeezing through a phase measurement followed by a feedback quadrature squeezing. In fact, such a kind of process is described by a CP map as well, and one can explicitly show that the phase uncertainty in the measurement eventually leads to an overall phase diffusion.

The same conclusions regarding the derivatives of $\langle e_+^k \rangle$ hold also for the cases of unbounded spectrum $S = \mathbb{Z}$ and bounded spectrum $S = \mathbb{Z}_q$ for a nondegenerate phase shift operator: also in these cases the phase uncertainty can decrease for any input state only if we reverse the arrow of time. For $S = \mathbb{Z}_q$ all

series in Eq. (12) are bounded and boundary terms appear in addition to the third one. For $S = \mathbb{Z}$ Eq. (12) rewrites:

$$-2\text{Re}\frac{\partial}{\partial t}\langle e_+^k \rangle$$

$$= \sum_{m=0}^{\infty} \sum_{j} \sum_{l=0}^{\infty} |\rho_{l,l+k}||g_{m,j}(m+l)$$

$$- g_{m,j}(m+l+k)|^2$$

$$+ \sum_{m=1}^{\infty} \sum_{j} \sum_{l=1}^{\infty} |\rho_{l,l+k}||h_{m,j}(l-m)$$

$$- h_{m,j}(l-m+k)|^2 \geq 0. \qquad (13)$$

In the general case a phase-covariant master equation does not evolve a phase-pure state into a phase-pure state, hence it may happen in principle that, after a finite time interval in which the phase-purity is lost, phase-purity is then recovered at the end with an overall decrease of $\delta\phi$. However, one cannot follow the evolution of $\delta\phi$ for finite time intervals if the definition itself of the phase is lost during the time evolution. A phase-covariant master Eq. (9) preserves phase-purity if and only if $\arg(g_{m,j}(F)) = \varphi_{m,j}$ and $\arg(h_{m,j}(F)) = \theta_{m,j}$ independent on F, and one can always choose $\varphi_{m,j} = \theta_{m,j} = 0$ identically, due to the bilinear form of the Lindblad superoperator.

In the case of degenerate F, we can find the optimal POVM for pure states $|\psi\rangle\langle\psi|$ as follows [17]: we select a vector $|n\rangle_{\parallel}$ for each degenerate eigenspace $\mathscr{H}_n$ corresponding to the eigenvalue n, such that $|n\rangle_{\parallel}$ is parallel to the projection of $|\psi\rangle$ on $\mathscr{H}_n$. So the Hilbert space $\mathscr{H}$ can be represented as $\mathscr{H}_{\parallel} \oplus \mathscr{H}_{\perp}$, where $\mathscr{H}_{\parallel}$ is the Hilbert space spanned by the vectors $|n\rangle_{\parallel}$ and $\mathscr{H}_{\perp}$ its orthogonal completion. Since $|\psi\rangle$ has null component in $\mathscr{H}_{\perp}$ the estimation problem reduces to a nondegenerate one in the Hilbert space $\mathscr{H}_{\parallel}$, and the optimal POVM is given by $d\mu(\phi) = d\mu_{\parallel}(\phi) \oplus d\mu_{\perp}(\phi)$, where $d\mu_{\parallel}$ is the optimal POVM for the nondegenerate estimation problem in $\mathscr{H}_{\parallel}$, while $d\mu_{\perp}(\phi)$ is an arbitrary POVM in $\mathscr{H}_{\perp}$. It is clear that the POVM obtained in this way is optimal also for phase-pure states that

are mixtures of pure states all with the same $\mathscr{H}_{\parallel}$. The most general phase-covariant master equation is again of the form in Eqs. (8) and (10), however, now there are infinitely many possible $B_{m,j}$ for fixed m,j corresponding to different operators e_+ which shift the eigenvalue of F while spreading the state in the whole $\mathscr{H}$ from $\mathscr{H}_{\parallel}$ in all possible ways. Therefore, a phase-covariant master equation does not keep the original state in $\mathscr{H}_{\parallel}$, apart from the case where one considers operators $B_{m,j}$ defined in terms of e_+ only of the form

$$
e_+ = \sum_n \mathrm{e}^{i(\chi_{n+1} - \chi_n)} |n+1\rangle_{\parallel\,\parallel}\langle n|. \tag{14}
$$

In the general case, however, a reduction of the phase uncertainty is possible in principle, due to the arbitrariness introduced by $d\mu_{\perp}(\phi)$ in the definition of phase in the degenerate case, the time derivative of $dp(\phi) = \mathrm{Tr}[\rho\, d\mu_{\parallel}(\phi)]$ generally depending on $d\mu_{\perp}(\phi)$.

We now focus attention back to the case of nondegenerate F. Looking at Eq. (12), one can see that it is possible to make all terms vanishing, getting a null derivative for the average cost. This is actually possible only for unbounded spectra as $S = \mathbb{N}$ and $S = \mathbb{Z}$, where one can find a master equation that preserves the phase uncertainty for any quantum state. For $S = \mathbb{N}$, one has the following conditions on the coefficients of Eq. (12): $g_{m,j}(F) = g_{m,j}$ constant and $h_{m,j}(F) = 0$. Upon defining $u_m = \sum_j |g_{m,j}|^2$ one has

$$
\frac{\partial \rho}{\partial t} = \sum_{m=1}^{\infty} u_m \left(e_+^m \rho e_-^m - \rho \right). \tag{15}
$$

In the case $S = \mathbb{Z}$ one has more generally $h_{m,j}(F) = h_{m,j}$ constant, and introducing $v_m = \sum_j |h_{m,j}|^2$, the phase-uncertainty preserving master equation takes the form

$$
\frac{\partial \rho}{\partial t} = \sum_{m=1}^{\infty} u_m \left(e_+^m \rho e_-^m - \rho \right) + v_m \left(e_-^m \rho e_+^m - \rho \right). \tag{16}
$$

The master Eqs. (15) and (16) are very interesting, since they represent a counterexample to the custom-

ary identification of 'decoherence' and 'dephasing'. The study of physical realizations of Eqs. (15) and (16) could provide insight in the understanding of decoherence and relaxation phenomena.

In conclusion, we have shown that isotropic squeezing of the phase is equivalent to reversing the arrow of time. This result is very general, as it holds for any definition of phase with nondegenerate shift operator, for any definition of phase-uncertainty in the Holevo's class, and for any initial phase-pure state. In this way we have related the concept of phase to the arrow of time statistically defined by the evolution of open quantum systems, thus enforcing the link between phase and time [20].

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