ACCURACY IN QUANTUM HOMODYNE TOMOGRAPHY

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We study the statistical errors in homodyne tomography of radiation density matrix in the photon number representation. We give an asymptotic estimate for large matrix indexes at different values of the quantum efficiency η of homodyne detectors. We show that for fixed η the errors increase exponentially as functions of the matrix index.

INTRODUCTION

Optical homodyne tomography is by now a well assessed method to measure the quantum state of radiation. The density operator $\hat{\rho}$ is measured in some representation by averaging the so-called kernel functions (or pattern functions) over homodyne data; in particular, the matrix element between generic states $|\varphi\rangle$ and $|\psi\rangle$ is evaluated under the following integral [1, 2]

$$\langle\varphi|\hat{\rho}|\psi\rangle = \int_0^\pi \frac{\mathrm{d}\phi}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}x\, p_\eta(x,\phi)\, \langle\varphi|K_\eta(x-\hat{x}_\phi)|\psi\rangle\,. \tag{1}$$

In Equation (1) $p_\eta(x,\phi)$ is the probability distribution of the field quadrature $\hat{x}_\phi = (a^\dagger e^{i\phi} + ae^{-i\phi})/2$ for overall efficiency η of the homodyne detector (ϕ is the phase of the field mode with respect to the local oscillator, $a^\dagger$ and a are the creation and annihilation operators of the mode). The kernel function $\langle\varphi|K_\eta(x-\hat{x}_\phi)|\psi\rangle$ is determined by the operator

$$K_\eta(x-\hat{x}_\phi) = \frac{1}{2}\mathrm{Re}\int_0^{+\infty} \mathrm{d}k\, k\, e^{\frac{1-\eta}{8\eta}k^2}\, e^{ik(x-\hat{x}_\phi)}\,. \tag{2}$$

The behavior of the kernel function depends both on the particular chosen representation (i.e. $|\varphi\rangle$ and $|\psi\rangle$) and on the value of η. The boundedness of the averaged kernel in Equation (1) sets the validity limits of the tomographic reconstruction. In a previous work[1] the most used representations were considered, corresponding to number, coherent, squeezed, and quadrature states. The bounds for η were established, below which

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the matrix elements cannot be measured: the coherent and the Fock representations turned out to be the best choice, with bound $\eta = 0.5$.

It is not clear if the bound $\eta = 0.5$ is the minimum for any representation, and "exotic" representations having some lower bound would be very interesting for tomography. Here we just mention that also unusual representations, such as the multi-photon representation,[3] have the same bound $\eta = 0.5$, whereas for the eigenvectors of the squeeze operator[4] $\Lambda = -i/2(a^2 - a^{\dagger 2})$ the bound is $\eta = 1$, as for the eigenvectors of the quadrature. On the other hand, representations based on eigenvectors of polynomials in a and $a^\dagger$ with degree larger than 2 are not analytical on the Fock space.[5]

The aim of this paper is to analyse the statistical errors in the measurement of the density operator in the Fock representation. We give an *a priori* asymptotic estimate of the error $\sigma(n,m)$ for large indexes n and m of the matrix element $\langle n|\hat{\rho}|m\rangle$. We show that for $0.5 < \eta < 1$ the error increases exponentially as functions of the matrix index. For $\eta = 0.5$ the errors diverge. Although this estimate is obtained after neglecting the correlations between different matrix elements, the asymptotic formula compares favorably with exact numerical evaluations. In the end, we give an analytic estimate of such correlations.

ACCURACY OF THE MEASUREMENT

Recently the statistical errors of the tomographic measurement of the density matrix in the number representation were numerically evaluated[6] on the basis of Equation (1). In the same paper it was also shown that for unit overall efficiency η of the homodyne detector the errors on the diagonal of the matrix saturate to a fixed value, independent of the radiation state. Now, in order to calculate the errors analytically, we exploit the inversion of the Bernoulli convolution,[7] that gives

$$\langle n|\hat{\rho}|m\rangle = \sum_{j=0}^{\infty} C^{\eta}_{n,m}(j)\, \overline{f_{n+j,m+j}}\,, \tag{3}$$

where we defined

$$C^{\eta}_{n,m}(j) = \eta^{-\frac{1}{2}(n+m)} \left[\begin{pmatrix} n+j \\ n \end{pmatrix} \begin{pmatrix} m+j \\ m \end{pmatrix} \right]^{\frac{1}{2}} (1-\eta^{-1})^j \tag{4}$$

$$f_{n+j,m+j}(x,\phi) = \langle n+j|K_1(x-\hat{x}_\phi)|m+j\rangle\,, \tag{5}$$

and

$$\overline{f_{n+j,m+j}} = \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dx\, p(x,\phi;\hat{\rho}_\eta)\, f_{n+j,m+j}(x,\phi)\,. \tag{6}$$

In Equation (6) $p(x,\phi;\hat{\rho}_\eta)$ is the homodyne probability distribution corresponding to the "dressed" state $\hat{\rho}_\eta$, that is related to the "bare" state $\hat{\rho}$ by the Liouvillian transformation*

$$\hat{\rho}_\eta = \sum_{s=0}^{\infty} \frac{(\eta-1)^s}{s!} a^s \eta^{-\frac{a^\dagger a}{2}}\, \hat{\rho}\, \eta^{-\frac{a^\dagger a}{2}} (a^\dagger)^s\,. \tag{7}$$

An *a priori* estimation of the measurement error is obtained by calculating the variances of the real and the imaginary part of the series (3). The variance of the real

*A discussion about the measurement of dressed states by homodyne tomography for non unit quantum efficiency at detectors has been recently reported.[8]

part is

$$\sigma_R^2(n,m) = \sum_{j,l=0}^{\infty} C_{n,m}^{\eta}(j) C_{n,m}^{\eta}(l)$$
$$\times \left[\overline{\mathrm{Re}[f_{n+j,m+j}] \cdot \mathrm{Re}[f_{n+l,m+l}]} - \overline{\mathrm{Re}[f_{n+j,m+j}]} \cdot \overline{\mathrm{Re}[f_{n+l,m+l}]}\right] . \tag{8}$$

The variance of the imaginary part $\sigma_I^2(n,m)$ is defined in analogous way. For a number of measurements N, the experimental error on the density matrix element $\langle n|\hat{\rho}|m\rangle$ is given by $\sigma(n,m)/\sqrt{N}$, with

$$\sigma(n,m) = \sqrt{\sigma_R^2(n,m) + \sigma_I^2(n,m)} . \tag{9}$$

Upon neglecting the statistical correlations for $j \neq l$ in Eq. (8), one obtains

$$\sigma^2(n,m) \simeq \sum_{j=0}^{\infty} [C_{n,m}^{\eta}(j)]^2 \, \Delta_{n+j,m+j}^2 , \tag{10}$$

where

$$\Delta_{n+j,m+j}^2 = \overline{|f_{n+j,m+j}|^2} - |\overline{f_{n+j,m+j}}|^2 . \tag{11}$$

For large indexes n,m the quantity $\Delta_{n+j,m+j}$ saturates to the fixed value $\sqrt{2}$ for any j. In this case the series in Equation (10) can be summed and the error behaves as follows:

$$\sigma^2(n,m) \simeq 2\,\eta^{-(n+m)} \Phi[n+1, m+1; 1; (1-\eta^{-1})^2] , \tag{12}$$

where $\Phi(\alpha, \beta; \gamma; z)$ is the customary hypergeometric function. The estimate (12) is independent of the radiation state for sufficiently large n and m. Notice that the convergence radius of the hypergeometric series in (12) is $(1-\eta^{-1})^2 < 1$, that means $\eta > 0.5$. Therefore, the errors diverge for $\eta = 0.5$, as also previously shown.[9]

For the diagonal errors, we can use a very good asymptotic approximation of the hypergeometric function $\Phi[n+1, n+1; 1; (1-\eta^{-1})^2]$ leading, for $n \gg (2\eta-1)/(1-\eta)$, to the following asymptotic expression

$$\sigma^2(n,n) \simeq \frac{\eta^{3/2}}{\sqrt{\pi(1-\eta)}} \, e^{\frac{1}{4n}\frac{(2\eta-1)^2}{\eta(1-\eta)}} \left(\frac{1}{2\eta-1}\right)^{2n+1} \frac{1}{\sqrt{n}} . \tag{13}$$

The estimate (13) shows the exponential growth of the error $\sigma(n,n)$ versus n, with rate $-\ln(2\eta-1)$.

In Fig. 1 we report the asymptotic expressions for the diagonal errors obtained from Eqs. (12) and (13) for tomography of a coherent state, in comparison to the exact numerical evaluation obtained from Eq. (1). The growth rate versus the matrix index is quite correctly reproduced. The difference between the asymptotic errors and the actual ones is due to the statistical correlations between different matrix elements.

Now we estimate the correlations that contribute to the diagonal errors. Since for large index n we have $\overline{f_{n+j,n+j}} = 0$, the contribution of the correlations is

$$\lambda(n,n) = \sum_{\substack{j,l=0 \\ j \neq l}}^{\infty} C_{n,n}^{\eta}(j) C_{n,n}^{\eta}(l) \overline{f_{n+j,n+j} f_{n+l,n+l}} \tag{14}$$

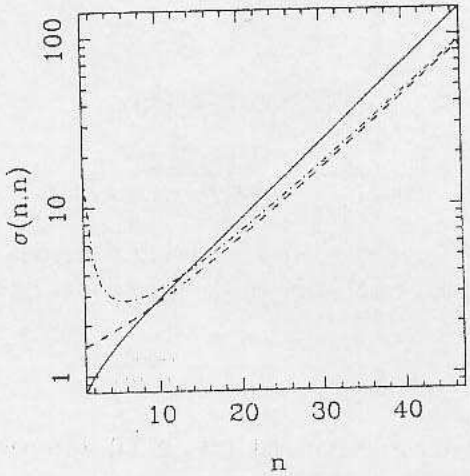

Figure 1. Diagonal statistical errors $\sigma(n,n)$ for a coherent state with 10 photons and $\eta = 0.95$ (log-linear scale). Solid line: actual errors; dashed line: estimate given by (12); dot-dashed line: estimate given by (13).

If we consider the asymptotic approximation for the kernel functions †

$$f_{n+j,n+j}(x) \simeq (-1)^{n+j} 2\cos(k_{n+j}x) \tag{15}$$

(with $k_{n+j} \propto \sqrt{n+j}$), Eq. (14) becomes

$$\lambda(n,n) = 2\sum_{j\neq l}(-1)^{j+l} C^{\eta}_{n,n}(j) C^{\eta}_{n,n}(l) \int_0^{\pi} \frac{\mathrm{d}\phi}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}x\, p(x,\phi;\hat{\rho}_{\eta}) \cos\left[\delta k_n(j,l)x\right], \tag{16}$$

where we neglected the fast oscillating part in $f_{n+j,n+j}(x) f_{n+l,n+l}(x)$ and we defined $\delta k_n(j,l) \doteq k_{n+j} - k_{n+l}$. Thus, the integration is easily performed and we have

$$\lambda(n,n) = 2\sum_{j\neq l}(-1)^{j+l} C^{\eta}_{n,n}(j) C^{\eta}_{n,n}(l)\, e^{-\frac{1}{8}\delta k_n(j,l)^2} \sum_{s=0}^{\infty} \langle s|\hat{\rho}_{\eta}|s\rangle\, L_s\left(\frac{\delta k_n(j,l)^2}{4}\right), \tag{17}$$

where $L_s(z)$ denote the Laguerre polynomials.

Let us give few examples. For highly excited states, if the photon probability distribution is considered approximately constant, the sum over s is zero. This is the only case where the correlations give a null contribution. For coherent states with average number of photons $|\alpha|^2$ we have

$$\lambda(n,n) = 2\sum_{j\neq l}(-1)^{j+l} C^{\eta}_{n,n}(j) C^{\eta}_{n,n}(l)\, e^{-\frac{1}{8}\delta k^2_n(j,l)} J_0\left(\sqrt{\eta\,|\alpha|^2\,\delta k_n(j,l)^2}\right), \tag{18}$$

where $J_0(z)$ denote the zeroth-order Bessel function. For thermal states with average number of photons $\bar{n}$ the correlations are given by

$$\lambda(n,n) = 2\sum_{j\neq l}(-1)^{j+l} C^{\eta}_{n,n}(j) C^{\eta}_{n,n}(l)\, e^{-\frac{1+2\eta\bar{n}}{8}\delta k^2_n(j,l)}. \tag{19}$$

From the definition (4) we see that in this case the correlations always give a positive contribution. Thus, for thermal states the expression in Eq. (12) is not only an asymptotic estimate, but also an asymptotic lower bound for the actual errors expected in homodyne measurement of the density matrix.

†We remind that the diagonal kernel functions are independent of ϕ.

CONCLUSIONS

We obtained an asymptotic analytical expression for the *a priori* estimated experimental errors in homodyne tomography of the density operator in the Fock representation. For fixed overall efficiency at homodyne detectors η, the diagonal errors are exponentially growing as functions of the matrix index, with rate $-\ln(2\eta - 1)$. In the end, we estimated the contribution to the errors of the statistical correlations between different matrix elements.

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