IL NUOVO CIMENTO Vol. 114 B, N. 7 Luglio 1999

Feedback-assisted measurement of the free-mass position under the Standard Quantum Limit(*)

G. M. D'Ariano, M. F. Sacchi and R. Seno

Dipartimento di Fisica "Alessandro Volta", Università degli Studi di Pavia
INFM, Unità di Pavia - via A. Bassi 6, I-27100 Pavia, Italy

(ricevuto il 14 Luglio 1998; approvato il 6 Maggio 1999)

Summary. — The Standard Quantum Limit (SQL) for the measurement of a free-mass position is illustrated, along with two necessary conditions for breaching it. A measurement scheme that overcomes the SQL is engineered. It can be achieved in three steps: i) a pre-squeezing stage; ii) a standard von Neumann measurement with momentum-position object-probe interaction and iii) a feedback. Advantages and limitations of this scheme are discussed. It is shown that all of the three steps are needed in order to overcome the SQL. In particular, the von Neumann interaction is crucial in getting the right state reduction, whereas other experimentally achievable Hamiltonians, as, for example, the radiation-pressure interaction, lead to state reductions that on the average cannot overcome the SQL.

PACS 03.65.Bz – Foundations, theory of measurement, miscellaneous theories (including Aharonov-Bohm effect, Bell inequalities, Berry's phase).
PACS 42.50.Dv – Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements.
PACS 42.50 – Quantum optics.

1. – Introduction

The problem of achieving a sequence of measurements of the position of a free mass with arbitrary precision has received much attention in the past, especially as a tool for monitoring the presence of an external classical field weakly interacting with the mass itself—typically, gravitational waves [1, 2]. Originally, a *Standard Quantum Limit* (SQL) was devised [1, 3] stating that *if a mass evolves freely for a time interval* t_f *between two measurements, the uncertainty of the second measurement cannot be lower than* $\Delta^2_{\mathrm{SQL}} = \hbar t_\mathrm{f}/m$, m being the inertial mass of the freely moving body.

(*) The authors of this paper have agreed to not receive the proofs for correction.

The SQL can be shortly illustrated as follows. Let us consider that the moving mass after the first measurement at $t=0$ is described by the following minimum uncertainty wave-packet (MUW) centered at q_0, and moving to the right with momentum $\hbar k_0$:

$$\psi(q,0)=\left(\frac{1}{2\pi\delta^2}\right)^{1/4}\exp\left[-\frac{(q-q_0)^2}{4\delta^2}+ik_0(q-q_0)\right]\ . \tag{1}$$

If such wave-packet undergoes a free evolution for a time t_f, its initial position variance $\delta^2=\langle\Delta\hat{q}^2(0)\rangle$ increases as follows [4]:

$$\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle=\delta^2\left(1+\frac{t_\mathrm{f}^2\hbar^2}{4\delta^4m^2}\right)=\delta^2+\frac{\Delta_{\mathrm{SQL}}^4}{4\delta^2}\ , \tag{2}$$

where $\langle\cdots\rangle\doteq\mathrm{Tr}[\cdots\hat{\varrho}]$ denotes the ensemble average and $\Delta\hat{O}\doteq\hat{O}-\langle\hat{O}\rangle$ for any operator $\hat{O}$. Minimizing $\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle$ with respect to δ^2 in eq. (2), one obtains $\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle\geq\Delta_{\mathrm{SQL}}^2$. Thus, there is a limit on the accuracy of a subsequent position measurement, which originates from the spreading of the free-mass wave-function.

Is it possible to beat the SQL by preparing the moving mass in a different wave-function? The answer was given by Yuen [5], who identified a class of states—the *contractive* states—that have position uncertainty that decreases *versus* time, and thus can go below the SQL. Yuen also presented some measurement models of the Arthurs-Kelly type [6] that realize in different ways the state reduction toward contractive states [7,8]. Ozawa subsequently proposed a different measurement scheme [9] that leaves the moving mass in a contractive state, and can overcome the SQL. Moreover, after identifying a necessary condition for breaching the SQL [10,11], Ozawa found a class of Hamiltonians that satisfy such condition [12].

In this paper we present a way of engineering *ab initio* a measurement scheme that beats the SQL. The scheme consists of three-steps: a pre-squeezing, a von Neumann measurement, and a feedback. It turns out that our measurement is equivalent (*i.e.* it has the same outcome probability distribution and the same state reduction) to a model belonging to a general class previously studied by Ozawa [12]. The object-probe interaction of the von Neumann measurement is $\hat{H}_I=\hat{q}\hat{P}$. In our notation, lower-case operators denote system observables (the moving mass) and capital operators denote observables of the probe which, in our case, is a single mode of the electromagnetic field. Thus $\hat{q}$ denotes the position operator, whereas the role of the linear momentum $\hat{P}$ for the field is played by the *quadrature* $\hat{P}\equiv\hat{X}_\phi\doteq(\hat{A}^\dagger e^{i\phi}+\hat{A}e^{-i\phi})/2$ of the probing mode, with annihilation and creation operators $\hat{A}$ and $\hat{A}^\dagger$. The $\hat{q}\hat{P}$ interaction may be difficult to achieve; hence we analyze also the case of a von Neumann measurement based on the interaction $\hat{H}_I=\hat{q}\hat{A}^\dagger\hat{A}$, which is just the radiation-pressure Hamiltonian of an interferometric measurement of a moving mirror position. However, in this case we show that we can only reach the SQL, but we cannot overcome it.

The outline of our paper is as follows. Section **2** is a brief review of the formal framework for repeated quantum measurements. After giving the concepts of probability operator-valued measures (POM's) and instruments, we recall the notions of *precision* and *posterior deviation* to describe the noise from the measurement device and the disturbance from the state reduction. With these two concepts in mind we can recall the precise statement of the SQL due to Ozawa [10], who has given a necessary condition to breach the SQL. In sect. **3** we see that the SQL can be overcome by a Gordon-Louisell

(GL) measurement [13], as also shown by Yuen [8,7]. After proving that every GL state reduction can be obtained by means of a suitable feedback mechanism, we show how in this way it is possible to engineer a measurement scheme that beats the SQL. As announced, the scheme consists of the sequence of a pre-squeezing, a von Neumann measurement, and a feedback. In sect. **4** we briefly recall Ozawa's measurement models that satisfy Ozawa's condition, and show how our scheme realizes some measurements in this class. In sect. **5** we analyze the case of a von Neumann measurement achieved with the interaction $\hat{H}_I = \hat{q}\hat{A}^\dagger\hat{A}$. Section **6** closes the paper with some concluding remarks.

2. – Repeated and approximated measurements

In this section we review the main points of the theory of repeated quantum measurements, and we resume the formulation of the SQL given by Ozawa [10, 11], based on a necessary condition to breach the SQL. We will not give a complete general treatment of the subject, but only introduce the main concepts and notation that will be used in the following sections: for more extensive and rigorous treatments see refs. [10, 11, 14] and references therein.

2·1. POM*'s and instruments.* – A complete description of a quantum measurement consists of both: i) the probability density $p(x|\hat{\varrho})\mathrm{d}x$ of the result x of the measurement when the quantum system is in the state described by the density matrix $\hat{\varrho}$; and ii) the state reduction $\hat{\varrho} \to \hat{\varrho}_x$ for the system immediately after the measurement with outcome x. Both $p(x|\hat{\varrho})\mathrm{d}x$ and $\hat{\varrho}_x$ can be expressed in terms of one map, the so-called *instrument* $\mathrm{d}I(x)$, which is a linear map on the space of trace class operators $\hat{\varrho} \to \mathrm{d}I(x)\hat{\varrho}$ given by

$$p(x|\hat{\varrho})\mathrm{d}x = \mathrm{Tr}[\mathrm{d}I(x)\hat{\varrho}] \ , \tag{3}$$

$$\hat{\varrho} \to \hat{\varrho}_x = \frac{\mathrm{d}I(x)\hat{\varrho}}{\mathrm{Tr}[\mathrm{d}I(x)\hat{\varrho}]} \ . \tag{4}$$

Hence, the instrument $\mathrm{d}I(x)$ gives a complete description of the quantum measurement. Sometimes, however, one is interested only in the probability density $p(x|\hat{\varrho})$ of the measure outcome x, ignoring the state reduction: in this case it is sufficient to know the *probability operator-valued measure* (POM) $\mathrm{d}\hat{\Pi}(x)$ of the measurement, which provides the probability distribution of the readout x for any state $\hat{\varrho}$ as follows:

$$p(x|\hat{\varrho})\mathrm{d}x = \mathrm{Tr}[\hat{\varrho}\mathrm{d}\hat{\Pi}(x)] \ . \tag{5}$$

By comparing eq. (5) with eq. (3) one can see that for every instrument $\mathrm{d}I(x)$ the corresponding POM $\mathrm{d}\hat{\Pi}(x)$ is defined through the trace-duality relation $\mathrm{Tr}[\hat{\varrho}\mathrm{d}\hat{\Pi}(x)] = \mathrm{Tr}[\mathrm{d}I(x)\hat{\varrho}]$. The correspondence between $\mathrm{d}I(x)$ and $\mathrm{d}\hat{\Pi}(x)$ is not one-to-one, because the same POM can be achieved by different instruments $\mathrm{d}I(x)$, namely with different state reductions.

Insofar we have given an abstract description of the quantum measurement, with no mention to the physical realization of the measuring apparatus. In order to have an output state that depends on the state before the measurement, the measurement apparatus must involve a probe that interacts with the system, and later is measured to yield information on the system. This *indirect* measurement scheme is completely specified once the following ingredients are given: i) the unitary operator $\hat{U}$ that describes the system-probe interaction; ii) the state $|\varphi\rangle$ of the probe before the interaction (we restrict our attention to the case of pure-state preparation of the probe); iii) the observable $\hat{X}$ which is measured on the probe. At the end of the system-probe interaction it is possible to consider a subsequent measurement of a (generally different) observable $\hat{Y}$ on the system (with outcome y). Then, it can be easily shown (see, for example, ref. [14]) that the conditional probability density $p(y|x)$ of getting the result y from the second measurement—x being the result of the first one—can be written in terms of Born's rule $p(y|x)\mathrm{d}y = \langle y|\hat{\varrho}_x|y\rangle$ upon defining a "reduced state" $\hat{\varrho}_x$ as in eq. (4), where the instrument and the POM are given by

$$\mathrm{d}\hat{\Pi}(x) = \mathrm{d}x\, \hat{\Omega}^\dagger(x)\hat{\Omega}(x)\ , \tag{6}$$

$$\mathrm{d}I(x)\hat{\varrho} = \mathrm{d}x\, \hat{\Omega}(x)\hat{\varrho}\hat{\Omega}^\dagger(x)\ , \tag{7}$$

and the operator $\hat{\Omega}(x)$, which acts on the Hilbert space of the system only, is defined by the following matrix element on the probe Hilbert space:

$$\hat{\Omega}(x) \doteq \langle x|\hat{U}|\varphi\rangle\ , \tag{8}$$

$|x\rangle$ being the eigenvector of the observable $\hat{X}$ corresponding to eigenvalue x. As regards the evolution operator $\hat{U}$, one can neglect, for simplicity, the free evolution during the measurement interaction time, and consider an *impulsive* interaction Hamiltonian that is switched on only for a very short time interval τ with a very large coupling constant K, such that $K\tau$ is finite. For simplicity of notation, in the following we will implicitly include $K\tau$ in the definition itself of the interaction Hamiltonian.

2˙2. *Precision and posterior deviation.* – In any scheme for a quantum measurement, in principle there are always two kinds of noise: i) the quantum noise of the observable that is intrinsic of the quantum state $\hat{\varrho}$, which is given by the variance $\langle\Delta\hat{q}^2\rangle$; ii) the noise due to the measuring apparatus, which is generally non-ideal, for example, because of a non-unit quantum efficiency, or as a consequence of the noise due to a joint measurement. This leads to an output probability distribution $p(x|\hat{\varrho})\mathrm{d}x$ that is broader than $\langle\Delta\hat{q}^2\rangle$. A POM $\mathrm{d}\hat{\Pi}(x)$ is said to be *compatible* with an observable $\hat{q}$ (or $\hat{q}$-compatible) if it satisfies the relation

$$[\mathrm{d}\hat{\Pi}(x), \hat{q}] = 0\ , \tag{9}$$

namely the POM has the same spectral decomposition of $\hat{q}$

$$\mathrm{d}\hat{\Pi}(x) = G(x,\hat{q})\mathrm{d}x = \mathrm{d}x \int G(x,q)|q\rangle\langle q|\,\mathrm{d}q\ , \tag{10}$$

$G(x,q)$ playing the role of a conditional probability density for the output x, given that the position of the system was q.

The extrinsic *instrumental noise* or *precision* $\epsilon^2[\varrho]$ of the apparatus with $\hat{q}$-compatible POM $\mathrm{d}\hat{\Pi}(x)$ for a measurement of the observable $\hat{q}$, estimates the broadening of the intrinsic noise due to the measurement, and is defined as follows:

$$\epsilon^2[\hat{\varrho}] \doteq \int\int (x-q)^2 \mathrm{Tr}[\hat{\varrho}\mathrm{d}\hat{\Pi}(x)|q\rangle\langle q|]\,\mathrm{d}q \equiv \int \mathrm{Tr}[(x-\hat{q})^2\hat{\varrho}\mathrm{d}\hat{\Pi}(x)]\ . \tag{11}$$

The overall noise or total uncertainty $\overline{\Delta x^2}[\hat{\varrho}]$ of the measurement is the variance of the experimental probability distribution, namely

$$\overline{\Delta x^2}[\hat{\varrho}] \equiv \mathrm{E}[\Delta x^2||\hat{\varrho}] \equiv \mathrm{E}[x^2||\hat{\varrho}] - \mathrm{E}[x||\hat{\varrho}]^2 \ , \tag{12}$$

where $\mathrm{E}[g(x)||\hat{\varrho}] \doteq \int g(x)p(x|\hat{\varrho})\mathrm{d}x$ denotes the experimental expectation value of the function $g(x)$ of the random outcome x. The total uncertainty in eq. (12) can be simply written as the sum of $\epsilon^2[\hat{\varrho}]$ and $\langle\Delta\hat{q}^2\rangle$ if the POM is *unbiased*, namely $\mathrm{E}[x||\hat{\varrho}] \equiv \langle\hat{q}\rangle$ for every state $\hat{\varrho}$. In that case one has

$$\langle\hat{q}\rangle \doteq \mathrm{Tr}[\hat{q}\hat{\varrho}] \equiv \int \mathrm{d}x\, x\, p(x|\hat{\varrho}) \ , \tag{13}$$

and hence $\hat{q}$ can be spectrally decomposed in terms of the POM itself, namely

$$\hat{q} = \int x\mathrm{d}\hat{\Pi}(x) \Longleftrightarrow \mathrm{E}[x||\hat{\varrho}] \equiv \langle\hat{q}\rangle \ , \tag{14}$$

for all states $\hat{\varrho}$ with $\overline{\Delta x^2}[\hat{\varrho}] < \infty$. The precision of a measurement realized with a $\hat{q}$-compatible POM is equal to zero if and only if $G(x,q) = \delta(x-q)$, namely if the measurement apparatus is *noiseless*. Hence

$$\epsilon^2[\hat{\varrho}] = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}\hat{\Pi}(x) = \mathrm{d}x|x\rangle\langle x| \ . \tag{15}$$

In this case the total uncertainty of the measurement is just the variance of the system state.

For a $\hat{q}$-compatible and unbiased POM, eq. (12) can be rewritten as

$$\overline{\Delta x^2}[\hat{\varrho}] = \epsilon^2[\hat{\varrho}] + \langle\Delta\hat{q}^2\rangle \ , \tag{16}$$

namely the intrinsic and the instrumental noises behave additively. The precision $\epsilon^2[\hat{\varrho}]$ of the measurement characterizes only the POM $\mathrm{d}\hat{\Pi}(x)$, namely the probability distribution of the measurement. On the other hand, the noise after the state reduction can be quantified by the *posterior deviation* $\sigma^2[\hat{\varrho}]$ of the instrument I, which is defined as follows:

$$\sigma^2[\hat{\varrho}] \doteq \int\int (q-x)^2\langle q|\hat{\varrho}_x|q\rangle p(x|\hat{\varrho})\,\mathrm{d}x\,\mathrm{d}q \equiv \int \mathrm{Tr}[(\hat{q}-x)^2\mathrm{d}I(x)\hat{\varrho}] \ . \tag{17}$$

In refs. [10] and [11] the quantity $\sigma^2[\hat{\varrho}]$ is named "resolution": here we suggest to adopt the nomenclature "posterior deviation", as it is clearly a property of the reduced state $\hat{\varrho}_x$. For $\hat{\varrho}$ satisfying $\overline{\Delta x^2}[\hat{\varrho}] < \infty$ and $\mathrm{Tr}\left[\Delta\hat{q}^2 \int \mathrm{d}I(x)\hat{\varrho}\right] < \infty$ one has [11]

$$\sigma^2[\hat{\varrho}] = \int \langle\Delta\hat{q}^2\rangle_x p(x|\hat{\varrho})\mathrm{d}x + \int \left[\langle\hat{q}\rangle_x - x\right]^2 p(x|\hat{\varrho})\mathrm{d}x \ , \tag{18}$$

where $\langle\cdots\rangle_x \doteq \mathrm{Tr}[\cdots\hat{\varrho}_x]$ denotes the conditional expectation. Therefore for *unbiased reduction*—*i.e.* the average value of $\hat{q}$ after the state reduction is still equal to the outcome

x of the measurement—$\sigma^2[\hat{\varrho}]$ is just the variance of the reduced state averaged over all the readouts x, namely

$$\mathrm{Tr}[\hat{q}\hat{\varrho}_x] = x \qquad \Longrightarrow \qquad \sigma^2[\hat{\varrho}] = \int \langle \Delta \hat{q}^2 \rangle_x p(x|\hat{\varrho}) \mathrm{d}x \ . \tag{19}$$

2·3. *Generalized Standard Quantum Limit.* – Let us consider a $\hat{q}$-compatible and unbiased POM. At $t = 0$ we measure the position of the system and we get outcome x and state reduction $\hat{\varrho}_x$, with probability distribution $p(x|\hat{\varrho})\mathrm{d}x$. Then we let the system evolve freely for a time interval t_f and perform a second measurement on $\hat{\varrho}_x(t_\mathrm{f})$. According to a mean-value strategy, one predicts the second measurement to have the outcome $h(x) \doteq \mathrm{Tr}[\hat{\varrho}_x(t_\mathrm{f})\hat{q}]$ with an uncertainty given by

$$\Delta(t_\mathrm{f}, \hat{\varrho}, x) = \left[\int [x' - h(x)]^2 p(x'|\hat{\varrho}_x(t_\mathrm{f}))\mathrm{d}x' \right]^{1/2} , \tag{20}$$

which, for an unbiased POM is just given by

$$\Delta^2(t_\mathrm{f}, \hat{\varrho}, x) = \overline{\Delta x^2}[\hat{\varrho}_x(t_\mathrm{f})] \ . \tag{21}$$

The "predictive uncertainty" $\Delta^2(t_\mathrm{f}, \hat{\varrho})$ of the repeated measurement [10] for prior state $\hat{\varrho}$ is defined as the average of $\Delta^2(t_\mathrm{f}, \hat{\varrho}, x)$ over all the outcomes x at $t = 0$, namely

$$\Delta^2(t_\mathrm{f}, \hat{\varrho}) \doteq \int \mathrm{d}x \, \Delta^2(t_\mathrm{f}, \hat{\varrho}, x) \, p(x|\hat{\varrho}) \ . \tag{22}$$

Ozawa [11] introduced a precise definition of the SQL as the lower bound of the predictive uncertainty $\Delta^2(t_\mathrm{f}, \hat{\varrho})$ with the hypothesis that the averaged precision of the evolved state—the precision of a second measurement on the reduced state at time t_f—is greater than the posterior deviation. Hence, for unbiased measurements satisfying the inequality

$$\int \mathrm{d}x \, p(x|\hat{\varrho}) \, \epsilon^2[\hat{\varrho}_x(t_\mathrm{f})] \geq \sigma^2[\hat{\varrho}] \ , \tag{23}$$

Ozawa proved the bound

$$\Delta^2(t_\mathrm{f}, \hat{\varrho}) \geq |\mathrm{Tr}\,[\hat{\varrho}_R[\hat{q}(0), \hat{q}(t_\mathrm{f})]]\,|^2 \doteq \Delta^2_{\mathrm{SQL}} \ , \tag{24}$$

where for the free-mass evolution $\Delta^2_{\mathrm{SQL}} = \hbar t_\mathrm{f}/m$. Equation (24) is the general version of the SQL due to Ozawa. Using eq. (21) one has

$$\Delta^2(t_\mathrm{f}, \hat{\varrho}) = \int \mathrm{d}x \, p(x|\hat{\varrho}) \, \epsilon^2[\hat{\varrho}_x(t_\mathrm{f})] + \int \mathrm{d}x \, p(x|\hat{\varrho}) \, \mathrm{Tr}[\Delta\hat{q}^2 \, \hat{\varrho}_x(t_\mathrm{f})]\rangle \geq \Delta^2_{\mathrm{SQL}} \ . \tag{25}$$

Therefore, a *necessary* condition for breaching the SQL corresponds to negating the hypothesis (23), namely that the precision of the measurement is less than the posterior deviation. We will call this condition ONC (Ozawa necessary condition). We recall that, while the precision (11) is a measure of the added noise and is related to the POM, the posterior deviation (17) is related to the state reduction. As an example, it is interesting to evaluate both $\sigma^2[\hat{\varrho}]$ and $\epsilon^2[\hat{\varrho}]$ for a generalized von Neumann measurement model with interaction Hamiltonian of the form $\hat{H}_I = \hat{q}\hat{O}$, where $\hat{O}$ is a generic observable of the probe, and $\hat{q}$ is just the quantity of the system that we want to measure—the position in our case (in the "standard" von Neumann model [15] $\hat{O} \equiv \hat{P}$, where $\hat{P}$ is the linear momentum of the probe). From eqs. (6), (7) and (8), it is apparent that both the POM and the instrument are only functions of the operator $\hat{q}$, so that the precision (11) and the posterior deviation (17) coincide. This means that condition (23) is verified only with the equal sign, and the von Neumann model just achieves the SQL.

3. – Evolution operator for beating the SQL

In this section we will resume the possibility to overcome the SQL by reducing the system to a contractive state. Then we will design an evolution operator $\hat{U}$ that realizes such reduction. Finally, we will generalize $\hat{U}$ to a class of operators which fulfill ONC.

3·1. *The contractive states.* – The original argument for the validity of the SQL due to Braginskĭi and Caves [1,3], was the following. Consider a mass m, whose position $\hat{q}$ has been measured once within a certain precision, and let it freely evolve for a time t_f. In the Heisenberg picture the position operator evolves *classically*, as $\hat{q}(t_\mathrm{f}) = \hat{q}(0) + \hat{p}(0)t_\mathrm{f}/m$. From the Heisenberg uncertainty principle, one obtains the following position variance constraint:

$$\begin{aligned}\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle &= \langle\Delta\hat{q}^2(0)\rangle + \langle\Delta\hat{p}^2(0)\rangle\left(\frac{t_\mathrm{f}}{m}\right)^2 \\ &\geq \langle\Delta\hat{q}^2(0)\rangle + \frac{\hbar^2}{4\langle\Delta\hat{q}^2(0)\rangle}\left(\frac{t_\mathrm{f}}{m}\right)^2 \geq \Delta^2_{\mathrm{SQL}}\,.\end{aligned} \tag{26}$$

However, as pointed out by Yuen [5, 8], this is not the correct general expression for $\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle$, because it neglects the correlation term $2\mathrm{Re}\langle\Delta\hat{q}(0)\Delta\hat{p}(0)\rangle \doteq \langle\Delta\hat{q}(0)\Delta\hat{p}(0) + \Delta\hat{p}(0)\Delta\hat{q}(0)\rangle$, hence implicitly assuming that it is greater than or equal to zero. The complete expression for the position uncertainty is

$$\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle = \langle\Delta\hat{q}^2(0)\rangle + 2\mathrm{Re}\langle\Delta\hat{q}(0)\Delta\hat{p}(0)\rangle\frac{t_\mathrm{f}}{m} + \langle\Delta\hat{p}^2(0)\rangle\left(\frac{t_\mathrm{f}}{m}\right)^2\,. \tag{27}$$

As a matter of fact, there is a class of states, the *contractive states* (CS), which have a negative correlation term in eq. (27). When the reduced state $\hat{\varrho}_x$ for the free mass after the first measurement is a CS, $\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle_x$ decreases in time before reaching a minimum value. If the measurement is sufficiently precise, the SQL can be overcome. In fact, if $\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle_x \leq \langle\Delta\hat{q}^2(0)\rangle_x$ for every outcome x, then this is true also on the average, namely

$$\int \mathrm{d}x\, p(x|\hat{\varrho})\,\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle_x \leq \int \mathrm{d}x\, p(x|\hat{\varrho})\,\langle\Delta\hat{q}^2(0)\rangle_x\,. \tag{28}$$

Equations (16) and (21) lead to

$$\Delta^2(t_\mathrm{f},\hat{\varrho}) = \int \mathrm{d}x\, p(x|\hat{\varrho})\,\epsilon^2[\hat{\varrho}_x(t_\mathrm{f})] + \int \mathrm{d}x\, p(x|\hat{\varrho})\,\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle_x\,, \tag{29}$$

which, for noiseless measurements $\epsilon^2[\hat{\varrho}_x(t_\mathrm{f})] = 0$, gives

$$\Delta^2(t_\mathrm{f}, \hat{\varrho}) = \int \mathrm{d}x\, p(x|\hat{\varrho})\, \langle \Delta\hat{q}^2(t_\mathrm{f})\rangle_x \leq \int \mathrm{d}x\, p(x|\hat{\varrho})\, \langle \Delta\hat{q}^2(0)\rangle_x \ . \tag{30}$$

A particular example of a CS are the squeezed states, or *twisted coherent state* (TCS) [16]. A TCS $|\mu\nu\alpha\omega\rangle$ for the mass has the following position representation:

$$\langle q|\mu\nu\alpha\omega\rangle = \left(\frac{mw}{\pi\hbar|\mu-\nu|^2}\right)^{1/4} \exp\left[-\frac{mw}{2\hbar}\frac{1+2i\xi}{|\mu-\nu|^2}(q-q_0)^2 + \frac{i}{\hbar}p_0(q-q_0)\right], \tag{31}$$

with fluctuations

$$\langle \Delta\hat{q}^2(t_\mathrm{f})\rangle = \frac{\hbar}{2m\omega}|\mu-\nu|^2 - \frac{2\hbar\xi}{m}t_\mathrm{f} + \frac{\hbar\omega}{2m}|\mu+\nu|^2 t_\mathrm{f}^2\ , \tag{32}$$

where $|\mu|^2 - |\nu|^2 = 1$, $\xi = \mathrm{Im}(\mu^*\nu) > 0$ and $\alpha = q_0 + ip_0$, q_0 and p_0 being real. The correlation function $\mathrm{Re}\langle\Delta\hat{q}(0)\Delta\hat{p}(0)\rangle = -\xi\hbar$, is negative for $\xi > 0$, so that the position uncertainty $\langle\Delta\hat{q}^2(t_\mathrm{f})\rangle$ decreases in time till, at $t_{M} = 2\xi/\omega|\mu+\nu|^2$ it reaches the minimum value $\langle\Delta\hat{q}^2(t_{M})\rangle = (4\xi)^{-1}(\hbar t_{M}/m) < \Delta^2_{\mathrm{SQL}}$ for sufficiently large ξ [10]. This implies that, for noiseless detection (eq. (15)) the total uncertainty of the measurement goes below the SQL. A detection scheme of this type is described by the measurement with state reduction $\hat{\varrho} \to \hat{\varrho}_Q = |\mu\nu Q\omega\rangle\langle\mu\nu Q\omega|$ and probability distribution $p(Q|\hat{\varrho}) = \langle Q|\hat{\varrho}|Q\rangle$, Q being the output of a single measurement. This is a GL measurement with reduction operator $\hat{\Omega}$ given by

$$\hat{\Omega}(Q) = |\mu\nu Q\omega\rangle\langle Q|\ . \tag{33}$$

It is easily proved that this scheme satisfies the request of unbiasedness for both the POM and the state reduction, and, as expected, eq. (23) is contradicted, because $\sigma^2[\hat{\varrho}] = \int \mathrm{d}Qp(Q|\hat{\varrho})\mathrm{Tr}[\Delta\hat{q}^2\hat{\varrho}_Q(0)] > \int \mathrm{d}Qp(Q|\hat{\varrho})\epsilon^2[\hat{\varrho}_Q(t_\mathrm{f})] \equiv 0$.

Yuen [7, 8] and Ozawa have given different interaction Hamiltonians that realize the operator $\hat{\Omega}(Q)$ in eq. (33). In the next section, we will engineer *ab initio* a measurement scheme that has the state reduction $|\mu\nu Q\omega\rangle$.

3·2. *Gordon-Louisell measurements*. – The definition of a general GL measurement is given by a state reduction operator of the form

$$\hat{\Omega}(x) = |\psi_x\rangle\langle\theta_x|\ , \tag{34}$$

where $|\theta_x\rangle$ is a complete set, *i.e.* $\int \mathrm{d}x|\theta_x\rangle\langle\theta_x| = \hat{1}$ and $|\psi_x\rangle$ is a normalized physical state. The operator $\hat{\Omega}(x)$ in eq. (34) abstractly represents an indirect measurement that leaves the system in the state $|\psi_x\rangle$, independently of the input state, and has probability distribution for the outcome x given by $p(x|\hat{\varrho}) = \langle\theta_x|\hat{\varrho}|\theta_x\rangle$. In the particular case that $|\theta_x\rangle$ is also an orthogonal set $\langle\theta_x|\theta'_x\rangle = \delta(x-x')$, there is a unitary operator $\hat{U}$ on the system-probe Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_P$ that gives $\hat{\Omega}(x)$ in eq. (34) through eq. (8). In fact, let us define the self-adjoint operator $\hat{\theta}$ such that $\hat{\theta}|\theta_x\rangle = \theta_x|\theta_x\rangle$. Then, let us consider an indirect measurement scheme for $\hat{\theta}$, with the probe prepared in the (pure) state $|\varphi\rangle$. Comparing eq. (8) with eq. (34), we see that $\hat{U}$ must satisfy the identity

$$\langle\theta_y| \otimes \langle\theta_x|\hat{U}|\theta_z\rangle \otimes |\varphi\rangle = \langle\theta_y|\psi_x\rangle\langle\theta_x|\theta_z\rangle = \langle\theta_y|\psi_x\rangle\delta(x-z)\ . \tag{35}$$

Hence, a unitary operator $\hat{U}$ that has the matrix elements (35) can be chosen as achieving the following linear transformation on $\mathcal{H}_S \otimes \mathcal{H}_P$:

$$\hat{U}|\theta_x\rangle \otimes |\varphi\rangle = |\psi_x\rangle \otimes |\theta_x\rangle\ . \tag{36}$$

How can we design such an operator $\hat{U}$? Consider the operator $\hat{R}$ that effects a rotation by $\pi/2$ in the (q,Q) plane, namely

$$\hat{R} = \exp\left[\frac{i\pi}{2\hbar}(\hat{p}\hat{Q} - \hat{q}\hat{P})\right]\ . \tag{37}$$

Apart from an inversion and a trivial overall phase factor, the operator R corresponds to a mode-permutation operator since one has

$$\hat{R}\psi_1(q)\psi_2(Q) = \psi_2(-q)\psi_1(Q)\ . \tag{38}$$

Then we introduce the notion of feedback operator $\hat{F}(x) \in \mathcal{H}_S$, namely a self-adjoint operator that parametrically depends on the measure outcome x. This can be any self-adjoint operator function of x: however, we need to restrict the class of operator functions $\hat{F}(x)$ such that the integral $\hat{F}_S(\hat{X}) \doteq \int \mathrm{d}x\, \hat{F}(x) \otimes |\theta_x\rangle\langle\theta_x|$ converges to a well-defined self-adjoint operator acting on $\mathcal{H}_S \otimes \mathcal{H}_P$, with $\hat{X}|\theta_x\rangle = x|\theta_x\rangle$ ([1]). We choose the feedback Hamiltonian such that it connects the vectors $|\varphi\rangle$ and $|\psi_x\rangle$, namely

$$|\psi_x\rangle \doteq \exp\left[-\frac{i}{\hbar}\hat{F}(x)\right]|\varphi\rangle\ . \tag{39}$$

Finally, we write the operator $\hat{U}$ as follows:

$$\hat{U} = \exp\left[-\frac{i}{\hbar}\hat{F}_S(\hat{X})\right]\exp\left[\frac{i\pi}{2\hbar}(\hat{p}\hat{Q} - \hat{q}\hat{P})\right]\ . \tag{40}$$

Now we specialize eq. (40) to the case described by eq. (33). The wave-functions $|\psi_x\rangle$ and $|\theta_x\rangle$ correspond to $|\mu\nu Q\omega\rangle$ and $|Q\rangle$ respectively, whereas the probe observable $\hat{\theta}$ is the position $\hat{Q}$, with outcome Q. This implies that the feedback operator has to shift the position of the system by a quantity equal to the output Q, namely $\hat{F}_S(\hat{Q}) = \hat{p}\hat{Q}$. Hence, the evolution operator has the form

$$\hat{U} = \exp\left[-\frac{i}{\hbar}\hat{p}\hat{Q}\right]\exp\left[\frac{i\pi}{2\hbar}(\hat{p}\hat{Q} - \hat{q}\hat{P})\right]\ . \tag{41}$$

([1]) For $\hat{F}(x) = \sum \hat{f}_n x^n$, one has $F_S(\hat{X}) = \sum \hat{f}_n \otimes \hat{X}^n$.

We emphasize that, as $\hat{U}$ contains a permutation operator, the initial probe state must be chosen to be a TCS. Hence, if the initial state of the system is $|\psi\rangle$, we get $\hat{R}|\psi\rangle\otimes|\mu\nu\alpha\omega\rangle \propto |\mu\nu(-\alpha)\omega\rangle\otimes|\psi\rangle$: in practice, the position of the object is indirectly squeezed by squeezing the probe position, and then exchanging the state of the system with that of the probe. For simplicity, we choose $\alpha = 0$, which means that the initial position of the probe is equal to zero. After that, the feedback operator shifts the position of the system so that it finally corresponds to the output of the measurement, that is $\exp\left[-\frac{i}{\hbar}\hat{F}(Q)\right]|\mu\nu 0\omega\rangle = |\mu\nu Q\omega\rangle$. Now we relax the value of the coupling constants in the interaction Hamiltonians and rewrite the evolution operator (41) in the general form

$$\hat{U} = \exp\left[-\frac{i}{\hbar}g_1\hat{p}\hat{Q}\right]\exp\left[\frac{i\pi}{2\hbar}g_2(s^{-1}\hat{p}\hat{Q} - s\hat{q}\hat{P})\right] . \tag{42}$$

As regards the concrete feasibility of the evolution described by eq. (42), the feedback part can be achieved by a position transducer, which displaces the mass by an amount $g_1 Q$ for every position outcome Q. Notice that we do not need to realize the feedback Hamiltonian $g_1\hat{p}\hat{Q}$, because as a consequence of eq. (8) this is equivalent to the system Hamiltonian $g_1\hat{p}Q$, where the coupling is rescaled by the eigenvalue Q in place of the operator $\hat{Q}$. The permutation operator will be discussed in the following section.

4. – Comparison with previous models and realization

In the first part of this section we compare the unitary evolutions of Ozawa's measurement models [12] satisfying ONC with our operator in eq. (42). In the second part we factorize our evolution into three steps, expressing the permutation operator by means of a pre-squeezing of both system and probe, followed by a von Neumann interaction $\hat{H}_I = \hat{q}\hat{P}$.

4˙1. *Ozawa's Hamiltonians and the conditions for beating the* SQL. – In ref. [12] Ozawa gives a class of measurement models that satisfy ONC. He explicitly writes the unitary system-probe evolution in an impulsive regime in terms of an interaction Hamiltonian $\hat{H}_I$, generalizing the standard von Neumann measurement model with $\hat{H}_I = \hat{q}\hat{P}$ [15]. The Hamiltonian is given by

$$\hat{H}_I = -k_+\hat{Q}\hat{p} - k_-\hat{q}\hat{P} + k_z(\hat{q}\hat{p} - \hat{Q}\hat{P}) , \tag{43}$$

with k_+ , k_- and k_z as real parameters. For probe initial state $|\varphi\rangle$, with $\varphi(Q) = \varphi(-Q)$ and $\langle\varphi|\hat{Q}|\varphi\rangle = 0$ (which is equivalent to condition $\alpha = 0$ in the previous section), and for the system initially in the state $|\psi\rangle$, eqs. (11) and (17) become

$$\epsilon^2[\psi] = (1-c)^2\langle\psi|\hat{q}^2|\psi\rangle + d^2\langle\varphi|\hat{Q}^2|\varphi\rangle , \tag{44}$$

$$\sigma^2[\psi] = (a-c)^2\langle\psi|\hat{q}^2|\psi\rangle + (b-d)^2\langle\varphi|\hat{Q}^2|\varphi\rangle . \tag{45}$$

Here the coefficients of the dynamics (a, b, c, d)—grouped in a matrix form that will be used in the following—are

$$\begin{bmatrix} a & -b \\ -c & d \end{bmatrix} = \begin{bmatrix} \cosh\mathcal{K} + k_z\frac{\sinh\mathcal{K}}{\mathcal{K}} & k_+\dfrac{\sinh\mathcal{K}}{\mathcal{K}} \\ k_-\dfrac{\sinh\mathcal{K}}{\mathcal{K}} & \cosh\mathcal{K} - k_z\dfrac{\sinh\mathcal{K}}{\mathcal{K}} \end{bmatrix} , \tag{46}$$

where $\mathcal{K} = \sqrt{k_z^2 + k_+ k_-}$ can be either real or pure imaginary.

The degrees of freedom of the above equations can be reduced by applying the criteria for a plausible object-probe interaction: one requests that if the uncertainty of the prior probe coordinate $\langle\varphi|\Delta\hat{Q}^2|\varphi\rangle$ tends to zero, then the precision and the posterior deviation tend to zero. From eqs. (44) and (45) it follows that one needs $a = c = 1$. It can be shown [12] that the condition $c = 1$ is also a consequence of the unbiasedness of the POM. Moreover, as $ad - bc = 1$, it turns out that $b = d - 1$, and using eqs. (46) one has $d = 2\cosh\mathcal{K} - 1$. With the above restrictions, the precision and the posterior deviation become $\epsilon^2 = d^2\langle\varphi|\Delta\hat{Q}^2|\varphi\rangle$ and $\sigma^2 = \langle\varphi|\Delta\hat{Q}^2|\varphi\rangle$, independently of the initial system state. Thus, the ONC for beating the SQL can be expressed as $|d| < 1$. In summary, we have

$$
\begin{aligned}
&a = 1\,, \quad b = d - 1\,, \quad c = 1\,, \\
&|d| = |2\cosh\mathcal{K} - 1| < 1\,, \quad \text{ONC} \Longleftrightarrow |d| < 1\,.
\end{aligned}
\tag{47}
$$

If $\mathcal{K}$ is real, then $d > 1$; but, if $\mathcal{K}$ is pure imaginary, $d = 2\cos|\mathcal{K}| - 1$, and $-3 < d < 1$. Thus, Ozawa's Hamiltonians satisfy ONC for certain values of the parameters (k_+, k_-, k_z) which can be found by comparing eqs. (47) with eqs. (46), namely

$$
k_+ = 2k_z\,, \qquad \cosh\mathcal{K} = 1 + \frac{k_z}{k_-}\,, \qquad -1 < \frac{k_z}{k_-} < 0\,.
\tag{48}
$$

In particular, among the Hamiltonians (43), the model described by Ozawa in ref. [9], for $k_z = \pi/3\sqrt{3}$ and $k_\pm = \pm 2k_z$, realizes the GL scheme $|\mu\nu Q\omega\rangle\langle Q|$ with $d = 0$. On the other hand, the original von Neumann model [15] corresponds to $d = 1$ and $\epsilon^2 = \sigma^2 = \langle\varphi|\Delta\hat{Q}^2|\varphi\rangle$. This means that there is a continuum of models for $|d| < 1$ that have a better precision than $\sigma^2 = \langle\varphi|\Delta\hat{Q}^2|\varphi\rangle$, and can possibly circumvent the SQL.

We now look for the relation between our model and the model just described. We use the following realization of the angular momentum (complex) Lie algebra $gl(2, C)$:

$$
\hat{J}_+ \doteq \frac{i\hat{Q}\hat{p}}{\hbar}\,, \qquad \hat{J}_- \doteq \frac{i\hat{q}\hat{P}}{\hbar}\,, \qquad \hat{J}_z \doteq \frac{i}{2\hbar}(\hat{Q}\hat{P} - \hat{q}\hat{p})\,.
\tag{49}
$$

One can easily verify that the above operators satisfy the $gl(2, C)$ commutation relations $[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z$ and $[\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm$, so that the exponential of their linear combinations can be faithfully represented by 2×2 Pauli matrices, independently of the value of the angular momentum J (the group $gl(2, C)$ is complex, and the fact that the realization (49) does not preserve Hermitian conjugation is irrelevant for the group multiplication law). Notice that the matrix in eq. (46) is nothing but the Pauli representation of the evolution operator $\hat{U} = \exp[-i\hat{H}_I/\hbar]$ in terms of the $gl(2, C)$ algebra realization (49). Both our operator $\hat{U}$ in eq. (41) and Ozawa's in eq. (46) for $k_z = \pi/3\sqrt{3}$ and $k_\pm = \pm 2k_z$, lead to the same coefficients (a, b, c, d), namely $(1, -1, 1, 0)$. In fact, we have just seen that they both realize $\hat{\Omega}(Q)$ in eq. (33). In the general case, the relation between the coefficients of the operator (42) and the ones in eq. (46) are

$$
\begin{aligned}
&a = \cos\left(\frac{\pi g_2}{2}\right) + g_1 s \sin\left(\frac{\pi g_2}{2}\right)\,, \qquad b = -\frac{1}{s}\sin\left(\frac{\pi g_2}{2}\right) + g_1\cos\left(\frac{\pi g_2}{2}\right)\,, \\
&c = s\sin\left(\frac{\pi g_2}{2}\right)\,, \qquad d = \cos\left(\frac{\pi g_2}{2}\right)\,.
\end{aligned}
\tag{50}
$$

The last of eqs. (50) clearly states that d belongs to the interval $[-1, 1]$, namely all possible models realizing the condition $\epsilon < \sigma$ are included.

4˙2. *Realization through $GL(2, C)$ elements.* – In this subsection we suggest a measurement scheme that realizes the evolution operator in eq. (42) or eq. (46) through three steps in the $GL(2, C)$ group. We are describing an indirect measurement of the system position $\hat{q}$ through detection of the probe position $\hat{Q}$, and the central element of our scheme is the von Neumann Hamiltonian $\hat{J}_- \doteq i\hat{q}\hat{P}/\hbar$, which entangles the system object with the probe by shifting $\hat{Q}$ by $\hat{q}$. Moreover, eqs. (42) and (49), suggest that *after* any detection with output Q, a feedback of the form $\hat{J}_+ \doteq i\hat{Q}\hat{p}/\hbar$ is requested. This implies that, among the six evolution operators that can be obtained by permuting the exponentials of $(\hat{J}_+, \hat{J}_-, \hat{J}_z)$, we will not consider the three permutations with $\exp\left[\zeta_+\hat{J}_+\right]$ applied before $\exp\left[\zeta_-\hat{J}_-\right]$: in the next subsection we will analyze the remaining three cases.

4˙3. *Feedback-assisted measurement.* – Consider the sequence

$$\exp\left[\zeta_+\hat{J}_+\right]\exp\left[\zeta_-\hat{J}_-\right]\exp\left[2\zeta_z\hat{J}_z\right] = \exp\left[k_+\hat{J}_+ + k_-\hat{J}_- + 2k_z\hat{J}_z\right] . \tag{51}$$

In this model, the operator $\exp\left[2\zeta_z\hat{J}_z\right] = \exp\left[i\zeta_z(\hat{Q}\hat{P} - \hat{q}\hat{p})/\hbar\right]$ does not entangle the system with the probe, but just *pre-squeezes* the states of both. Then, $\exp\left[\zeta_-\hat{J}_-\right] = \exp\left[i\zeta_-\hat{q}\hat{P}/\hbar\right]$ entangles the probe with the system. Finally, the operator $\exp\left[\zeta_+\hat{J}_+\right] = \exp\left[i\zeta_+\hat{Q}\hat{p}/\hbar\right]$ corresponds to a feedback mechanism that shifts the system position by $\zeta_+ Q$. By means of the Lie algebra decomposition formulas [17], the relations between the parameters $\zeta_+, \zeta_-, \zeta_z$ and Ozawa's write as follows:

$$a = (1 + \zeta_+\zeta_-)e^{\zeta_z} , \qquad -b = \zeta_+ e^{-\zeta_z} , \qquad -c = \zeta_- e^{\zeta_z} , \qquad d = e^{-\zeta_z} . \tag{52}$$

The range of the coefficients $(\zeta_+, \zeta_-, \zeta_z)$ in which the SQL can be overcome is (see eq. (47))

$$\zeta_- = -e^{-\zeta_z} , \qquad \zeta_+ = e^{\zeta_z} - 1 , \qquad \zeta_z > 0 . \tag{53}$$

Now a problem arises, regarding the class of Ozawa's Hamiltonians represented by this model. In fact, as $d = e^{-\zeta_z} > 0$, it is clear that a part of the evolutions that circumvent the SQL are excluded by this parameterization. In particular, rewriting (k_+, k_-, k_z) in terms of $(\zeta_+, \zeta_-, \zeta_z)$ as in (53), one gets

$$\begin{aligned} k_z &= \left|\frac{e^{-\zeta_z} - 1}{e^{-\zeta_z} + 3}\right|^{1/2} \arcsin\left[\frac{1 - e^{-\zeta_z}}{2}\left|\frac{e^{-\zeta_z} + 3}{e^{-\zeta_z} - 1}\right|^{1/2}\right] , \\ k_+ &= 2k_z , \qquad k_- = \frac{2k_z}{e^{-\zeta_z} - 1} , \quad -\frac{1}{2} < \frac{k_z}{k_-} < 0 , \end{aligned} \tag{54}$$

where a narrower interval for k_z/k_- than in eq. (48) is obtained. One is lead to think that there must be some other schemes that give Ozawa's Hamiltonian for $d > 0$. However, an outlook at the remaining two permutations with $\exp\left[\zeta_- \hat{J}_-\right]$ acting before $\exp\left[\zeta_+ \hat{J}_+\right]$ reveals that also in these cases $d = e^{-\zeta_z} > 0$, which implies that a substantial part of Ozawa's Hamiltonians cannot be realized with our scheme. Notice, however, that the Hamiltonian with coefficients $k_z = \pi/3\sqrt{3}$, $k_+ = 2k_z$, $k_- = -2k_z$, which beats the SQL, is included in our scheme (51), and can be achieved in the limit of infinite ζ_z, that is for very high squeezing for both probe and object preparations. The realization of the squeezing of a mass position remains as a challenge for experimentalists. We stress the fact that, if either ζ_z or ζ_+ in eq. (51) are set equal to zero and the conditions for breaching the SQL are imposed, the remaining two coefficients become equal to zero, too. That is, both the feedback and the dilatation are essential in order to achieve a detection scheme that possibly beats the SQL.

Now we calculate the operator $\hat{\Omega}(Q)$ relative to the evolution operator $\hat{U}$ just described. Because of the peculiar form of the operator $\hat{U}$, also the operator $\hat{\Omega}(Q)$ can be factorized into three parts. By separating the exponentials that act only on the system from those acting also on the probe, we obtain

$$\begin{aligned}\hat{\Omega}(Q) = \exp\left[i\zeta_+ Q\hat{p}/\hbar\right]\langle Q|\exp\left[i\zeta_- \hat{q}\hat{P}/\hbar\right]\exp\left[i\zeta_z \hat{Q}\hat{P}/\hbar\right]|\varphi\rangle \times \\ \times \exp\left[-i\zeta_z \hat{q}\hat{p}/\hbar\right] .\end{aligned} \tag{55}$$

By imposing the conditions (53) for a plausible measurement, the operator (55) becomes

$$\hat{\Omega}(Q) = \exp\left[i\left(e^{\zeta_z} - 1\right)Q\hat{p}/\hbar\right]\exp\left[-i\zeta_z \hat{q}\hat{p}/\hbar\right]\varphi[e^{\zeta_z}(Q - \hat{q})] . \tag{56}$$

In particular, for probe initial state $|\varphi\rangle$ chosen as a TCS $|\varphi\rangle = |\mu\nu 0\omega\rangle$, we obtain

$$\begin{aligned}\hat{\Omega}(Q) = \exp\left[i\left(e^{\zeta_z} - 1\right)Q\hat{p}/\hbar\right]\exp\left[-i\zeta_z \hat{q}\hat{p}/\hbar\right] \\ \left(\frac{mw}{\pi\hbar|\mu-\nu|^2}\right)^{1/4}\exp\left[-\frac{mw}{2\hbar}\frac{1+2i\xi}{|\mu-\nu|^2}[e^{\zeta_z}(Q-\hat{q})]^2\right] .\end{aligned} \tag{57}$$

Thus, the POM becomes

$$\mathrm{d}\hat{\Pi}(Q) = \left(\frac{mw}{\pi\hbar|\mu-\nu|^2}\right)^{1/2}\exp\left[-\frac{mw}{\hbar}\frac{1}{|\mu-\nu|^2}[e^{\zeta_z}(Q-\hat{q})]^2\right]\mathrm{d}Q , \tag{58}$$

which is Gaussian and unbiased. Hence, the precision $\epsilon^2[\hat{\varrho}]$, is independent of the state $\hat{\varrho}$, and is given by the variance of the Gaussian, namely $\epsilon^2[\hat{\varrho}] = e^{-2\zeta_z}\langle\varphi|\Delta\hat{Q}^2|\varphi\rangle$. This is precisely what we expected, because, as we have seen in subsect. **4**˙1, $\epsilon^2[\hat{\varrho}] = d^2\langle\varphi|\Delta\hat{Q}^2|\varphi\rangle$. We recall that, for very high squeezing, the operator (57) reduces to the GL operator (33), and $\epsilon^2[\hat{\varrho}]$ tends to zero.

5. – Radiation-mirror interaction

In our knowledge there is no viable way to achieve the von Neumann interaction $\hat{H} \propto \hat{q}\hat{P}$. In order to approach its behavior, other Hamiltonians have been suggested. In particular, Walls *et al.* [18] described the interaction $\hat{H}_I = \hat{A}^\dagger\hat{A}\hat{q}$, where a harmonic oscillator, the end mirror of a cavity, is interacting with the radiation in the cavity through radiation pressure. In this section, we examine the situation in which the mirror can be considered as a free mass, similarly to the case of a typical gravitational waves interferometer, where the wave detector that is attached to the mirror is a very massive bar ($m \sim 100$ kg [3]), which has negligible oscillation frequency. We rewrite the radiation interaction Hamiltonian in the form

$$\hat{H}_I = -\hbar K_m \hat{q} \hat{A}^\dagger \hat{A} \ . \tag{59}$$

For the mirror at one hand of a cavity of length L, the coupling constant K_m can be derived [18] as $K_m = \omega_0/L$, where ω_0 is the resonance frequency of the cavity. With the impulsive approximation $K_m \tau \sim 1$ the unitary evolution operator becomes simply

$$\hat{U} = \exp[-i\hat{H}_I \tau/\hbar] = \exp[i\underline{\hat{q}}\hat{A}^\dagger \hat{A}] \ , \tag{60}$$

where the mass position $\underline{\hat{q}} \doteq \hat{q}/l_\tau$ is rescaled by $l_\tau = L/\omega_0\tau$. We know that the von Neumann-type measurement schemes allow only to reach—not to beat—the SQL, and from the previous sections we learned that in order to overcome the SQL we need to add a feedback and a pre-squeezing to the present scheme. Therefore, we propose the following simple detection scheme. The probe is prepared in a highly excited coherent state $|\varphi\rangle = |\alpha\rangle$, where $\alpha = -i|\alpha|$, and then we detect (by homodyning it) the scaled quadrature $\hat{X}/|\alpha| = (\hat{A}^\dagger + \hat{A})/2|\alpha|$. The operator $\hat{\Omega}$ in eq. (8) becomes

$$\hat{\Omega}_{\hat{q}}(X) = \left(\frac{2|\alpha|^2}{\pi}\right)^{1/4} \exp\left[-|\alpha|^2 (X + ie^{i\underline{\hat{q}}})^2 - \frac{|\alpha|^2}{2}(1 + e^{2i\underline{\hat{q}}})\right] \ , \tag{61}$$

and for the POM we have

$$\mathrm{d}\hat{\Pi}_{\hat{q}}(X) = \mathrm{d}X \left(\frac{2|\alpha|^2}{\pi}\right)^{1/2} \exp\left[-2|\alpha|^2 (X - \sin\underline{\hat{q}})^2\right] \ . \tag{62}$$

For states with $\langle \underline{\hat{q}} \rangle \ll 1$ and $\langle \Delta \underline{\hat{q}}^2 \rangle \ll 1$, such that $\sin\underline{\hat{q}}$ can be approximated as $\sin\underline{\hat{q}} \simeq \underline{\hat{q}}$, one has the unbiased POM

$$\mathrm{d}\hat{\Pi}_{\hat{q}}(X) = \mathrm{d}X \left(\frac{2|\alpha|^2}{\pi}\right)^{1/2} \exp\left[-2|\alpha|^2 (X - \underline{\hat{q}})^2\right] \ . \tag{63}$$

Hence, for small mirror displacements, we can check if there are violations of the SQL. Notice that, within the small q approximation, the operator (61) becomes

$$\hat{\Omega}_{\hat{q}}(X) = \left(\frac{2|\alpha|^2}{\pi}\right)^{1/4} \exp\left[-|\alpha|^2 (X - \underline{\hat{q}})^2 + i|\alpha|^2 (\hat{q}^2 X + \hat{q} - 2X)\right] \ . \tag{64}$$

The feedback shifts the object position by a quantity proportional to the output X. The operators $\hat{\Omega}_{\hat{q}}(X)$ becomes

$$\hat{\Omega}(X) = \exp[i(\tau'/\tau)X\underline{\hat{p}}]\,\hat{\Omega}_{\hat{q}}(X) \ , \tag{65}$$

where $\underline{\hat{p}} = \hat{p}/p_\tau$, with $p_\tau \doteq \hbar l_\tau^{-1}$. As the feedback is represented by a unitary operator, it does not modify the POM, but changes only the state reduction, so that the posterior deviation $\sigma[\hat{\varrho}]^2$ changes and condition (23) can be beaten.

Now we check if there is violation of the SQL. From eq. (63), the precision of the measurement can be calculated as $\epsilon[\hat{\varrho}]^2 = l_\tau^2/(4|\alpha|^2)$, which is independent of the state of the system: however, the position fluctuation depends on it. Thus, we must choose the object initial wave-function $|\psi\rangle$ at time $t = -\tau$ before the interaction, then calculate the reduced state after the first measurement at $t = 0$, and then evolve it for a time interval t_f. We choose $\psi(q, -\tau)$ to be a MUW as in eq. (1). The calculation can be made in two main steps. We first multiply the function $\Omega_{\hat{q}}(X)$ by $\psi(q, -\tau)$ (the reduction operator $\hat{\Omega}_{\hat{q}}(X)$ depends only on the position operator, and $\psi(q, -\tau)$ is written in the position representation). The second step consists in applying the feedback operator $\exp[i(\tau'/\tau)X\underline{\hat{p}}]$ to the result, shifting the q-coordinate by $l_\tau(\tau'/\tau)X$. The reduced wave-function $\psi(q, 0)$ turns out to be of the form

$$\begin{aligned}\psi(q,0) \propto \exp\Bigg[&-q^2\left[\frac{|\alpha|^2}{l_\tau^2}(1-iX)+\frac{1}{4\delta_0^2}\right]+\\ &+2q\left[\frac{|\alpha|^2}{l_\tau}\left(X+\frac{i}{2}\right)+\frac{q_0}{4\delta_0^2}+\frac{ik_0}{2}-\frac{|\alpha|^3}{l_\tau}\frac{\tau'}{\tau}X(1-iX)-\frac{|\alpha|}{4}\frac{\tau'}{\tau}\frac{l_\tau X}{\delta_0^2}\right]\Bigg]\ . \end{aligned} \tag{66}$$

By remembering that the probe is prepared in a highly excited coherent state, and by keeping only the higher-order terms in $|\alpha|$, we get

$$\psi(q,0) \propto \exp\left[-\frac{|\alpha|^2}{l_\tau^2}(1-iX)\left(q+\frac{\tau'}{\tau}|\alpha| l_\tau X\right)^2\right]\ . \tag{67}$$

The comparison between eq. (67) and eqs. (31) and (32) shows that $\psi(q, 0)$ can be a TCS or not, depending on the value of the output X of the measurement. In fact, the variance $\langle\psi|\Delta\hat{q}^2(0)|\psi\rangle = l_\tau^2/(4|\alpha|^2)$ can be set equal to $\hbar|\mu-\nu|^2/(2m\omega)$, with $|\mu|^2 - |\nu|^2 = 1$, if $|\alpha|$ and l_τ are appropriately chosen. The time evolution for $\langle\psi|\Delta\hat{q}^2(t)|\psi\rangle$ is described by eq. (32) with $\xi m\omega/(\hbar|\mu-\nu|^2) = -|\alpha|^2 X/l_\tau^2$, which means that $\xi \equiv \mathrm{Im}(\mu^*\nu) = -X/2$. This implies that the reduced state (67) is a TCS only when the measurement result gives a negative value $X < 0$. However, this occurs only with 50% probability, because the probability density for X is Gaussian and centered in $X = 0$. Hence, on the average—*i.e.* in eq. (29)—the SQL is not beaten.

Even a pre-squeezing of the free mass before the measurement does not solve the matter in the hand. In fact, the operator $\hat{\Omega}(X)$ in (65) would become

$$\hat{\Omega}(X) = \exp[i(\tau'/\tau)X\underline{\hat{p}}]\,\hat{\Omega}_{\hat{q}}(X)\,\exp[-i(\tau''/\tau)(\underline{\hat{q}}\hat{p}+\underline{\hat{p}}\hat{q})]\ . \tag{68}$$

The squeezing operator $\exp[-i(\tau''/\tau)(\underline{\hat{q}}\hat{p}+\underline{\hat{p}}\hat{q})]$ squeezes the $\hat{q}$-coordinate as $\hat{q} \to \hat{q}e^{(\tau''/\tau)}$. This reflects on the POM $\mathrm{d}\hat{\Pi}_{\hat{q}}(X)$, which changes to $\mathrm{d}\hat{\Pi}_{\hat{q}e^{\tau''/\tau}}(X)$ so that the precision becomes $\epsilon^2[\hat{\varrho}] = l_\tau^2/(4|\alpha|^2 e^{2\tau''/\tau})$. This precision becomes very small both if the initial state for the probe is very excited and if the pre-squeezing on the system is very high. However, the squeezing operator does not modify the functional form of the initial MUW, because it only rescales its average values: in fact $\exp[-i(\tau''/\tau)(\underline{\hat{q}}\hat{p}+\underline{\hat{p}}\hat{q})]|q\rangle = e^{\tau''/2\tau}|e^{\tau''/\tau}q\rangle$, which implies that $\psi(q, -\tau) \to \psi(qe^{\tau''/\tau}, -\tau)$. Thus, the reduced state of the system after the pre-squeezing, the measurement and the feedback is again of the form (67), and the SQL is not overcome. This result suggests some considerations about the way of beating the SQL. We know that, for generalized von Neumann measurements, the averaged precision equals the posterior deviation, which, if the condition (19) is satisfied, coincides with the variance of the reduced wave-function averaged on the previous readout. Thus, when starting from von Neumann schemes, in order to beat the SQL, we must change $\langle \Delta\hat{q}^2(0)\rangle$. We accomplished that by applying a pre-squeezing and a feedback to the system, and we have seen that, if the probe is initially in a coherent state, at best the SQL can be reached. The reason is that, even though the precision tends to zero for very high squeezing, the uncertainty of the reduced state does not decrease with time, in general. This suggests us to squeeze also the probe [9,18] as we have done in sect. **3**. This should reflect on the state of the free mass by reducing it to a contractive state, independently on the output of the measurement. Thus, we now calculate the operator $\hat{\Omega}$ when the probe is prepared in a squeezed coherent state $|-i|\alpha|,-r\rangle \equiv D(-i|\alpha|)S(-r)|0\rangle$. The reason why we do not take a squeezed vacuum as in ref. [18] is apparent from the form of the evolution operator (60): it represents a rotation in the complex phase space, and if applied to the vacuum it does not modify the state. The evaluation of the operator $\hat{\Omega}_{\hat{q}}(X)$ needs a quite long derivation: we only report the result

$$
\begin{aligned}
(69)\quad \hat{\Omega}_{\underline{\hat{q}}}(X) = &\left(\frac{2|\alpha|^2}{\pi\hat{k}^2}\right)^{1/4} e^{-\frac{i}{2}(\hat{\phi}+\underline{\hat{q}})}\times \\
&\times\exp\left[-i|\alpha|^2\cos\underline{\hat{q}}(2X-\sin\underline{\hat{q}}) - |\alpha|^2(X-\sin\underline{\hat{q}})^2\left[\frac{i}{\tan\underline{\hat{q}}}\left(\frac{1}{\hat{k}^2e^{2r}}-1\right)+\frac{1}{\hat{k}^2}\right]\right],
\end{aligned}
$$

where

$$
(70)\qquad \hat{k}^2 = e^{2r}\sin^2\underline{\hat{q}} + e^{-2r}\cos^2\underline{\hat{q}}\ ,
$$

$$
(71)\qquad \hat{\phi} = -\arctan\left(e^{2r}\tan\underline{\hat{q}}\right)\ .
$$

If we suppose that the mirror displacements are very small, which means that $\langle\hat{q}\rangle \ll l_\tau$, we have the first-order approximation for $\hat{\Omega}_{\underline{\hat{q}}}(X)$

$$
\begin{aligned}
(72)\qquad \hat{\Omega}_{\underline{\hat{q}}}(X) = &\left(\frac{2|\alpha|^2e^{2r}}{\pi}\right)^{1/4}\times \\
&\times\exp\left[i|\alpha|^2\left(X\underline{\hat{q}}^2+\underline{\hat{q}}-2X\right) - |\alpha|^2e^{2r}\left(X-\underline{\hat{q}}\right)^2 + \frac{i\underline{\hat{q}}}{2}(e^{2r}-1)\right]\ .
\end{aligned}
$$

As we have seen in the previous derivation, the pre-squeezing does not change the form of the $\hat{q}$-representation of the system wave-function, but only rescales both the average and the variance by squeezing factors. Hence, we choose an interaction described only by a measurement followed by a feedback, like in eq. (65). Once again, we first multiply the initial wave-function by $\hat{\Omega}_{\underline{\hat{q}}}(X)$, then shift q by $|\alpha|l_\tau X/\tau$ and finally impose that $|\alpha|$ is very high. We get

$$
(73)\qquad \psi(q,0) \propto \exp\left[-\frac{|\alpha|^2e^{2r}}{l_\tau^2}\left(1-\frac{iX}{e^{2r}}\right)\left(q+\frac{\tau'}{\tau}|\alpha|l_\tau X\right)^2\right]\ ,
$$

which is of the same functional form of the state reduction (67), and is contractive only for negative values of the output X of the measurement, that still occurs with 50% probability. Therefore, by replacing the standard von Neumann Hamiltonian $\hat{H}_I = \hat{q}\hat{P}$ with the interaction of the free mass with the radiation pressure, we can beat the SQL for certain values of the output, but on the average we only get a very narrow final state. This holds true even for a probe prepared in a highly squeezed state, and using a pre-squeezing and a feedback.

6. – Conclusions

In this paper, we have engineered *ab initio* a measurement scheme that allows to beat the SQL. We have shown that our scheme belongs to a class of measurement models previously studied by Ozawa. The measurement can be performed in three-steps, involving a pre-squeezing stage, a von Neumann interaction $\hat{H}_I = \hat{q}\hat{P}$, and a feedback. For large squeezing, and with the probe prepared in a TCS, the state reduction puts the system into a TCS too, and the SQL is breached. When the standard von Neumann interaction is replaced by a mirror-radiation interaction $\hat{H}_I = \hat{q}\hat{A}^\dagger\hat{A}$, the SQL cannot be beaten, even though all other conditions are kept the same. This is due to the fact that in the limit of small free mass displacements the radiation-pressure Hamiltonian gives the same probability distribution than the von Neumann one, but not the same state reduction. In fact, the von Neumann interaction has the capability of transferring the shape of the wave-function from the probe to the system, so that the reduced system state can be made contractive, and thus the SQL is breached. On the other hand, this is no longer true for the radiation-pressure Hamiltonian, where the state reduction of a MUW is a very narrow state, which can be contractive with 50% probability, but on the average the SQL is not beaten. Thus we conclude that the experimental realization of the precise form of the von Neumann Hamiltonian is in order, if the measurement apparatus is designed to beat the SQL.

REFERENCES

[1] Braginskĭi V. B. and Vorontsov Yu. I., *Sov. Phys.-Usp.*, **17** (1975).
[2] Edelstein W. A., Hough J., Pugh J. R. and Martin W., *J. Phys. E*, **11** (1978) 710.
[3] Caves C. M., Thorne K. S., Drever R. W. P., Sandberg V. D. and Zimmermann M., *Rev. Mod. Phys.*, **52**(1980) 341; Caves C. M., *Phys. Rev. Lett.*, **54** (1985) 2465.
[4] See, *e.g.*, Schiff L. I., *Quantum Mechanics* (McGraw-Hill, New York) 1966, pp. 833-840.
[5] Yuen H. P., *Phys. Rev. Lett.*, **51** (1983) 719.
[6] Arthurs E. and Kelly J. L., *Bell. Syst. Tech. J.*, **44** (1965) 725.
[7] Lynch R., *Phys. Rev. Lett.*, **52** (1984) 1730 and Yuen's response.
[8] Yuen H. P., unpublished.
[9] Ozawa M., *Phys. Rev. Lett.*, **60** (1988) 385.
[10] Ozawa M., in *Squeezed and Nonclassical Light*, edited by by Tombesi P. and Pike E. R. (Plenum Press, New York) 1989.
[11] Ozawa M., in *Quantum Aspects of Optical Communications*, edited by Bendjaballah C., Hirota O. and Reynaud S. (Springer-Verlag, Berlin Heidelberg) 1991.
[12] Ozawa M., *Phys. Rev. A*, **41** (1990) 1735.
[13] Gordon J. P. and Louisell W. H., in *Physics of Quantum Electronics*, edited by Kelly P. L., Lax B. and Tannenwald P. E. (Mc Graw-Hill, New York) 1966, pp. 833-840.
[14] D'Ariano G. M., *Quantum Estimation Theory and Optical Detection*, in *Concepts and Advances in Quantum Optics and Spectroscopy of Solids*, edited by T. Hakioğlu and Shumovsky A. S. (Kluwer Academic Publishers, Amsterdam) 1996.

[15] von Neumann J., *Mathematical Foundations of Quantum Mechanics* (Princeton Univ. Press, Princeton, NJ) 1955, pp. 442-445.
[16] Yuen H. P., *Phys. Rev. A*, **13** (1976) 2226.
[17] Ban M., *J. Opt. Soc. Am. B*, **10** (1993) 1347.
[18] Pace A. F., Collet M. J. and Walls D. F., *Phys. Rev. A*, **47** (1996) 3173.