THE ISING MODEL ON FINITELY GENERATED GROUPS AND THE BRAID GROUP

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Among the several different methods of solution of the 2-dimensional Ising Model, one turns out to be particularly promising and suitable for generalization to 3-dimensional cases: so-called Pfaffian method [1]. The latter generalization has been implemented in particular in those situations when the model is defined over a finite lattice L_0 isomorphic with the graph of some finitely presented group G_0.
Let us first review the far-reaching properties that such an assumption implies, schematically.
Let G_0 have presentation

(1) $$G_0 = \langle a_k ;\ k = 1,\dots,n \mid R_j\ (\{a_k\})\ ;\ j = 1,\dots,m \rangle$$

where the a_k's denote the generators of G_0 and R_j its defining relators.
The Cayley graph C of G_0 is constructed in the following way. Let M be a topological space (either a set or a metric space), and for each element $g_s \in G_0$ select a point $P_s \in M$; $s = 1,\dots, |G_0|$, where $|G_0|$ is the order of G_0; in such a way that the elements of G_0 are in one-to-one correspondence with the points selected.
Join then pairs of points P_s, P_t, $s,t = 1,\dots, |G_0|$ in M, with a $n|G_0|$ -fold set of edges defined as follows.

(2) $a_k\, g_s = g_t \quad , \quad a_k^{-1}\, g_s = g_t'$

then the points P_s and P_t, and P_t' and P_s are connected by an edge of type k, respectively beginning at P_s and ending at P_t, beginning at P_t' and ending at P_s.
Thus exactly one positively oriented edge of each of the n types begins at every point P_s, and one ends at P_s.
Moreover the edges do not intersect mutually except at the points P_s.
The graph C is the collection of all points P_s, $s = 1,\ldots,|G_0|$ and edges $E_W^{(k)} = (g_s, g_t^{(\lambda|k)})$, where the index w is indeed a multi-index, $w = (s,t,\lambda)$ and $g_t^{(\lambda|k)} = a_k^{\lambda}\, g_s$, $\lambda = \pm 1$, $w = 1,\ldots,2_n|G_0|$.
Modulo a trivial conjugation in G_0, any point in M can be associated with the identity element $\mathcal{J}$, let it be P_1. Any word W in the generating symbols $\{a_k^{\lambda}\}$ is uniquely represented by a path of oriented edges in M, starting from P_1. If such a path is closed, the corresponding word is equivalent to the identity of G_0.
If the path is closed and has no subpaths of measure zero, namely path-components which correspond to subwords of the form $W' = f\, f^{-1}$, where f is a word, the word it represents coincides with one of the elements R_j of the complete set of defining relators in (1).
Given the graph C, the group G_0 can then be reconstructed as the equivalence class group of the presentation given in terms of the generating symbols induced by the edges $E_w^{(k)}$ and the relators induced by closed paths.
L_0 is assumed to be isomorphic to C.
Notice that C is connected, hence L_0 is connected: indeed any element $g_s \in G_0$ can be written

(3) $g_s = a_{k_1}^{\lambda_1}\; a_{k_2}^{\lambda_2} \ldots a_{k_s}^{\lambda_s}$

and, if $E_r = E_{w_r}^{(k_r)}$, $w_r = (r-1, r, \lambda_r)$, then the path $E_{r_1} e_{r_2} \ldots E_{r_s}$ joins the identity to g_s (or P_1 to P_s).
The general theory of the Pfaffian method [2] is based on the property that, upon writing - for an Ising model defined over the lattice L_0 - the Hamiltonian in the form

(4) $H = - \sum_{s=1}^{N_0} \sum_{k=1}^{n} J_k\; \sigma_{g_s}\; \sigma_{a_k g_s}$

where the dynamical variables $\sigma_g \in \mathcal{Z}_2$, $g \in G_0$ have been explicitly labelled by elements of G_0 instead than by the points P_s of L_0; J_k are the coupling constants characteristic of each type of (unoriented) edge of L_0, and we have set $|G_0| = N_0$ in order to remind that $|G_0|$ is indeed the number of sites of L_0; the partition function

(5) $$Z \doteq \sum_{\{\sigma_g ; g \in G_0\}} \exp(-\beta H)$$

where β is the inverse temperature, is given by

(6) $$Z = [2 \prod_{k=1}^{n} (\cosh u_k)]^{N_0} F(\{t_k\})$$

where

(7) $$u_k = \beta J_k \quad , \quad t_k = \tanh u_k$$

and $F(\{t_k\})$ is the generating function - in the indeterminates t_k - of the number of closed loops with specified number of sides along the edges of different types k, one can draw on L_0.

$F(\{t_k\})$ is computed according to the following scheme. First L_0 is decorated [3], i.e. each site of L_0 is blown up into a set of $n_q = 3(q - 2)$ points, where q denotes the coordination number of L_0, each of coordination 3.

Let us name L the decorated lattice; the number of sites of L is $N = n_q N_0$.

Each site of L is now in one-to-one correspondence with a pair (g_s, i), with $g_s \in G_o$ and $i \in I \equiv \{1, \ldots, n_q\}$.

$F(\{t_k\})$ is identical with the dimer covering generating function for L, when weight t_k is given to the bonds in $L \cap L_0$ of type k, and weight 1 to the $(4q - 9)$ bonds $b_a \in L \setminus L_0$ added by the decoration procedure.

If L is a 3-dimensional lattice - or a 2-dimensional one endowed with periodic boundary conditions - L has genus $c \geq 1$ (i.e. L can be embedded in a 2-dimensional, orientable, closed surface S_c of topological genus c, but not in one of genus $(c - 1)$).

One introduces then a new lattice L_c, which is the 2^{2c}-fold covering of L, in which the bonds are restricted to the subsets of those of L corresponding to all possible orientations required by the combinatorial problem.

It turns out that if L_c is homogeneous under the group G which is:

i) locally the central extension of G_0 obtained by addition of the (multiplicative) element $- \mathcal{J}$ and the consequent modification of the relators so that these satisfy the requirements of Kasteleyn's theorem [4],

ii) globally the extension of the group G_0 by the homology group of S_c, $H_1(S_c)$ (namely the set of images of the fundamental group of S_c, $\pi_1(S_c)$, with respect to the homomorphism $h: \pi_1(S_c) \dashrightarrow \mathbb{Z}_2$; the fundamental group of S_c is free abelian over $2c$ generators and the homology group is a finitely generated module over the Noetherian ring $\mathbb{Z}_2$);

then $F(\{t_k\})$ is given by a Pfaffian:

$$\ln F(\{t_k\}) = 2^{-2c-1} \operatorname{Tr} \ln A \quad . \tag{8}$$

Here $A = \{A_{ts}\}$ is the antisymmetric ($A_{st} = - A_{ts}$) weighted incidence matrix of L_c, namely

$$A_{ts} = \begin{cases} 0 & \text{if } P_t, P_s \in L_c \text{ are not nearest neighbours in } L_c \\ \operatorname{sgn}(t,s) & \text{if the bond } P_tP_s \in L_c \text{ is the image under } G \text{ of a bond } b_\Delta \in L \\ \operatorname{sgn}(t,s)\,|t_k| & \text{if the bond } P_tP_s \in L_c \text{ is the image of a bond of type } k \text{ in } L. \end{cases}$$

where $\operatorname{sgn}(t,s)$ is the signature of the oriented bond $P_tP_s \in L_c$ homogeneous under G.
By the Baer-Nielsen theorem [5], the description of all the finite extensions of the fundamental group of an orientable surface of genus c, S_c, is equivalent to the determination of the (finite) group of mapping classes, $MC(S_c)$. Thus G is globally isomorphic to the group

$$MC(S_c) = \text{Homeo } S_c \;/\; \text{Isot } S_c \tag{9}$$

namely the group of homeomorphisms of S_c which preserve isotopy (or at most a maximal subgroup thereof). On the other hand, S_c is isomorphic to the sphere with c handles T_c, and

$$MC(S_c) \sim \pi_0 \operatorname{Diff}^+(T_c) \tag{10}$$

the group of isotopy classes of orientation preserving diffeomorphisms of T_c.
If R denotes the regular representation of G, A can be written in general in the form

$$(11) \qquad A = \sum_{k=1}^{n} \left\{ z_k^{(+)} R(a_k) + z_k^{(-)} R(a_k^{-1}) \right\}$$

where the coefficients $z_k^{(+)} = z_k^{(-)}(\{t_k\})$, and $z_k^{(-)} = z_k^{(-)}(\{t_k\})$ depend on the presentation of G_0 and the homology of S_c only; and - since no ambiguity arises - we designated the generating symbols of G with the same notation as for those of G_0.

We have two alternative ways of reducing (8) to an algorithm which could be solved.

If $D^{(J)}(g)$, $g \in G$, denotes the J-th irreducible representation of G, of dimension $/J/$, recalling that

$$(12) \qquad R(G) = \bigoplus_J [J]\, D^{(J)}(G)$$

we can write the free energy per site f in the form

$$(13) \qquad -\beta f = \ln 2 + \sum_{k=1}^{n} \ln \cosh u_k + \frac{2^{-2c-1}}{N_0} \sum_J \mathrm{Tr} \ln A^{(J)}$$

where $A^{(J)}$ is the J-th irreducible block-diagonal component of A; the sum is over all the irreps.

Thus, on the one hand, we are faced with the problem of constructing the irreps of G, and the algorithm is reduced to the calculation of the finite set of finite determinants $\det A^{(J)}$, for all J's.

On the other hand, from (8), (11)

$$\ln F(\{t_k\}) = -2^{-2c-1} \sum_{p=1}^{\infty} \frac{1}{p} \sum_{\{k_i\}_1^p = 1}^{n} \sum_{\{\lambda_i\}_1^p = \pm 1} \left[\prod_{i=1}^{p} z_{k_i}^{(\lambda_i)} \right]$$

$$(14) \qquad \times\, \mathrm{Tr} \left\{ R(a_{k_1}^{\lambda_1}) \ldots R(a_{k_p}^{\lambda_p}) \right\} \quad .$$

Notice that if we denote by $W^{(p)}_{(k_i,\, \lambda_i)}$ the word in G corresponding to the element $g = a_{k_1}^{\lambda_1} \ldots a_{k_p}^{\lambda_p}$, in (14) we have indeed to evaluate $\mathrm{Tr} \left\{ R(W^{(p)}_{(k_i,\, \lambda_i)}) \right\}$. Being in the regular representation implies that only those words for which g is the identity element in G give a non-vanishing contribution to the r.h.s. of (14).

In fact once more the latter correspond to closed paths, now in L_c.

The problem of deciding for a group G defined by a given presentation

$$(15) \qquad G = \langle a_k;\ k = 1, \ldots, n \mid \bar{R}_j(\{a_k\}),\ j = 1, \ldots, d \rangle$$

$d > m$, for an arbitrary word W in the generators, in a finite number of steps, wether W defines the identity element of G or not, has a long standing in combinatorial group theory and is known as Dehn's word problem [6].

In our second alternative we are thus led to the word problem for G. For both ways of solution, the presentation of G is an indispensable ingredient. Hereafter we will briefly discuss the general properties one can state for G, and how they bear on both the word problem and the construction of the irreps.

S_c can be thought of as a 2-dimensional compact manifold with a Riemannian metric of constant Gaussian curvature -1, as well as a homogeneous space of G, say G/K, where K is a normal, compact subgroup of G of finite index. The fundamental group acts as a discontinuous discrete subgroup of automorphisms, and L_c is the orbit of a point in the fundamental region of G under the action of G itself. The universal cover of S_c has constant negative curvature as well, so it is isometric to the open unit disk endowed with Poincaré metric, $H^{(2)}$.

Since the covering translations of the universal cover act by isometries, S_c can also be represented by $H^{(2)}/T$, where T is a subgroup of the isometry group B of $H^{(2)}$.

The latter is isomorphic to the Fuchsian group $\pi_1(S_c)$. Since the topological genus of S_c is c, the space of hyperbolic surfaces S_c together with a fixed isomorphism of $\pi_1(S_c)$ to T (where two surfaces are equivalent if there is an isometry between them respecting this isomorphism) is the Teichmüller space of genus c, $\mathcal{T}_c$ [7].

$\mathcal{T}_c$ can be thought of as the subset of discrete representations of $\pi_1(S_c)$ into B up to conjugacy.

Then $MC(S_c)$ is isomorphic to the group of outer automorphisms of $\pi_1(S_c)$, say $\mathcal{O}_\pi$.

The diffeomorphism group of S_c acts on $\mathcal{T}_c$ by pulling back metrics, and the action descends to an action of G since the points of $\mathcal{T}_c$ are isotopy classes of metrics.

On the other hand the automorphism group of $\pi_1(S_c)$ acts on the space of discrete representations of $\pi_1(S_c)$ into B, and the action descends to an action of the group $\mathcal{O}_\pi$ on $\mathcal{T}_c$. Such an action is properly discontinuous and faithful. The quotient space is the moduli space of genus c. Every finite subgroup K of G, acting on $\mathcal{T}_c$, has a fixed point and can therefore be realized as a subgroup of isometries of some hyperbolic structure on a surface of genus c. Indeed, let $S_c^{(o)}$ be the fixed point of K when acting on $\mathcal{T}_c$. For each $\bar{k} \in K$ there is an isometry of $S_c^{(o)}$ to itself in the isotopy class of $\bar{k}$.

Notice that such isometry is unique, because if there were two, one could generate - by composing one with the inverse of the other - an isometry of $S_c^{(o)}$ isotopic to the identity but not equal to the identity, which would have a lift to $H^{(2)}$ commuting with every element of $\pi_1(S_c)$ (acting on $H^{(2)}$). Then all the elements of $\pi_1(S_c)$ should have the same endpoints, which is absurd.

The group of isometries $\bar{K} \subset B$ generated by choosing such a unique isometry in each class of $\bar{k} \in K$ is isomorphic to K itself, because any word in $\bar{K}$ which represents the trivial word in K is an isometry isotopic to the identity (hence equal to the identity). The map

$$(16) \qquad \psi : \text{Diff}\ (S_c) \dashrightarrow G$$

exists for any manifold S_c : the question wether or not it is possible to lift G back into Diff (S_c), namely to chose a single representative in Diff (S_c) for each element in G, so that any word in the lifted elements isotopic to the identity is equal indeed to the identity, is referred to as the lifting problem.

Now the lifting problem for ψ is certainly solvable for surfaces S_c of arbitrary genus c if $K \subset G$ is finite. For finite genus, the Fuchsian group T is a finitely generated, discrete subgroup of B, which is not cyclic. G can then be viewed as the group of outer automorphisms of T, $\mathcal{O}_T$, acting on the space of representations, induced from homeomorphisms of S_c.

The center of T is trivial (in fact the centralizer of every element is cyclic). Upon defining the finite extension $\bar{T}$ of T by K, corresponding to the exact sequence:

$$(17) \qquad 1 \dashrightarrow T \dashrightarrow \bar{T} \dashrightarrow K \dashrightarrow 1$$

there is a homomorphism from $\bar{T}$ to $\mathcal{O}_T$ defined by sending any collection L of simple curves filling up S_c (such a collection is G-invariant, since the orbit GL fills up the surface as well) to the automorphism of T induced by conjugation by L.

$\bar{T}$ is again Fuchsian, and since K can be thought of as a finite subgroup of $\mathcal{O}_T$, realizable as a group of isometries acting on $H^{(2)}/T \sim S_c$, such a Fuchsian group is associated to the quotient space G/K.

As for the presentation of G, one can identify [8] K with the subgroup of diffeomorphisms of S_c which preserve a cut system. The latter is defined as follows.

Let $\{C_p ; p = 1,\dots,c\}$ be a set of disjoint cycles on $T_c \sim S_c$: $\{C_p\} = C_1 \cup C_2 \cup \dots \cup C_c$. $T_c \setminus \{C_p\}$ is then a 2c-punctered sphere.

An isotopy class $<C_p; p = 1,\dots,c>$ of $\{C_p\}$ is a cut system. K is then the group which permutes the C_p's and reverses their orientations. Denoting by $Q \in G$ the element of G supported locally by an homology exchange between two intersecting cycles, we have:

i) G is generated by Q and K.

ii) There exists an exact sequence

$$(18) \qquad \mathcal{Z} \longrightarrow \mathcal{Z}^c \oplus \mathcal{B}_{2c-1} \longrightarrow K \longrightarrow \pm \mathcal{S}_c \longrightarrow 0$$

where $\mathcal{B}_n$ denotes the braid group on n strands, and $\mathcal{S}_n$ the group of permutations of n objects.

iii) The elements $g_s \in G$ are represented by words W_s whose letters belonging to $\{Q^p; p \in \mathcal{Z}\} \cup K$ are indeed elements of K. Thus the relations of G are generated by words of the form $W_s g_s^{-1}$.

There follows that since K is finitely presented, so is G. Moreover, all the relations of G follow from relations supported on subsurfaces of S_c of genus at most 2.

Finally there exists a finite matrix representation of G which has - due to (18) - the structure of a wreath product and which can be obtained by induction from the (finite) matrix representation of K.

Thus $F(\{t_k\})$ can be written as an automorphic function, depending on the representations of K and the characters of $\mathcal{S}_c$ [7].

In turn the representations of K derive, by induction, from those of $\mathcal{B}_{2c-1}$. Let us then recall, in conclusion, some of the basic properties and definitions of the braid group $\mathcal{B}_n$.

$\mathcal{B}_n$ can be thought of as the fundamental group of the space of unordered sets of n distinct points in a plane. If $P^{(n)}$ is the space of polynomials of degree n,

$$(19) \qquad \mathcal{B}_n \sim \pi_1(P^{(n)}) \quad , \quad \pi_i(P^{(n)}) = 0 \ , \ i > 1$$

and thinking of $P^{(n)}$ as the space of hyperelliptic curves of degree n one can in fact obtain a representation of $\mathcal{B}_n$ in the group of symplectic integral matrices, namely the matrices of automorphisms of the homologies of the curves induced by circuits in the coefficient plane. This somewhat seems to bridge present approach with the connection - recently established by Sato, Jimbo and Miwa [9] in their holonomic quantum field theory - between the problem of evaluating the 2-point

Green's function for the 2-dimensional Ising model and the Schlesinger isomonodromy problem.
$\mathcal{B}_n$ has (n-1) generators $s_i, i = 1,\dots,n-1$ and (n-1)(n-2)/2 relations of the form [10]:

$$s_i \, s_j = s_j \, s_i \quad , \quad i-j \geq 2 \tag{19}$$

$$s_i \, s_{i+1} \, s_i = s_{i+1} \, s_i \, s_{i+1} \quad , \quad 1 \leq i \leq (n-2) \quad . \tag{20}$$

An interesting link between the relations of the braid group and the Yang-Baxter-Zamolodchikow factorization equations, leading to the formulation of the latter within the scheme of an infinite dimensional Lie algebra (euclidean), whereby the connected combinatorics is reconducted to the generalized Roger-Ramanujan identities, has been recently pointed out [11].
Work is in progress along these lines.

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