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Quantum Walks, Weyl Equation and the Lorentz Group

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Abstract Quantum cellular automata and quantum walks provide a framework for the foundations of quantum field theory, since the equations of motion of free relativistic quantum fields can be derived as the small wave-vector limit of quantum automata and walks starting from very general principles. The intrinsic discreteness of this framework is reconciled with the continuous Lorentz symmetry by reformulating the notion of inertial reference frame in terms of the constants of motion of the quantum walk dynamics. In particular, among the symmetries of the quantum walk which recovers the Weyl equation—the so called Weyl walk—one finds a non linear realisation of the Poincaré group, which recovers the usual linear representation in the small wave-vector limit. In this paper we characterise the full symmetry group of the Weyl walk which is shown to be a non linear realization of a group which is the semidirect product of the Poincaré group and the group of dilations.

Keywords Quantum walk · Doubly special relativity · Quantum cellular automata · Quantum field theory · Lorentz transformations

1 Introduction

The conjecture, originally advanced by Feynman [20], that the laws of physics can be ultimately modelled by finite algorithms is a very inspirational proposal [29]. There are many reasons why this might prove to be the case and, thus, for adopting this conjecture as a standpoint for a research program. The primary reason is stated by Feynman himself: "It always bothers me that according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to

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figure out what goes on in no matter how tiny a region of space and no matter how tiny a region of time". A similar concern is that in an arbitrarily small region of a continuous space-time it is in principle possible to store an infinite amount of bits of information. The only alternative to this situation is that the dynamics of systems in a finite region of space-time is perfectly computed by a finite algorithm running on a finite memory. Furthermore, the idea that the dynamical laws could be reconstructed within a (quantum) computational framework appears as a natural continuation of the research on quantum foundations from the information perspective (see e.g. Refs. [15,16,21,23] and for a comprehensive historical overview see Refs.[19,25,26]).

As long as we accept that the best microscopic theory at our disposal is quantum theory, the most natural computational model for the description of physical laws is a *quantum cellular automaton* [20,22,36]. The approach to the foundations of quantum field theory based on quantum cellular automata was explored for various decades [10,34,35,39] and it is gathering increasing interest [6–9]. Nevertheless, the idea that a discrete quantum computer can exactly compute the evolution of elementary physical systems is seemingly at clash with continuous symmetries [37].

In recent years, free relativistic field equations were derived starting from the requirements of homogeneity, locality, linearity and isotropy [11,14,17,18]. The free quantum field theory (Weyl, Dirac, and Maxwell) is achieved by restricting to evolutions that are linear in the field—i.e. a quantum walk—in the limit of small wave-vectors, namely for states so delocalised that the discrete underlying structure cannot be resolved. It is remarkable that Lorentz-invariant equations can be derived without imposing the relativity principle, and not even mechanical notions. However, the Lorentz symmetry has no direct interpretation in the above framework, where the geometry of space-time is not assumed a priori. The achievement of Weyl, Dirac and Maxwell's equations is a clear indication that an alleged conflict between discrete dynamics and continuous symmetries was drawn based only on naive intuition.

In Ref. [13] the notion of inertial reference frame has been formulated in terms of representation of the dynamics parameterised by the values of the constants of motion. Such notion is suitable to the study of dynamical symmetries, without the need of resorting to a space-time background. In this way the Galileo principle of relativity is formulated by identifying the notion of change of inertial frame with the change of representation that leaves the eigenvalue equation of the quantum walk invariant. In the same Ref. [13] it has been shown that such changes of representations for the Weyl quantum walk encompass a non-linear realization of the Poincaré group. This result, besides embodying a microscopic model of doubly special relativity (DSR) [4,5,31], represents a proof of principle of the coexistence of a discrete quantum dynamics with the symmetries of classical space-time.

In this paper we review and extend the results of Ref. [13] classifying the full symmetry group of the Weyl quantum walk, which is a semidirect product of the group of diffeomorphic dilations of the null mass shell by the Poincaré group.

2 Weyl Quantum Walk

A quantum cellular automaton gives the evolution of a denumerable set of cells, each one corresponding to a quantum system. We consider the case in which each quantum system is described by the algebra generated by a set of field operators. Following the definition of Ref. [36], a quantum cellular automaton is an automorphism of the quasi-local algebra. The restriction to non interacting dynamics corresponds to consider algebra automorphism that are linear in the field operators (i.e. each field operator is mapped to a linear combination of field operators). In the same way the dynamics of a free field is specified by its single particle sector, a linear quantum cellular automaton is specified by a quantum walk describing the evolution of a single particle. A quantum walk [2,24] on a discrete lattice Γ of sites $\mathbf{x} \in \Gamma$ is given by a unitary operator $A \in \mathcal{L}(\mathcal{H})$ where $\mathcal{H} := \ell^2(\Gamma) \otimes \mathbb{C}^{\mathsf{s}}$ where $\ell^2(\Gamma)$ is the space of square summable functions on Γ and $\mathbb{C}^{\mathsf{s}}$ corresponds to some internal degree of freedom. If $|\mathbf{x}\rangle$, $|i\rangle$ are orthonormal basis for $\ell^2(\Gamma)$ and $\mathbb{C}^{\mathsf{s}}$ respectively, a (pure)state in $\mathcal{H}$ is a vector $|\psi\rangle = \sum_{\mathbf{x}\in\Gamma, i\in\mathsf{s}} \psi(\mathbf{x}, i)|\mathbf{x}\rangle|i\rangle$ where $\sum_{\mathbf{x}\in\Gamma, i\in\mathsf{s}} |\psi(\mathbf{x}, i)|^2 = 1$. The quantum walk A is usually assumed to be local, i.e., for any $\mathbf{x}$, we have that $\langle\mathbf{x}|\langle i|A|\mathbf{x}'\rangle|i'\rangle \neq 0$ only if $\mathbf{x}'$ belongs to a finite *neighboring set*[1].

As it shown in Ref. [18] (which we refer to for a complete discussion), in the three-dimensional case with minimal dimension $\mathsf{s} = 2$ the assumptions of *locality*, *homegeneity*, and *isotropy* single out only one lattice, the body centered cubic one, and four admissible quantum walks (modulo a local change of basis) $A^{(\pm)}$, $B^{(\pm)}$. These quantum walks are given by the following unitary operators

$$
\begin{aligned}
A^{(\pm)} &= \sum_{\mathbf{h}\in S} T_{\mathbf{h}} \otimes A^{(\pm)}_{\mathbf{h}} \\
B^{(\pm)} &= \sum_{\mathbf{h}\in S} T_{\mathbf{h}} \otimes B^{(\pm)}_{\mathbf{h}} \quad B^{(\pm)}_{\mathbf{h}} = \left(A^{(\pm)}_{\mathbf{h}}\right)^T
\end{aligned}
\tag{1}
$$

where S is a set of generators of the BCC lattice $S := \{\pm\mathbf{h}_1, \pm\mathbf{h}_2, \pm\mathbf{h}_3, \pm\mathbf{h}_3\}$ with

$$
\mathbf{h}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \ \mathbf{h}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}, \quad \mathbf{h}_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\-1\end{pmatrix}, \ \mathbf{h}_4 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\-1\\1\end{pmatrix}, \tag{2}
$$

$T_{\mathbf{h}}$ are the translation operators $T_{\mathbf{h}}|\mathbf{x}\rangle = |\mathbf{x} - \mathbf{h}\rangle$, and the matrices $A^{(\pm)}_{\mathbf{h}}$ are defined as follows:

$$
\begin{gathered}
A^{(\pm)}_{\mathbf{h}_1} = \begin{pmatrix}\zeta^* & 0\\ \zeta^* & 0\end{pmatrix}, A^{(\pm)}_{\mathbf{h}_2} = \begin{pmatrix}0 & \zeta^*\\ 0 & \zeta^*\end{pmatrix}, \qquad A^{(\pm)}_{\mathbf{h}_3} = \begin{pmatrix}0 & -\zeta^*\\ 0 & \zeta^*\end{pmatrix}, A^{(\pm)}_{\mathbf{h}_4} = \begin{pmatrix}\zeta^* & 0\\ -\zeta^* & 0\end{pmatrix}, \\
A^{(\pm)}_{-\mathbf{h}_1} = \begin{pmatrix}0 & -\zeta\\ 0 & \zeta\end{pmatrix}, A^{(\pm)}_{-\mathbf{h}_2} = \begin{pmatrix}\zeta & 0\\ -\zeta & 0\end{pmatrix}, \qquad A^{(\pm)}_{-\mathbf{h}_3} = \begin{pmatrix}\zeta & 0\\ \zeta & 0\end{pmatrix}, A^{(\pm)}_{-\mathbf{h}_4} = \begin{pmatrix}0 & \zeta\\ 0 & \zeta\end{pmatrix} \\
\zeta = \frac{1 \pm i}{4}.
\end{gathered}
\tag{3}
$$

[1] For example, if Γ is the one dimensional lattice which we identify with the set of integers $\mathbb{Z}$, we may require $|x - x'| \geq n \Rightarrow \langle x|\langle i|A|x'\rangle|i'\rangle = 0$ for some $n \geq 1$. More synthetically we can say that the unitary matrix A is block-sparse.

From Eq. (1) one immediately sees that the quantum walk commutes with the lattice translations generated by the vectors $\mathbf{h}_i$, i.e. $[A^{\pm}, T_{\mathbf{h}_i} \otimes I] = [B^{\pm}, T_{\mathbf{h}_i} \otimes I] = 0$. It is therefore convenient to consider the Fourier transform basis

$$\begin{aligned} &|\mathbf{k}\rangle = \frac{1}{\sqrt{|B|}} \sum_{\mathbf{x}\in\Gamma} e^{-i\mathbf{k}\cdot\mathbf{x}} |\mathbf{x}\rangle, \qquad |\mathbf{x}\rangle = \frac{1}{\sqrt{|B|}} \int_B d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} |\mathbf{k}\rangle, \\ &\mathbf{k} = \sum_{j=1}^{3} k_j \tilde{\mathbf{h}}_j, \qquad \tilde{\mathbf{h}}_j \cdot \mathbf{h}_l = \delta_{jl}. \end{aligned} \tag{4}$$

where B is the first Brillouin zone of the BCC lattice (see Fig. 1). In the Fourier basis the quantum walks of Eq. (1) becomes

$$\begin{aligned} &A^{(\pm)} = \int_B d\mathbf{k} |\mathbf{k}\rangle\langle\mathbf{k}| \otimes A^{(\pm)}_{\mathbf{k}}, \qquad A_{\mathbf{k}} = I\lambda^{(\pm)}(\mathbf{k}) - i\mathbf{n}^{(\pm)}(\mathbf{k}) \cdot \boldsymbol{\sigma}^{(\pm)} \\ &\lambda^{(\pm)}(\mathbf{k}) := c_x c_y c_z \mp s_x s_y s_z \qquad \mathbf{n}^{(\pm)}(\mathbf{k}) = \begin{pmatrix} n^{(\pm)}_x \\ n^{(\pm)}_y \\ n^{(\pm)}_z \end{pmatrix} := \begin{pmatrix} s_x c_y c_z \pm c_x s_y s_z \\ c_x s_y c_z \mp s_x c_y s_z \\ c_x c_y s_z \pm s_x s_y c_z \end{pmatrix} \\ &c_i = \cos\left(\frac{k_i}{\sqrt{3}}\right) \quad s_i = \sin\left(\frac{k_i}{\sqrt{3}}\right) \qquad \boldsymbol{\sigma}^{(\pm)} := (\sigma_x, \mp\sigma_y, \sigma_z)^T . \end{aligned} \tag{5}$$

It is possible to show that the matrices $A^{(\pm)}$ can be written as

$$A^{\pm}_{\mathbf{k}} = e^{-i\frac{k_x}{\sqrt{3}}\sigma_x} e^{\mp i\frac{k_y}{\sqrt{3}}\sigma_y} e^{-i\frac{k_z}{\sqrt{3}}\sigma_z}. \tag{6}$$

from which one can immediately see that, in the limit of small wave-vector $\mathbf{k} \to 0$, the quantum walk $A^{(+)}$ recovers (up to a rescaling $\frac{\mathbf{k}}{\sqrt{3}} \to \mathbf{k}$) the Weyl equation for right-handed spinors, i.e. $(i\partial_t - \mathbf{k}\cdot\sigma)\psi = 0$. Therefore, in order to lighten the notation, it is useful to make the rescaling

$$\frac{\mathbf{k}}{\sqrt{3}} \to \mathbf{k}. \tag{7}$$

We can also verify that, in the limit $\mathbf{k} \to 0$, the quantum walk $A^{(-)}$ recovers, up to the change of basis induced by the conjugation with the σ_y matrix, the Weyl equation for left-handed spinors i.e. $(i\partial_t + \mathbf{k}\cdot\sigma)\psi = 0$. For this reason, the quantum walks $A^{(\pm)}$, $B^{(\pm)}$ are called *Weyl quantum walks*. The Weyl equation is also recovered when $|\mathbf{k} - \mathbf{k}_i| \to 0$ where $\mathbf{k}_1 := \frac{\pi}{2}(1, 1, 1)$, $\mathbf{k}_2 := -\frac{\pi}{2}(1, 1, 1)$, $\mathbf{k}_3 := \pi(1, 0, 0)$. For $\mathbf{k} \to \mathbf{k}_2$ we have the same chirality as for $\mathbf{k} \to \mathbf{k}_0 := 0$ while for $\mathbf{k} \to \mathbf{k}_1, \mathbf{k}_3$ the chirality changes. We have then that a single quantum walk describes four different kind of massless particles, two left-handed and two right-handed. This fact can be interpreted as an instance of the known phenomenon of fermion doubling [38] but with a different discrete framework. In the following we will use the expression "small wave-vector" to denote the neighborhoods of the vectors $\mathbf{k}_i$, $i = 0, \ldots 3$.

2.1 The Map $\mathbf{n}(\mathbf{k})$

Before discussing the symmetries and the change of inertial frame for the Weyl Quantum Walks, we are going to describe some features of the maps $\mathbf{n}^{(\pm)}(\mathbf{k})$ defined in Eq. (5). The results we are going to show, will be used for the characterization of the symmetry transformations of the Weyl Quantum Walks. For sake of simplicity, we focus on the map $\mathbf{n}^{(+)}(\mathbf{k}) =: \mathbf{n}(\mathbf{k})$ but the same analysis can be carried out for the map $\mathbf{n}^{(-)}$. Moreover the map $\mathbf{n}(\mathbf{k})$ is a smooth analytic map from the Brillouin zone B to $\mathbb{R}^3$. Its Jacobian $J_{\mathbf{n}}(\mathbf{k})$ is given by

$$J_{\mathbf{n}}(\mathbf{k}) := \det[\partial_i n_j(\mathbf{k})] = \cos(2k_y)\lambda(\mathbf{k}), \tag{8}$$

and it vanishes on the set $\mathsf{F} := \mathsf{G} \cup \mathsf{X}$, where

$$\mathsf{X} := \{\mathbf{k} \in B |\ \cos(2k_y) = 0\}, \quad \mathsf{G} := \{\mathbf{k} \in B |\ \lambda(\mathbf{k}) = 0\}.$$

Let us then define the open sets

$$\begin{aligned} B'_0 &:= \{\mathbf{k} \in B |\ \lambda(\mathbf{k}) > 0, \cos(2k_y) > 0\}, \\ B'_1 &:= \{\mathbf{k} \in B |\ \lambda(\mathbf{k}) < 0, \cos(2k_y) > 0\}, \\ B'_2 &:= \{\mathbf{k} \in B |\ \lambda(\mathbf{k}) > 0, \cos(2k_y) < 0\}, \\ B'_3 &:= \{\mathbf{k} \in B |\ \lambda(\mathbf{k}) < 0, \cos(2k_y) < 0\}. \end{aligned} \tag{9}$$

and let us denote with $\mathbf{n}_i(\mathbf{k})$ the restriction of $\mathbf{n}(\mathbf{k})$ to the set B'_i. Since $J_{\mathbf{n}}(\mathbf{k}) \neq 0$ for $\mathbf{k} \in B'_i$ the map $\mathbf{n}_i(\mathbf{k})$ defines an analytic diffeomorphism between B'_i and its image $\mathbf{n}_i(B'_i)$. An expression for the inverse map $\mathbf{n}_i^{-1} : \mathbb{R}^3 \to B'_i$ can be obtained exploting the following identities:

$$\begin{aligned} &2(\lambda n_x - n_y n_z) = \sin 2k_x \cos 2k_y, && 2(\lambda n_z - n_y n_x) = \sin 2k_z \cos 2k_y \\ &1 - 2(n_x^2 + n_y^2) = \cos 2k_y \cos 2k_x, && 1 - 2(n_z^2 + n_y^2) = \cos 2k_y \cos 2k_z \\ &2(\lambda n_y + n_x n_z) = \sin 2k_y, && \lambda^2 = 1 - n_x^2 - n_y^2 - n_z^2. \end{aligned} \tag{10}$$

The ambiguities emerging from the inverse trigonometric functions are solved by the requirement that $\mathbf{n}_i^{-1}(\mathbf{n}) \in B'_i$. One can see that the domain of the inverse function coincides with the unit ball in $\mathbb{R}^3$ except for the image $\mathbf{n}(\mathsf{F})$ of the critical points of $\mathbf{n}$. This set is easily characterized as follows:

$$\begin{aligned} &\mathrm{H}' := \mathrm{U} \backslash \mathrm{n}(\mathrm{F}), \\ &n(\mathrm{F}) = \left\{ \mathrm{m} \in U | m_x = \pm m_z, 2m_x^2 + 2m_y^2 = 1 \right\}, \\ &\mathrm{U} := \left\{ \mathrm{m} \in \mathbb{R}^3 |\ ||\mathrm{m}||^2 < 1 \right\}, , \end{aligned} \tag{11}$$

namely the unit ball minus two ellipses (see Fig. 1). The map $\mathbf{n}_i$ then defines an analytic diffeomorphism between B'_i and H'. We can easily see that H' is connected but not

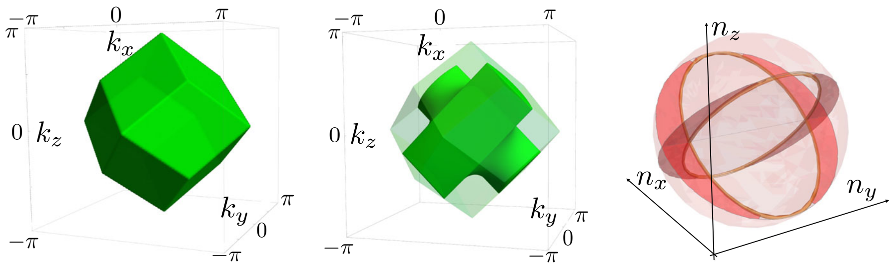

Fig. 1 (*Colors online*) *Left* the Brillouin zone for the BCC lattice. The region os defined as: $B := \{\mathbf{k}| -\frac{\pi}{2} \leq \mathbf{k} \cdot \tilde{\mathbf{h}}_i \leq \frac{\pi}{2},\ 1 \leq i \leq 6\}$, which in Cartesian coordinates reads $-\pi \leq k_i \pm k_j \leq \pi,\ i \neq j \in \{x, y, z\}$. *Middle* The set $B_0 := \mathbf{n}_0^{-1}(\mathsf{H})$ embedded in the Brillouin zone. *Right* the star shaped region H. The set H has been obtained by removing the set F' (*dark red region*) from the unit ball.

simply connected. For our purposes we will need to restrict the range of the function $\mathbf{n}$ to a star-shaped (and then simply connected) region. The largest star-shaped region including H' is

$$\begin{aligned} &\mathrm{H} := \mathrm{U}\backslash\mathrm{F}', \\ &\mathrm{F}' = \left\{\mathrm{m} \in \mathrm{U} | m_x = \pm m_z, 2m_x^2 + 2m_y^2 \geq 1\right\}, \end{aligned} \tag{12}$$

and we also restrict the domain of $\mathbf{n}_i$ (see Fig. 1) to the counter image

$$B_i := \mathbf{n}_i^{-1}(\mathsf{H}). \tag{13}$$

Let us summarize what we have shown so far. We have defined four different sets B_i such that their union is the whole Brillouin zone B except a null-measure set. We introduced the set H which is star shaped and differs from the unit ball in $\mathbb{R}^3$ by a null measure set. For each $i = 0, \ldots, 3$, the map $\mathbf{n}_i(\mathbf{k})$ defines an analytic diffeomorphism between B_i and H. We can verify that each of the vectors $\mathbf{k}_i$, which were defined at the end of the previous section, belongs to a different set B_i, namely $\mathbf{k}_i \in B_i$. In the following we will see that we can interpret the four regions B_i as the momentum space of four different massless fermionic particles.

3 Change of Inertial Frame

It is now convenient to express the dynamics of the Weyl quantum walk through its eigenvalue equation

$$A_{\mathbf{k}}\psi(\omega, \mathbf{k}) = e^{i\omega}\psi(\omega, \mathbf{k}), \tag{14}$$

whose solution set provides an equivalent way to present the walk operator A. In order to lighten the notation we will focus only on the walk $A_{\mathbf{k}} := A_{\mathbf{k}}^{(+)}$. However, the following derivation holds for any of the admissible Weyl quantum walks.

If we consider the real and imaginary part of $A_{\mathbf{k}}$ separately, Eq. (14) splits into two equations as follows:

$$\begin{cases} [\cos\omega - \lambda(\mathbf{k})]\psi(\omega,\mathbf{k}) = 0, \\ [\sin\omega I - \mathbf{n}(\mathbf{k})\cdot\boldsymbol{\sigma}]\psi(\omega,\mathbf{k}) = 0, \end{cases} \tag{15}$$

where $\lambda(\mathbf{k})$ and $\mathbf{n}(\mathbf{k})$ were defined in Eq. (5). Notice that the two equations are not independent, as one can easily verify by applying $\sin\omega I + \mathbf{n}(\mathbf{k})\cdot\boldsymbol{\sigma}$ to the left of the second equation, and then reminding that by unitarity $\lambda(\mathbf{k}) = 1 - \|\mathbf{n}(\mathbf{k})\|^2$. The second equation can be easily rewritten in relativistic notation as follows

$$n_\mu(k)\sigma^\mu\psi(k) = 0, \tag{16}$$

where we introduced the four-vectors $k := (\omega,\mathbf{k})$, $n(k) := (\sin\omega,\mathbf{n}(\mathbf{k}))$, and we defined $\sigma := (I,\boldsymbol{\sigma})$. The eigenvalues ω of Eq. (16) then necessarily obey the dispersion relation

$$\cos\omega = \lambda(\mathbf{k}), \tag{17}$$

with two branches of eigenvalues, namely $\omega = \pm\arccos\lambda(\mathbf{k})$. In the small wave-vector limit, Eq. (16) is approximated by the usual relativistic dispersion relation $\omega^2 = \|\mathbf{k}\|^2$. Following the analogy with quantum field theory, we can interpret and the two solutions of Eq. (17) as particles for $\omega > 0$ and anti-particles for $\omega < 0$.

Let us now restrict the domain of the function $\mathbf{n}(\mathbf{k})$ to one of the four region B_i defined in Eq. (13). Since the following considerations won't be affected by the choice of B_i we will omit the subscript i. The solutions of Eq. (16) are preserved if we multiply the left hand side by an arbitrary function $f(k)$ such that $f(k)\mathbf{n}(\mathbf{k})$ can be inverted as a function on B_i. In particular, we choose an arbitrary rescaling function $f(k)$ such that $f(k)\mathbf{n}(\mathbf{k})$ maps B_i to the full $\mathbb{R}^3$. This is achieved by any rescaling function f that, besides preserving invertibility of $f(k)\mathbf{n}(\mathbf{k})$ on the regions B_i, is singular at the border of the region H. In particular, we consider C^∞ functions f. The eigenvalue equation thus becomes

$$p^{(f)}_\mu(k)\sigma^\mu\psi(k) = 0, \quad p^{(f)} = \mathcal{D}^{(f)}(k) := f(k)n(k). \tag{18}$$

The values $\mathbf{k}$ and ω provide a representation of the state space in terms of constants of motion of the quantum walk dynamics. We now define a change of inertial frame as a change of representation that preserve the set of solutions of the eigenvalue equation. We conveniently use the expression of the eigenvalue equation in Eq. (18).

A change of representation of the dynamics in terms of the constants of motion is given by a function

$$k' : k = \begin{pmatrix}\omega \\ \mathbf{k}\end{pmatrix} \mapsto k'(k) := \begin{pmatrix}\omega' \\ \mathbf{k}'\end{pmatrix}.$$

We remark that by definition, since $p^{(f)}_\mu(k) = f(k)n_\mu(k)$ and $n_\mu(k)n^\mu(k) = \det(n_\mu(k)\sigma^\mu) = \sin^2\omega - \|\mathbf{n}(\mathbf{k})\|^2$, for $\omega = \pm\arccos\lambda(\mathbf{k})$ one has $p^{(f)}_\mu(k)p^{(f)\mu}(k) = 0$. On the other hand, for $\omega \neq \pm\arccos\lambda(\mathbf{k})$ the eigenvalue equation must have trivial solution $\psi(k) = 0$, and then one has $p^{(f)}_\mu(k)p^{(f)\mu}(k) \neq 0$. Thus, for every invertible map k' one can define $M(k) \in \mathbb{GL}(2,\mathbb{C})$ such that $M(k)\psi(k) = \alpha(k')\psi(k')$, with $\alpha(k) \in \mathbb{C}$. For values of k on the mass shell $k = (\omega(\mathbf{k}), \mathbf{k})^T$, this linear transformation can be expressed in the space $\ell^2(\Gamma) \otimes \mathbb{C}^2$ as

$$T := \int_B \mathrm{d}\,\mathbf{k} |\mathbf{k}'(\mathbf{k})\rangle\langle\mathbf{k}| \otimes M(\mathbf{k}). \tag{19}$$

Let us restrict ourselves to those transformations $k'(k)$ for which there exists an $M \in \mathbb{GL}(2,\mathbb{C})$ independent of k and a rescaling $\alpha(k)$ such that $M\psi(k) = \alpha(k')\psi(k')$.

The above arguments motivate the following definition:

Definition 1 (*Change of inertial reference frame for the Weyl walk*) A change of inertial reference frame for the Weyl walk is a quadruple $(k', a, M, \tilde{M})$ where

$$\begin{aligned} &k' : k = \begin{pmatrix}\omega \\ \mathbf{k}\end{pmatrix} \mapsto k'(k) := \begin{pmatrix}\omega' \\ \mathbf{k}'\end{pmatrix} \\ &a : B \times [-\pi,\pi] \to [-\pi,\pi] \\ &M, \tilde{M} \in \mathbb{GL}(2,\mathbb{C}) \end{aligned} \tag{20}$$

such that the eigenvalue equation (18) is preserved, i.e.

$$p^{(f')}_\mu[k'(k)]\sigma^\mu = \tilde{M}\, p^{(f)}_\mu(k)\sigma^\mu\, M^{-1}, \tag{21}$$

and the eigenvectors are transformed as

$$\psi'(k') = e^{ia(k)} M\psi(k). \tag{22}$$

Notice that the change of f to f' in Eq. (21) allows to take $\alpha(k')$ as a phase $e^{ia(k)}$. A special case of change of inertial frame is given by the trivial map $k' = k$ along with the matrices $M = \tilde{M} = I$. As we will discuss in the next section, the above subgroup of changes of inertial frame, that only involves the phases $e^{ia(k)}$, recovers the group of translations in the relativistic limit. The set of all the admissible changes of inertial frame forms a group, which is the largest group of symmetries of the Weyl walk.

In order to classify this group, we now observe that a map acting as in Eq. (20) transforms the four Pauli matrices linearly $\sigma^\mu \mapsto L^\mu_\nu\sigma^\nu$, and in turn this implies that $p^{(f')}_\mu(k') = L^\nu_\mu p^{(f)}_\nu(k)$. Moreover, the set of invertible linear transformations represented by L^ν_μ must preserve the mass-shell $p^{(f)}_\nu p^{(f)\nu} = 0$. By the Alexandrov-Zeeman theorem [1,40] this implies that the transformations L^ν_μ must be a representation of the Lorentz group. Thus, a general change of inertial frame $(k', a, M, \tilde{M})$ for the right-handed Weyl walks must be of the form

$$\begin{aligned} k'(k) &= \mathcal{D}^{(g)^{-1}} \circ L_\beta \circ \mathcal{D}^{(f)}, \\ M &= \Lambda_\beta, \quad \tilde{M} = \tilde{\Lambda}_\beta, \end{aligned} \tag{23}$$

where L_β, Λ_β and $\tilde{\Lambda}_\beta$ are the $(\frac{1}{2}, \frac{1}{2})$, $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ representations of the Lorentz group, respectively. The only difference in the case of left-handed Weyl walks is that the representations Λ_β and $\tilde{\Lambda}_\beta$ are exchanged. Notice that

$$\mathcal{D}^{(f)} \circ \mathcal{D}^{(g)^{-1}} = M_f \circ n \circ n^{-1} \circ M_g^{-1} = M_f \circ M_g^{-1}, \tag{24}$$

where

$$M_f(m) = f(n^{-1}(m))m \tag{25}$$

one has

$$\mathcal{D}^{(f)} \circ \mathcal{D}^{(g)^{-1}}(m) = h(m)m, \tag{26}$$

and thus

$$\begin{aligned} &(\mathcal{D}^{(g')^{-1}} \circ L_{\beta'} \circ \mathcal{D}^{(f')}) \circ (\mathcal{D}^{(g)^{-1}} \circ L_\beta \circ \mathcal{D}^{(f)}) \\ &= (\mathcal{D}^{(g')^{-1}} \circ L_{\beta'} \circ \mathcal{D}^{(f')} \circ \mathcal{D}^{(g)^{-1}} \circ L_{\beta'}^{-1}) \circ L_{\beta'} \circ L_\beta \circ \mathcal{D}^{(f)} \\ &= \mathcal{D}^{(g'')} \circ L_{\beta' \circ \beta} \circ \mathcal{D}^{(f)} \\ &\mathcal{D}^{(g'')} := \mathcal{D}^{(g')^{-1}} \circ L_{\beta'} \circ \mathcal{D}^{(f')} \circ \mathcal{D}^{(g)^{-1}} \circ L_{\beta'}^{-1} \end{aligned} \tag{27}$$

It is then sufficient to prove that a function f with the desired properties exists, otherwise the group of symmetries of the walk would be trivial. We have already shown in Sect. 2.1 that the restriction $\mathbf{n}_i(\mathbf{k})$ of $\mathbf{n}(\mathbf{k})$ to B_i define an analytic diffeomorphism between B_i and the manifold $\mathsf{H} \subset \mathsf{U}$. Let us consider the solutions of Eq. (18), and define the function $g(\omega, r\mathbf{m}) := f(\omega, \mathbf{n}^{-1}(r\mathbf{m}))$, where g is monotonic versus $r \geq 0$ for every $\mathbf{m} \in \mathsf{H}$. We notice that the function $g(\omega, r\mathbf{m})$ is well defined since H is star-shaped. Furthermore, if $g(\omega, r\mathbf{m})$ diverges on the boundary of H, we have that the map $\mathcal{D}^{(f)}(k)$ defines a diffeomorphism between the set $C_i := \{k = (\omega, \mathbf{k}) | \mathbf{k} \in B_i, \cos\omega = \lambda\mathbf{k}\}$ and the null mass shell $\mathsf{K} := \{p \in \mathbb{R}^4,\ \text{s.t. } p^\mu p_\mu = 0\}$. A possible choice of $f(k)$ which satisfies all the previous requirements is given by

$$\begin{aligned} &f(\omega, \mathbf{k}) := f'(\mathbf{n}(\mathbf{k})), \\ &\tilde{f}'(r, \theta, \phi) := 1 + r\int_0^r ds \left(\frac{1}{a(s)} + \frac{1}{b(s, \theta, \phi)} \right), \\ &a(r) := 1 - r^2, \quad b(r, \theta, \phi) := \cos^2 2\phi + (\tfrac{1}{2} - r^2(1 - \cos^2\theta \sin^2\phi))^2 \end{aligned} \tag{28}$$

where we used spherical coordinates $n_x = r\cos\theta\cos\phi$, $n_y = r\sin\theta$, $n_z = r\cos\theta\sin\phi$ for the argument in the definition of the function $f' : \mathsf{H} \to \mathbb{R}$, with the convention that for $\mathbf{n} = \mathbf{0}$ one has $\phi = 0$.

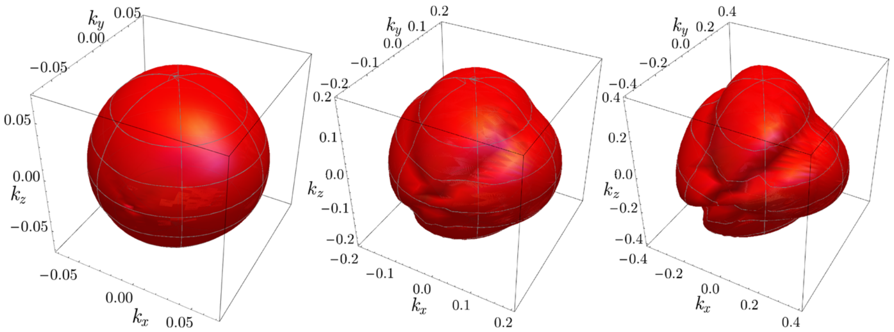

Fig. 2 (*Colors online*) The *red* surfaces represent the orbit of a wave-vector $\mathbf{k} = (k_x, 0, 0)$ under the action of the deformed rotations $\mathcal{R} = \mathcal{D}^{(f)^{-1}} \circ R \circ \mathcal{D}^{(f)}$ where f is the function defined in Eq. (28). *Left* $k_x = 0.07$. *Middle* $k_x = 0.2$ *Right* $k_x = 0.4$

In order to classify the most general transformation leaving the walk invariant, it is still possible to allow for transformations of the kind

$$k' = \sum_i n^{-1}_{j(i)} n_i, \tag{29}$$

$$a = 0, \quad M = \tilde{M} = I, \tag{30}$$

where the region B_i is mapped to the region $B_{j(i)}$. Notice that this corresponds to a permutation of the four regions B_i, which however must fulfil the constraint that i and $j(i)$ must labe lregions corresponding to walks with the same chirality ($\{B_0, B_2\}$ and $\{B_3, B_4\}$). This part of the group thus corresponds to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

By considering the case $f = g$ in Eq. (23), we have

$$\mathcal{L}_\beta := \mathcal{D}^{(f)^{-1}} \circ L_\beta \circ \mathcal{D}^{(f)} \tag{31}$$

which is a non linear representation of the Lorentz group as the ones considered within the context of doubly special relativity [3,4,32]. It is easy to observe that, if $f'(0) = 1$ and $\partial_\mu f' = 0$ where $f(\omega, \mathbf{k}) = f'(\sin\omega, \mathbf{n}(\mathbf{k}))$ as in Eq. (28), the Jacobian $J_{\mathcal{L}_\beta}$ of $\mathcal{L}_\beta$ coincides with L_β. In the limit of small wave-vector we have that $\mathcal{L}_\beta = L_\beta + O(|\mathbf{k}|^2)$ that is the non linear Lorentz transformations recover the usual linear one. In Fig. 2 we show the numerical evaluation of some wave-vector orbits under the subgroup of rotations of the nonlinear representation of the Lorentz group. We see how the distortion effects, which are negligible for small wave-vector, become evident at larger wave-vectors.

4 Conclusion

The analysis of the previous section can be in priniciple applied to any quantum walk dynamics for which we know a complete set of constant of motion. In particular we could consider the Dirac quantum walk of Ref. [18], whose eigenvalue equation is

$(p_\mu(\omega, k, m)\gamma^\mu - mI)\psi(\omega, k, m) = 0$ where γ_μ are the Dirac γ matrices in the chiral representation, m is the particle mass and $p(\omega, k, m) := (\sin\omega, \sqrt{1-m^2}\mathbf{n}(\mathbf{k}))$. In this case we may generalize Definition 1 and allow for maps that change the value of m. We can then consider the invariance of the whole family of Dirac quantum walks parametrized by m. One could prove that the symmetry group of the Dirac walks include a non-linear representation of the De Sitter group $\mathbb{SO}(1, 4)$.

Since the frequency (or energy) ω and the wave-vector (or momentum) $\mathbf{k}$ are the constant of motion of the quantum walk dynamics, the scenario we discussed so far deals with the changes of reference frame in the energy-momentum $(\omega, \mathbf{k})$ space. In particular we saw that the Lorentz group is recovered and one could wonder how to give a time-position description of the deformed relativity framework that we obtained in energy-momentum space. It is believed that the nonlinear deformations of the Lorentz group in momentum space have profound consequences on our notion of space-time. In particular we may have the emergence of relative locality [5], i.e. the coincidence of events in space-time becomes observer dependent. This would imply that not only the coordinates on space-time are observer dependent, as in ordinary special relativity, but also that different observer may infer different space-time manifolds for the same dynamics. Non-commutative space-time and Hopf algebra symmetries [12,27,28,30,33] have been also considered for a time-position space formulation of deformed relativity.

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