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On the general problem of quantum phase estimation

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Abstract

The problem of estimating a generic phase-shift experienced by a quantum state is addressed for a generally degenerate phase-shift operator. The optimal positive operator-valued measure is derived along with the optimal input state. Two relevant examples are analyzed: (i) a multi-mode phase-shift operator for multi-path interferometry; (ii) the two-mode heterodyne phase detection. © 1998 Elsevier Science B.V.

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1. Introduction

The problem of estimating the phase shift experienced by a radiation beam has been the object of hundreds of studies in the last forty years [1]. The problem arises because for a single mode of the electromagnetic field there is no self-adjoint operator for the phase. This is due to the semiboundedness of the number operator [2,3], which is canonically conjugated to the phase as a Fourier-transform pair [4]. The most general and, at the same time, concrete approach to the problem of the phase measurement is quantum estimation theory [5], a framework that has become popular only in the last ten years in the field of quantum information. The most powerful method for deriving the optimal phase measurement was given by Holevo [6] in the covariant case. In this way the optimal positive operator-valued measure (POM) for phase estimation has been derived for a single-mode field. Regarding the multi-mode case, only little theoretical effort has been spent [3], mostly devoting attention to the Lie algebraic structure for two modes [3,7,8]. For two modes, one can adopt the difference between their photon numbers as the phase-shift operator, which thus is no longer bounded from below. This opens the route toward an exact phase measurement based on a self-adjoint operator [9], with a concrete experimental setup using unconventional heterodyne detection [10,11]. The problem is, however, complicated by the (infinite) degeneracy of the shift operator, and for this reason the optimal states for this case have never been derived.

In this paper the general problem of estimating the phase shift ϕ is addressed for any degenerate shift operator with discrete spectrum, either $S = \mathbb{Z}$ (unbounded), or $S = \mathbb{N}$ (bounded from below), or $S = \mathbb{Z}_q$ (bounded), generalizing the Holevo method for the covariant estimation problem. We find the optimal POM for estimating the phase shift of a state $|\psi_0\rangle$, and then we optimize the state itself. The degeneracy of the shifting operator is removed through a simple projection technique. The case of mixed input state, which is generally very difficult, is considered in some special situations. Two sections are devoted to the analysis of

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two relevant examples: one concerning a multi-mode phase estimation problem that arises in multi-path interferometry; the other involving a shift operator that is the difference between the number of photons of two modes, corresponding to unconventional heterodyne detection of the phase.

2. Optimal POM for the phase-shift estimation

We address the problem of estimating the phase-shift ϕ pertaining to the unitary transformation

$$\rho_\phi = \mathrm{e}^{-\mathrm{i}\phi\hat{H}}\,\rho_0\,\mathrm{e}^{\mathrm{i}\phi\hat{H}}, \tag{1}$$

where $\hat{H}$ is a self-adjoint operator degenerate on the Hilbert space $\mathcal{H}$ with discrete (un)bounded spectrum $S = \mathbb{Z}$, or $S = \mathbb{N}$, or $S = \mathbb{Z}_q$, $q > 0$, and ρ_0 is a generic initial state (actually in the following we will mostly restrict to the pure state case). The estimation problem is posed in the most general framework of quantum estimation theory [5] on the basis of a cost function $C(\phi_*, \phi)$, which weights the errors for the estimate ϕ_* given the true value ϕ. For a given a priori probability density $p_0(\phi)$ for the true value ϕ the estimation problem consists in minimizing the average cost,

$$\bar{C} = \int_0^{2\pi} \mathrm{d}\phi\, p_0(\phi) \int_0^{2\pi} \mathrm{d}\phi_* C(\phi_*, \phi)\, p(\phi_*|\phi), \tag{2}$$

where $p(\phi_*|\phi)$ is the conditional probability of estimating ϕ_* given the true value ϕ. The average cost is minimized by optimizing the positive operator-valued measure (POM) [5] $\mathrm{d}\mu(\phi_*)$ which gives the conditional probability by the Born rule

$$p(\phi_*|\phi)\,\mathrm{d}\phi_* = \mathrm{Tr}[\mathrm{d}\mu(\phi_*)\,\mathrm{e}^{-\mathrm{i}\phi\hat{H}}\rho_0\,\mathrm{e}^{\mathrm{i}\phi\hat{H}}]. \tag{3}$$

We consider the general situation in which ϕ is a priori uniformly distributed, i.e. with probability density $p_0(\phi) = 1/2\pi$. Moreover, we want to weight errors independently on the value ϕ of the phase, but only versus the size of the error $\phi_* - \phi$, so that the cost function becomes an even function of only one variable, i.e. $C(\phi_*, \phi) \equiv C(\phi_* - \phi)$. It follows that also the optimal conditional probability will depend only on $\phi_* - \phi$, and the optimal POM can be obtained restricting attention only to phase-covariant POMs, i.e. of the form

$$\mathrm{d}\mu(\phi_*) = \mathrm{e}^{-\mathrm{i}\hat{H}\phi_*}\,\xi\,\mathrm{e}^{\mathrm{i}\hat{H}\phi_*}\frac{\mathrm{d}\phi_*}{2\pi}, \tag{4}$$

where ξ is a positive operator. satisfying the completeness constraints needed for the normalization of the POM $\int_0^{2\pi} \mathrm{d}\mu(\phi) = 1$. In fact, using Eq. (3) and the invariance of trace under cyclic permutations one can easily recognize that $p(\phi_*|\phi) \equiv p(\phi_* - \phi)$ if and only if $\mathrm{d}\mu(\phi_*)$ is covariant. Hence the optimization problem resorts to finding the best positive operator ξ for a given cost function $C(\phi)$ and a generic given state ρ_0. As we will see, the POM obtained in this way is optimal for a whole class of cost functions and initial states ρ_0. Once the best POM is obtained, one further optimizes the state ρ_0. This resorts to solving a linear eigenvalue problem. In fact, the average cost can be written as the expectation value of the cost operator $\hat{C}$, i.e.

$$\bar{C} = \mathrm{Tr}[\hat{C}\rho_0], \tag{5}$$

where

$$\hat{C} = \int \mathrm{d}\mu(\phi) C(\phi). \tag{6}$$

Using the Lagrange multipliers method to account for normalization and mean energy one has to minimize the function

$$\mathcal{L}[\rho_0] = \mathrm{Tr}[\hat{C}\rho_0] - \lambda\,\mathrm{Tr}[\rho_0], \tag{7}$$

which for a pure state $|\psi_0\rangle\langle\psi_0|$ is a quadratic form whose minimum is given by the eigenvalue equation

$$\hat{C}|\psi_0\rangle = \lambda|\psi_0\rangle \tag{8}$$

with the Lagrange parameter λ playing the role of an eigenvalue. The linear problem can be easily extended to account also for finite mean energy.

In summary, our problem is to minimize the cost $\bar{C}$ for a given cost function $C(\phi)$ in Eq. (2). This is done in two steps: (i) by optimizing the positive operator ξ for given generic fixed state ρ_0: this will give a POM which is optimal for an equivalence class of states $\mathcal{E}(\rho_0)$; (ii) by further optimizing the state in the equivalence class $\mathcal{E}(\rho_0)$. Since the original state was arbitrarily chosen, this will give the absolute minimum

cost and the corresponding set of optimal states and POM's.

The solution of the optimization problem is conveniently posed in the representation where $\hat{H}$ is diagonal. The operator $\hat{H}$ is generally degenerate, and we will denote by $|n\rangle_\nu$ a choice of (normalized) eigenvectors corresponding to eigenvalue n, ν being a degeneracy index, and by Π_n the projector onto the corresponding degenerate eigenspace. The problem for an input generally mixed state ρ_0 is too difficult to address: therefore, we focus our attention on the case of pure state $\rho_0 = |\psi_0\rangle\langle\psi_0|$, and we will leave some general assertions on the mixed state case for the following. The problem is restricted to the Hilbert space $\mathcal{H}_\|$ spanned by the (normalized) vectors $|n\rangle \propto \Pi_n|\psi_0\rangle \neq 0$ with the choice of the arbitrary phases such that $\langle n|\psi_0\rangle > 0$. Hence the POM can be chosen of the block diagonal form on $\mathcal{H} = \mathcal{H}_\| \otimes \mathcal{H}_\perp$, i.e. $\mathrm{d}\mu(\phi) = \mathrm{d}\mu_\|(\phi) \oplus \mathrm{d}\mu_\perp(\phi)$ with $\mathrm{d}\mu_\perp(\phi)$ any arbitrary POM on $\mathcal{H}_\perp$. For the optimization of the POM we consider $\Pi_n|\psi_0\rangle \neq 0\ \forall n \in S$, as it is clear that the resulting POM will be optimal also for states having zero projection for some $n \in S$. In this fashion the problem is reduced to the "canonical" phase estimation problem restricted to $\mathcal{H}_\|$: $|\psi_0\rangle \to \exp(\mathrm{i}H_\|\phi)|\psi_0\rangle$, where $H_\| = \sum_{n\in S} n|n\rangle\langle n|$ and $|\psi_0\rangle = \sum_{n\in S} w_n|n\rangle$. Now the problem is to find the positive operator $\xi_\|$ that minimizes the cost $\bar{C}$ in Eq. (2). On the $|n\rangle$ basis the operator $\xi_\|$ is written as

$$\xi_\| = \sum_{n,m\in S} |n\rangle\langle m|\xi_{nm}. \tag{9}$$

For a generic even 2π-periodic function $C(\phi) = -\sum_{l=0}^{\infty} c_l \cos l\phi$ the average cost is given by

$$\bar{C} = -c_0 - \tfrac{1}{2}\sum_{l=1}^{\infty} c_l \sum_{|n-m|=l} \langle\psi_0|n\rangle\langle m|\psi_0\rangle\xi_{nm}. \tag{10}$$

Positivity of ξ implies the generalized Schwartz inequalities

$$|\xi_{nm}| \leqslant \sqrt{\xi_{nn}\xi_{mm}} = 1, \tag{11}$$

where the last equality comes from the POM completeness $\int \mathrm{d}\mu_\|(\phi) = 1_\|$. One can write

$$\mathrm{sign}(c_l) \sum_{|n-m|=l} \langle\psi_0|n\rangle\langle m|\psi_0\rangle\xi_{nm} \leqslant \sum_{|n-m|=l} |\langle\psi_0|n\rangle||\langle m|\psi_0\rangle|, \tag{12}$$

and the equality is obtained only for

$$\xi_{nm} = \mathrm{sign}(c_{|n-m|})$$

(notice that we chose $\langle\psi_0|n\rangle > 0\ \forall n \in S$). The minimum cost is

$$\bar{C} = -c_0 - \tfrac{1}{2}\sum_{l=1}^{\infty} |c_l| \sum_{|n-m|=l} |\langle\psi_0|n\rangle||\langle m|\psi_0\rangle|, \tag{13}$$

where we put $\mathrm{sign}(0) = 1$, since the cost $\bar{C}$ is independent of ξ_{nm} for $c_{|n-m|} = 0$. Notice that positivity of $\xi_\|$ is not generally guaranteed for any set of $\mathrm{sign}(c_l)$. However, one can easily check that $\xi_\| > 0$ if $\mathrm{sign}(c_{|n-m|}) = \exp[\mathrm{i}\pi(\epsilon_n - \epsilon_m)]$, ϵ_n being any integer valued function of n. In fact, this choice corresponds to a unitary transformation of the operator $\xi_\|$ optimized with all $c_l \geqslant 0\ \forall l \geqslant 1$ (the parameter c_0 is irrelevant). The particular choice $c_l \geqslant 0\ \forall l \geqslant 1$ has been considered by Holevo [6], and includes a large class of cost functions corresponding to the most popular optimization criteria, as (i) the likelihood criterion for $C(\phi) = -\delta_{2\pi}(\phi)$; (ii) the 2π-periodic "variance" for $C(\phi) = 4\sin^2(\phi/2)$; (iii) the fidelity optimization $C(\phi) = 1 - |\langle\psi_0|\mathrm{e}^{\mathrm{i}\hat{H}\phi}|\psi_0\rangle|^2$ (here $c_l = 2\sum_{|n-m|=l} |w_n|^2|w_m|^2$). For the Holevo class of cost functions the optimal POM becomes

$$\mathrm{d}\mu_\|(\phi) = \frac{\mathrm{d}\phi}{2\pi}|e(\phi)\rangle\langle e(\phi)|, \tag{14}$$

where the (Dirac) normalizable vectors $|e(\phi)\rangle$ are given by

$$|e(\phi)\rangle = \sum_{n\in S} \mathrm{e}^{\mathrm{i}n\phi}|n\rangle. \tag{15}$$

The vectors $|e(\phi)\rangle$ generalize the Susskind–Glogower representation $|\mathrm{e}^{\mathrm{i}\phi}\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n\phi}|n\rangle$ for generic integer spectrum. Therefore, the optimal POM $\mathrm{d}\mu(\phi)$ is the projector on the state $|e(\phi)\rangle$ in the Hilbert space $\mathcal{H}_\|$, and it is orthogonal for either $S = \mathbb{Z}$, or $S = \mathbb{Z}_q$, whereas it is not for $S = \mathbb{N}$. Notice that the POM (14) is also optimal for a density matrix ρ_0 which is a mixture of states in $\mathcal{H}_\|$, with the

additional constraint of having constant phase along the diagonals. This can be easily proved by re-phasing the basis $|n\rangle$ in such a way that all matrix elements of ρ_0 become positive. Then the assertion easily follows in a way similar to the derivation from Eq. (10) to Eq. (13). Moreover, it is easy to see that the pure state case minimizes the cost, which for the optimal POM is given by $\bar{C} = -\sum_{l=1}^{\infty} c_l \sum_{n\in S}\langle n|\rho_0|n+l\rangle$ (remember that $\rho_0 > 0$ implies that $|\langle n|\rho_0|m\rangle|^2 \leqslant \langle n|\rho_0|n\rangle\langle m|\rho_0|m\rangle$, and the bound is achieved by the pure state case $\langle n|\rho_0|m\rangle = w_n^* w_m$). Finally, we want to emphasize that for the bounded spectrum $S = \mathbb{Z}_q$ there is no need for considering a continuous phase $\mathrm{d}\mu(\phi)$. In fact, it is easy to show [12] that the same average cost is achieved by restricting ϕ to the set of discrete values $\{\phi_s = 2\pi s/q,\ s \in \mathbb{Z}_q\}$, $(q \equiv \dim(\mathcal{H}_{\|}))$, and using as the optimal POM the orthogonal projector-valued operator $|e(\phi_s)\rangle\langle e(\phi_s)|$.

Once the form of the optimal POM is fixed, one can optimize the state $|\psi_0\rangle$ solving the linear problem in Eq. (8). In the following we show two examples of estimation of the phase shift pertaining to highly degenerate integer operators (finite dimensional cases are considered in Ref. [12]). In the first example we consider the operator $\hat{H} = \sum_{l=1}^{M} l\, a_l^\dagger a_l$ that describes a multi-path interferometer, involving M different modes of radiation. In the second, we focus our attention on the two-mode phase estimation using unconventional heterodyne detection, where the phase-shift operator $\hat{H} = a^\dagger a - b^\dagger b$ is given by the difference of photon numbers of the two modes.

3. Optimal POM for multi-path interferometer

We consider the operator

$$\hat{H} = \sum_{l=1}^{M} l\, a_l^\dagger a_l \tag{16}$$

as the generator of the phase shift in Eq. (1). Such phase shift affects an M-mode state of radiation in a multi-path interferometer, where contiguous paths suffer a fixed relative phase shift ϕ [13] (this is also a schematic representation of the phase shift accumulated by successive reflections in a Fabry–Perot cavity). The operator $\hat{H}$ in Eq. (1) has an integer degenerate spectrum $S = \mathbb{N}$. We can take into account the degeneracy by renaming the number of photons of different modes as follows,

$$\hat{H}|n\rangle_\nu = n|n\rangle_\nu, \tag{17}$$

with $\nu = (\nu_2, \nu_3, \ldots, \nu_M)$, and

$$|n\rangle_\nu \doteq \left|n - \sum_{l=2}^{M} l\nu_l\right\rangle \otimes |\nu_2\rangle \otimes |\nu_3\rangle \otimes \ldots \otimes |\nu_M\rangle. \tag{18}$$

The allowed values of ν are restricted to the set $\mathcal{E}_k$ given by

$$\mathcal{E}_k \doteq \left\{\nu_2 = 0, 1, \ldots, \left[\frac{k}{2}\right], \nu_3 = 0, 1, \ldots, \left[\frac{k-2\nu_2}{3}\right], \ldots, \nu_M = \left[\frac{k - \sum_{l=2}^{M-1} l\nu_l}{M}\right]\right\}, \tag{19}$$

where $[x]$ denotes the integer part of x.

For the unshifted initial state $|\psi_0\rangle$ we choose a linear symmetrized superposition of eigenvectors in Eq. (17), namely

$$|\psi_0\rangle = \sum_{n=0}^{\infty} w_n |n\rangle_{\mathrm{sym}}, \tag{20}$$

where

$$|n\rangle_{\mathrm{sym}} = \frac{1}{\sqrt{N_n}} \times \sum_{\{\nu_l\}} \delta\left(\sum_{l=1}^{M} l\nu_l - n\right) |\nu_1\rangle \otimes |\nu_2\rangle \otimes \ldots \otimes |\nu_M\rangle, \tag{21}$$

N_n being the number of elements $\nu \in \mathcal{E}_n$. Without loss of generality, the basis $|n\rangle_{\mathrm{sym}}$ has been chosen such that the coefficients w_n in Eq. (20) are real and positive. According to Eqs. (14) and (15) the optimal POM readily reads as follows,

$$\mathrm{d}\mu(\phi) = \frac{\mathrm{d}\phi}{2\pi} \sum_{n,m=0}^{\infty} \mathrm{e}^{\mathrm{i}(n-m)\phi}\, |n\rangle_{\mathrm{sym}\ \mathrm{sym}}\langle m|. \tag{22}$$

One can now choose a cost function and then minimize the average cost for the POM (22) upon varying the coefficients w_n of the state (20). By choosing the cost function $C(\phi) = 4\sin^2(\phi/2)$ and by imposing the normalization constraint through the Lagrange

multiplier λ, the eigenvalue equation (8) gives the recursion for the coefficients w_n of the form

$$w_n + w_{n+2} - 2\lambda w_{n+1} = 0. \tag{23}$$

The solutions of Eq. (23) can be found in terms of the Chebyshev's polynomials, and the corresponding optimal state written as follows,

$$|\psi\rangle = \left(\frac{2}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \sin[(n+1)\theta]\,|n\rangle_{\rm sym},$$
$$\theta = \arccos\lambda. \tag{24}$$

The state in Eq. (24) is Dirac normalizable. It is formally equivalent to the eigenstate of the cosine operator $\hat{C}$ of the phase of a single mode [14]. The Dirac normalizability comes from the non-existence of normalizable states that minimize the uncertainty relation for cosine and sine operators,

$$\Delta\hat{C}\,\Delta\hat{S} \geqslant \tfrac{1}{2}|\langle[\hat{C},\hat{S}]\rangle| = \tfrac{1}{4}\langle|0\rangle\langle 0|\rangle, \tag{25}$$

as proved in Ref. [15].

4. Phase difference of two-mode fields

In the previous example, $\hat{H}$ was bounded from below and $S \equiv \mathbb{N}$, such that the degenerate case is reduced to the standard Holevo's problem. For the difference operator $\hat{H} = a^\dagger a - b^\dagger b$ one has $S \equiv \mathbb{Z}$, and the set of eigenvectors $|d\rangle_\nu$ can be written in terms of the joint eigenvector $|n\rangle|m\rangle$ for the number operators $a^\dagger a$ and $b^\dagger b$ with eigenvalues n and m as follows,

$$|d\rangle_\nu = |d+\nu\rangle|\nu\rangle,$$
$$d \in \mathbb{Z}, \quad \nu \in [\max(0,-d), +\infty). \tag{26}$$

We consider an initial state $|\psi_0\rangle$ of the form

$$|\psi_0\rangle = h_0|0\rangle|0\rangle + \sum_{n=1}^{+\infty}(h_n|n\rangle|0\rangle + h_{-n}|0\rangle|n\rangle), \tag{27}$$

where the basis has been chosen to have $h_n \geqslant 0$, $\forall n$. The optimal POM can be written in the form of Eq. (14) in terms of the vectors $|\lambda_n\rangle$, $n \in \mathbb{Z}$, where

$$|\lambda_n\rangle = |n\rangle_0 \equiv |n\rangle|0\rangle, \qquad n \geqslant 0,$$
$$= |n\rangle_{|n|} \equiv |0\rangle||n|\rangle, \quad n \leqslant 0. \tag{28}$$

Here, the generalized Susskind–Glogower vector $|e(\phi)\rangle$ is given by

$$|e(\phi)\rangle = \sum_{n\in\mathbb{Z}} e^{in\phi}|\lambda_n\rangle \equiv |0\rangle|0\rangle + \sum_{d=1}^{+\infty}(e^{i\,d\phi}\,|d\rangle|0\rangle + e^{-i\,d\phi}|0\rangle|d\rangle). \tag{29}$$

Notice that, differently from the usual case of spectrum $S = \mathbb{N}$, now the POM is orthogonal (in the Dirac sense),

$$\langle e(\phi)|e(\phi')\rangle = \sum_{n=-\infty}^{+\infty} e^{in(\phi-\phi')} = \delta_{2\pi}(\phi-\phi'), \tag{30}$$

where $\delta_{2\pi}(\phi)$ is the Dirac comb. This means that in this case it is possible to define a self-adjoint phase operator

$$\hat{\phi} = \int_{-\pi}^{+\pi} d\phi|e(\phi)\rangle\langle e(\phi)|\phi, \tag{31}$$

as already noticed by Hradil and Shapiro [9,10].

We now address the problem of finding the normalized state of the form (27) with a finite mean photon number that minimizes the average cost evaluated through the ideal POM (14). As a cost function we choose again $C(\phi) = 4\sin^2(\phi/2)$ (periodicized-variance criterion), corresponding to the cost operator

$$\hat{C} = 2 - e^+ - e^-, \tag{32}$$

where

$$e^+ = \sum_{n\in\mathbb{Z}} |\lambda_{n+1}\rangle\langle\lambda_n|, \quad e^- = (e^+)^\dagger. \tag{33}$$

Introducing the energy operator $\hat{E} = a^\dagger a + b^\dagger b$ and an additional Lagrange parameter accounting for finite mean energy $\langle\hat{E}\rangle$, the eigenvalue problem in Eq. (8) can be rewritten as follows,

$$[\hat{C} - \lambda' - \mu'(a^\dagger a + b^\dagger b)]\,|\psi_0\rangle = 0, \tag{34}$$

where λ' and μ' are the Lagrange multipliers for normalization and mean energy, respectively. The following recursion relations for the coefficients h_n are obtained,

$$h_{n+1} + h_{n-1} - \mu(\lambda + |n|)h_n = 0, \tag{35}$$

with $\lambda = (\lambda' - 2)/\mu'$ and $\mu = -\mu'$. The solution of Eq. (35) is given in terms of Bessel functions of the first kind in the following form,

$$h_n = k(\lambda, \mu)\, J_{\lambda+|n|}(2/\mu), \tag{36}$$

$k(\lambda, \mu)$ being the constant of normalization,

$$k(\lambda, \mu) = \left(\sum_{n=-\infty}^{+\infty} J^2_{\lambda+|n|}(2/\mu) \right)^{-1/2}. \tag{37}$$

The matching of the recursion for positive and negative indices leads to the condition

$$\lambda J_\lambda(2/\mu) - (2/\mu) J_{\lambda+1}(2/\mu) = (2/\mu) \frac{\mathrm{d}}{\mathrm{d}\,(2/\mu)} J_\lambda(2/\mu) = 0. \tag{38}$$

Eq. (38) has infinitely many solutions $\mu = \mu(\lambda)$, and one needs to further minimize the average cost in Eq. (2) versus the average photon number N parameterized by λ and $\mu = \mu(\lambda)$

$$N = 2k(\lambda, \mu)^2 \left(\sum_{n=0}^{+\infty} n\, J^2_{\lambda+n}(2/\mu) \right). \tag{39}$$

In this way one can find the normalized and finite-energy states that achieve the minimum cost for the optimal POM.

The solution (36) of the recursive relation (35) has some similarity with the solution for the minimum phase-uncertainty states of a single-mode field [14,15]. The proof of convergence of the series in Eq. (37) can be found in Ref. [15]. However, the matching condition (38) (instead of the vanishing condition for h_n with $n < 0$ for one mode) makes the two-mode phase estimation problem more difficult, since one cannot exploit the properties of the zeros of the Bessel functions in an asymptotic approximation, as done in Ref. [16] for the single-mode case.

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