COMMENTS

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Loss-error compensation in quantum-state measurements

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In the two papers [T. Kiss, U. Herzog, and U. Leonhardt, Phys. Rev. A **52**, 2433 (1995); U. Herzog, Phys. Rev. A **53**, 1245 (1996)] with titles similar to the one given above, the authors assert that in some cases it is possible to compensate a quantum efficiency $\eta \leqslant 1/2$ in quantum-state measurements, violating the lower bound 1/2 proved in a preceding paper [G. M. D'Ariano, U. Leonhardt, and H. Paul, Phys. Rev. A **52**, R1801 (1995)]. Here we reestablish the bound for homodyning any quantum state, and show how the proposed loss-compensation method would fail in a real measurement outside the $\eta > 1/2$ regime.

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In Ref. [1] the homodyne tomography method—so far the only experimental scheme to measure the density matrix of the quantum state of radiation—was shown to be robust to the smearing effect of nonunit quantum efficiency η at detectors. It was proved that the measurement is possible only above a lower bound for η that depends on the chosen matrix representation of the state $\hat{\varrho}$. In particular, for the Fock basis, such a bound is $\eta = 1/2$, and in the same Ref. [1] a reconstruction algorithm that depends parametrically on η was provided. As noticed in Ref. [2], the existence of a lower bound for quantum efficiency in state measurements is a relevant issue for the foundations of quantum mechanics, as it prevents the measurement of the wave function of an individual quantum system through a series of repeated weak measurements. Therefore, if a scheme for loss-error compensation is devised, then, in principle, such a scheme can open the door to prove an inconsistency within the logical framework of quantum mechanics.

In Ref. [3] it was proposed to perform the state measurement by using the same algorithm of Ref. [1] for $\eta = 1$, and treating the effect of a low quantum efficiency as a lossy evolution of radiation, thus separating the ''bare'' detection with $\eta = 1$ from the loss-compensation procedure. More explicitly, the idea is to regard any state measurement with $\eta < 1$ on the ''signal'' state $\hat{\varrho}_{\rm sig}$ as the corresponding hypothetical bare measurement with $\eta = 1$ on a ''dressed'' damped state $\hat{\varrho}_{\rm meas}$. Also, stated in different words, the effect of nonunit quantum efficiency is referred to the quantum state itself rather than to the detector, regarding nonunit quantum efficiency in a Schrödinger-like picture, with the state evolving from $\hat{\varrho}_{\rm sig}$ to $\hat{\varrho}_{\rm meas}$, and the quantum efficiency playing the role of a time parameter $t = -\ln \eta$. The core of the method is the inversion of the generalized Bernoulli loss transformation, which relates the matrix elements of the signal state $\hat{\varrho}_{\rm sig}$ with those of the dressed state $\hat{\varrho}_{\rm meas}$ in the Fock basis. This is given by

$$\langle n|\hat{\varrho}_{\rm sig}|n+d\rangle = \frac{\eta^{-1/2(2n+d)}}{\sqrt{n!(n+d)!}} \sum_{j=0}^{\infty} \frac{\sqrt{(n+j)!(n+d+j)!}}{j!} \times (1-\eta^{-1})^j \langle n+j|\hat{\varrho}_{\rm meas}|n+d+j\rangle. \quad (1)$$

The argument of Ref. [3] states that above the bound $\eta = 1/2$ the convergence of series (1) is guaranteed for any state, but the convergence radius for η depends on the matrix elements of $\hat{\varrho}_{\rm meas}$, and, in principle, for some particular states it is possible for the series to converge also for $\eta \leqslant 1/2$. Thus, for example, for a thermal state with $\bar{n}$ photons, the matrix elements of $\hat{\varrho}_{\rm meas}$ decay as $[\bar{n}/(\bar{n}+1)]^j$ versus the summation index j, and one concludes that the series (1) converges for $\eta > (2+1/\bar{n})^{-1}$, which violates the bound $\eta = 1/2$. In this Comment we show that this argument does not apply to a measurement, because in this case the series (1) must be evaluated with coefficients given by the measured values of the matrix elements of $\hat{\varrho}_{\rm meas}$ in place of their expectation values $\langle n|\hat{\varrho}_{\rm meas}|m\rangle$. In order to have a successful experiment based on N repeated measurements, the series with measured coefficients must converge for increasingly large truncation index to a result within an error that vanishes for large N. For statistically uncorrelated coefficients c_j the variance of the (unconditionally) convergent series $\Sigma_{j=0}^{\infty} z^j c_j$ is given by the series of variances $\Sigma_{j=0}^{\infty} z^{2j} \langle \Delta c_j^2 \rangle$, where $z = 1-\eta^{-1}$. Then, an *a priori* estimation of the measurement error is given by $\epsilon = \sqrt{\Sigma_{j=0}^{\infty} z^{2j} \epsilon_j^2}$, with $\epsilon_j = \sqrt{\langle \Delta c_j^2 \rangle / N}$. Now, it is clear that statistical errors depend on the particular detection scheme used to measure $c_j \propto \langle n+j|\hat{\varrho}_{\rm meas}|n+d+j\rangle$. For direct photodetection—which, however, is not a quantum-state measurement—the error ϵ_j associated to a diagonal element $p_j = \langle n+j|\hat{\varrho}_{\rm meas}|n+j\rangle$ is $\epsilon_j \simeq \sqrt{(1-p_j)p_j/N}$, and ϵ_j^2 vanishes linearly with p_j itself for $j \to \infty$. This is the logical basis of the argument of Refs. [3,4] (which we are comment-

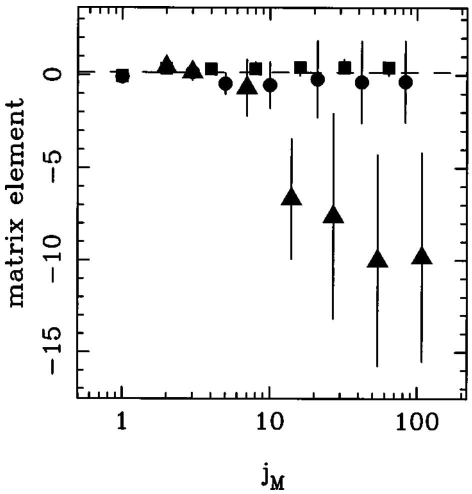

FIG. 1. Matrix element $\langle 2|\hat{\varrho}_{\text{sig}}|2\rangle$ for a thermal state with two photons evaluated as the truncated series (1) with homodyne detected matrix elements $\langle j+2|\varrho_{\text{meas}}|j+2\rangle$ versus the truncation index j_M. The squares correspond to $\eta=0.6$, the circles to $\eta=0.55$, the triangles to $\eta=0.53$. All points are obtained with Monte Carlo experiments using 24 000 homodyne data.

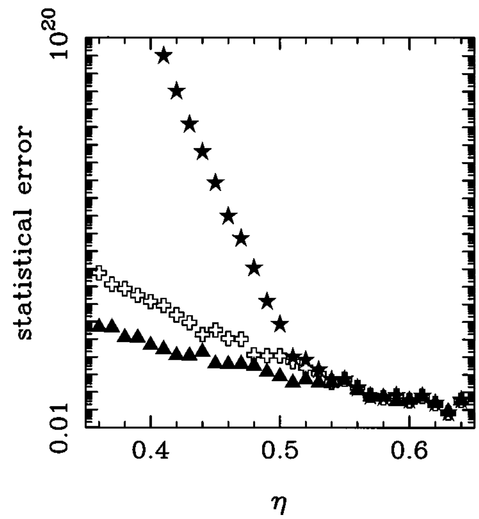

FIG. 2. Statistical error corresponding to the same matrix element and the same state as in Fig. 1, evaluated as the truncated series (1), as a function of η for different values of the truncation index j_M. The triangles correspond to $j_M=10$, the crosses to $j_M=20$, and the stars to $j_M=100$. All points are obtained with Monte Carlo experiments using 8000 homodyne data.

ing on) [5], where, however, the effects of statistical errors in homodyne tomography were not considered. Actually, when the density matrix elements are measured by homodyne tomography, the statistical error ϵ_j does not vanish as the respective matrix element $p_j=\langle n+j|\hat{\varrho}_{\text{meas}}|n+d+j\rangle$ (for fixed n and d), but ''saturates'' to the value $\sqrt{2/N}$, independently of d, n, and $\hat{\varrho}_{\text{meas}}$ [6,7]. Therefore, the above argument no longer holds, and the convergence radius equals the lower bound $\eta=1/2$. Thus for $\eta\leqslant 1/2$ the loss-compensation procedure is meaningless for any state, even though the series (1) converges analytically, as for the case of the thermal states considered before. The same is also true for the iterated analytical continuation procedure given by Eq. (14) of Ref. [4]. Here, one has a number of infinite series that involve matrix elements of the bare state with shifted indices. However, in an actual measurement, a truncation value for each series must be given, and this is determined by the largest index of the measured density matrix. With such truncation one can reorder terms in the series and prove very easily that Eq. (14) of Ref. [4] and Eq. (1) are exactly the same. For this reason, in the following we will always refer only to the loss-error compensation procedure (1) of Ref. [3].

We illustrate the effect of experimental errors in the loss-compensation procedure by means of Monte Carlo simulated experiments of homodyne tomography with $\hat{\varrho}_{\text{sig}}$ as a thermal state with $\overline{n}=2$. According to Refs. [3,4] this state should be accessible for $\eta>2/5$, hence outside the allowed region $\eta>1/2$.

In Fig. 1 we report the matrix element $\langle 2|\varrho_{\text{sig}}|2\rangle$ evaluated as the truncated series (1) with homodyne detected matrix elements $\langle j+2|\varrho_{\text{meas}}|j+2\rangle$ versus the truncation index j_M of the series, and for different values of η. One can see that for η well above the bound $\eta=1/2$ ($\eta=0.6$ and $\eta=0.55$) the series converges versus j_M to the correct theoretical value, whereas for η approaching the bound the size of the error bar increases dramatically, with the sum departing more and more from the theoretical value. For $\eta=1/2$ or below, the result is out of the scale of the figure and oscillates unboundedly versus j_M. In Fig. 2 the statistical error for the same matrix element is studied versus the quantum efficiency η for different values of the truncation index j_M: it is apparent that the error converges for all values $\eta>1/2$. When the bound $\eta=1/2$ is approached, the error starts growing as a function of the truncation index j_M, and diverges for all values $\eta\leqslant 1/2$ (the finite slope versus η is only due to the finiteness of j_M). As we can see from the figure, $\eta=1/2$ is manifestly a ''transition value'' from convergent to divergent behavior of the statistical error.

Actually, in a real experiment homodyne data are contaminated by additional sources of noise other than quantum efficiency. However, for tomography some kinds of noise can still be treated as a negative contribution to the overall quantum efficiency [8].

Before concluding, we want to stress that to our knowledge there is no method—alternative to homodyne tomography—that has statistical errors which make the series (1) convergent for $\eta<1/2$. For all methods recently proposed by several authors [9–11], an analysis of statistical errors is still lacking. Moreover, in the set of papers [9] the radiation is measured indirectly through measurements on atoms that interacted with it, and in such circumstance the quantum noise or loss of the apparatus cannot be described just in terms of an overall quantum efficiency.

In conclusion, we have shown that the loss-compensation procedures proposed in Refs. [3,4] cannot be used to measure the quantum state through homodyne tomography below the bound $\eta=1/2$ proved in Ref. [1]. These procedures are still valid for $\eta<1/2$ when the diagonal matrix elements are measured by direct detection.

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[1] G. M. D'Ariano, U. Leonhardt, and H. Paul, Phys. Rev. A **52**, R1801 (1995).

[2] G. M. D'Ariano and H. P. Yuen, Phys. Rev. Lett. **76**, 2832 (1996).

[3] T. Kiss, U. Herzog, and U. Leonhardt, Phys. Rev. A **52**, 2433 (1995).

[4] U. Herzog, Phys. Rev. A **53**, 1245 (1996).

[5] Notice, however, that the convergence radius $z<\sqrt{R}$ is the square root of the convergence radius of the series (1).

[6] U. Leonhardt, M. Munroe, T. Kiss, Th. Richter, and M. G. Raymer, Opt. Commun. **127**, 144 (1996).

[7] An exhaustive analysis of the statistical errors arising in homodyne tomography is given in G. M. D'Ariano, C. Macchiavello, and N. Sterpi, Quantum Semiclassic Opt. **9**, 929 (1997) (e-print quant-ph/9701010). A precise estimate of the saturation values for the statistical errors is obtained by taking into account the fast oscillating behavior of the kernel functions for large indexes of the matrix elements, and is given by

$$\epsilon_n \simeq \left\{ \frac{1}{N} \int_0^{\pi} \frac{d\phi}{\pi} \int_{-\infty}^{+\infty} dx p(x,\phi) 4 \cos^2(k_n x) \right\}^{1/2}.$$

As $k_n \to \infty$ for large values of n, $p(x,\phi)$ can be considered constant over a cycle $\Delta x = \pi/k_n$ and the integral over x gives $\epsilon_n \simeq \sqrt{2/N}$. A similar argument holds also for off-diagonal elements. This saturation value is overestimated by a factor $\sqrt{2}$ in Ref. [6].

[8] G. M. D'Ariano, in *Quantum Communication, Computing and Measurement*, edited by O. Hirota, A. S. Holevo, and C. M. Caves (Plenum, New York, 1997), p. 253. In this paper it is shown that half a photon of thermal noise has the same effect of a quantum efficiency at the bound $\eta = 1/2$.

[9] P. J. Bardoff, E. Mayr, and W. P. Schleich, Phys. Rev. A **51**, 4963 (1995); P. J. Bardoff, E. Mayr, W. P. Schleich, P. Domokos, M. Brune, J. M. Raimond, and S. Haroche, *ibid.* **53**, 2736 (1996); R. Walser, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **77**, 2658 (1996).

[10] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996).

[11] H. Paul, P. Törmä, T. Kiss, and I. Jex, Phys. Rev. Lett. **76**, 2464 (1996); S. Wallentovitz and W. Vogel, Phys. Rev. A **53**, 4528 (1996).