Trace forms for the generalized Wigner functions

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Summary. — We derive simple formulas connecting the generalized Wigner functions for s-ordering with the density matrix, and vice versa. These formulas proved very useful for quantum-mechanical applications, as, for example, for connecting master equations with Fokker-Planck equations, or for evaluating the quantum state from Monte Carlo simulations of Fokker-Planck equations, and finally for studying positivity of the generalized Wigner functions in the complex plane.

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Since Wigner's pioneering work [1], generalized phase-space techniques have proved very useful in various branches of physics [2]. As a method for expressing the density operator in terms of c-number functions, the Wigner functions often lead to considerable simplification of the quantum equations of motion, as for example, by transforming operator master equations into more amenable Fokker-Planck differential equations (see, for example, ref. [3]). By the Wigner function one can express quantum-mechanical expectation values in form of averages over the complex plane (the classical phase space), the Wigner function playing the role of a c-number quasi-probability distribution, which generally can also have negative values. More precisely, the original Wigner function allows to easily evaluate expectations of symmetrically ordered products of the field operators, corresponding to Weyl's quantization procedure [4]. However, with a slight change of the original definition, one defines generalized s-ordered Wigner function $W_s(\alpha, \overline{\alpha})$, as follows [5]:

$$(1) \qquad W_s(\alpha, \overline{\alpha}) \doteq \int \frac{\mathrm{d}^2\lambda}{\pi^2} \, e^{\alpha\overline{\lambda} - \overline{\alpha}\lambda + \frac{s}{2}|\lambda|^2} \mathrm{Tr}[D(\lambda)\widehat{\varrho}],$$

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where the integration is performed on the complex plane with measure $d^2\lambda = d\operatorname{Re}\lambda \, d\operatorname{Im}\lambda$, $D(\alpha) = e^{\alpha a^\dagger - \bar{\alpha}a}$ denotes the displacement operator, and a and $a^\dagger$ ($[a, a^\dagger] = 1$) are the annihilation and creation operators of the field mode of interest. Then, using the Wigner function in eq. (1) one can evaluate s-ordered expectation values of the field operators through the following relation:

$$(2) \qquad\qquad \operatorname{Tr}[:(a^\dagger)^n a^m:{}_s\widehat{\varrho}] = \int d^2\alpha \, W_s(\alpha, \bar{\alpha}) \, \bar{\alpha}^n \alpha^m \,.$$

It is easy to show that the particular cases $s = -1, 0, 1$ lead to *antinormal*, *symmetrical*, and *normal* ordering, respectively, in which cases the generalized Wigner function $W_s(\alpha, \bar{\alpha})$ historically was denoted with the following symbols and names:

$$(3) \qquad W_s(\alpha, \bar{\alpha}) = \begin{cases} \dfrac{1}{\pi} \, Q(\alpha, \bar{\alpha}) & \text{for } s = -1 \text{ "Q-function"} , \\[2mm] W(\alpha, \bar{\alpha}) & \text{for } s = 0 \text{ (usual Wigner function)} , \\[2mm] P(\alpha, \bar{\alpha}) & \text{for } s = 1 \text{ "P-function"} . \end{cases}$$

For the normal ($s = 1$) and antinormal ($s = -1$) orderings, the following two simple relations between the generalized Wigner function and the density matrix are well known:

$$(4) \qquad\qquad Q(\alpha, \bar{\alpha}) \equiv \langle \alpha | \widehat{\varrho} | \alpha \rangle \,,$$

$$(5) \qquad\qquad \widehat{\varrho} = \int d^2\alpha \, P(\alpha, \bar{\alpha}) \, |\alpha\rangle\langle\alpha| \,,$$

where $|\alpha\rangle$ denotes the customary coherent state $|\alpha\rangle = D(\alpha)|0\rangle$, $|0\rangle$ being the vacuum state of the field. Among the three particular representations (3), it is also well known that the Q-function is positively definite and infinitely differentiable (it actually represents the probability distribution for ideal joint measurements of position and momentum of the harmonic oscillator: see, for example, ref. [6]). On the other hand, the P-function is known to be possibly highly singular, and the only pure states for which it is positive are the coherent states [7]. Finally, the usual Wigner function has the remarkable property of providing the probability distributions of the quadratures of the field in form of marginal distributions, namely

$$(6) \qquad\qquad \int d\operatorname{Im}\alpha \, W(\alpha e^{i\phi}, \bar{\alpha}e^{-i\phi}) = {}_\phi\langle \operatorname{Re}\alpha | \widehat{\varrho} | \operatorname{Re}\alpha \rangle_\phi \,,$$

where $|x\rangle_\phi$ stands for the eigenstates of the field quadrature $\widehat{X}_\phi = (a^\dagger e^{i\phi} + \text{h. c.})/2$ (any couple of conjugated quadratures $\widehat{X}_\phi$, $\widehat{X}_{\phi+\pi/2}$, with $[\widehat{X}_\phi, \widehat{X}_{\phi+\pi/2}] = i/2$, are equivalent to the position and momentum of a harmonic oscillator). Usually, negative values of the Wigner function are viewed as signature of a nonclassical state (one of the more eloquent examples is given by the Schrödinger-cat states [8] whose Wigner function is characterized by rapid oscillations around the origin of the complex plane). From eq. (1) one can see that all s-ordered Wigner functions are related to each other

through the convolution relation

$$(7) \qquad W_s(\alpha, \bar\alpha) = \int d^2\beta \, W_{s'}(\beta, \bar\beta) \, \frac{2}{\pi(s'-s)} \exp\left[-\frac{2}{s'-s}|\alpha-\beta|^2\right],$$

$$(8) \qquad W_s(\alpha, \bar\alpha) = \exp\left[\frac{s'-s}{2}\frac{\partial^2}{\partial\alpha\partial\bar\alpha}\right] W_{s'}(\alpha, \bar\alpha), \qquad (s' > s).$$

Equation (7) shows the positiveness of the generalized Wigner function for $s < -1$, as a consequence of the positiveness of the Q-function. From a qualitative point of view, the maximum value of s keeping the generalized Wigner functions as positive can be considered as an indication of the classical nature of the physical state.

In this paper we present three equivalent trace forms that connect s-ordered Wigner functions with the density matrix. They are the following:

$$(9) \qquad W_s(\alpha, \bar\alpha) = \frac{2}{\pi(1-s)} \, e^{-\frac{2}{1-s}|\alpha|^2} \, \mathrm{Tr}\left[\left(\frac{s+1}{s-1}\right)^{a^\dagger a} e^{\frac{2}{1-s}\bar\alpha a} \, \hat\varrho \, e^{\frac{2}{1-s}\alpha a^\dagger}\right],$$

$$(10) \qquad W_s(\alpha, \bar\alpha) = \frac{2}{\pi(1-s)} \, e^{\frac{2}{1+s}|\alpha|^2} \, \mathrm{Tr}\left[\left(\frac{s+1}{s-1}\right)^{a^\dagger a} e^{-\frac{2}{1+s}\alpha a^\dagger} \, \hat\varrho \, e^{-\frac{2}{1+s}\bar\alpha a}\right],$$

$$(11) \qquad W_s(\alpha, \bar\alpha) =$$

$$= \frac{2}{\pi(1-s)} \, e^{-\frac{2s}{1-s^2}|\alpha|^2} \, \mathrm{Tr}\left[\left(\frac{s+1}{1-s}\right)^{\frac{1}{2}a^\dagger a} D\left(\frac{2\alpha}{\sqrt{1-s^2}}\right)\left(\frac{s+1}{1-s}\right)^{\frac{1}{2}a^\dagger a}(-)^{a^\dagger a}\hat\varrho\right].$$

Equations (9), (10) can be compared with the Cahill-Glauber formula [5]

$$(12) \qquad W_s(\alpha, \bar\alpha) = \frac{2}{\pi(1-s)} \, \mathrm{Tr}\left\{:\exp\left[-\frac{2}{1-s}(\bar\alpha - a^\dagger)(\alpha - a)\right]:\hat\varrho\right\},$$

where the colons denote the usual normal ordering; eq. (11) represents a generalization of the formula [9]

$$(13) \qquad W(\alpha, \bar\alpha) = \frac{2}{\pi} \mathrm{Tr}\left[\hat\varrho D(2\alpha) \exp[i\pi a^\dagger a]\right].$$

Vice versa, the density matrix can be recovered from the generalized Wigner functions using the following expression:

$$(14) \qquad \hat\varrho = \frac{2}{1+s} \int d^2\alpha \, W_s(\alpha, \bar\alpha) e^{-\frac{2}{1+s}|\alpha|^2} \exp\left[\frac{2\alpha}{1+s}a^\dagger\right]\left[\frac{s-1}{s+1}\right]^{a^\dagger a}\exp\left[\frac{2\bar\alpha}{1+s}a\right].$$

The proof of our statements requires the following identity:

$$(15) \qquad e^{a^\dagger \partial_{\bar\alpha}}|0\rangle\langle 0|e^{a\partial_\alpha}\Big|_{\alpha=\bar\alpha=0} e^{|\alpha|^2 + \bar\alpha\lambda - \alpha\bar\lambda - \frac{1}{2}|\lambda|^2} = D(\lambda),$$

which is proved in the appendix. Then, through the following steps:

$$W_s(\alpha, \bar{\alpha}) = \int \frac{d^2\lambda}{\pi^2} e^{\alpha\bar{\lambda} - \bar{\alpha}\lambda + \frac{s}{2}|\lambda|^2} \mathrm{Tr}\,[D(\lambda)\hat{\varrho}] =$$

$$= \int \frac{d^2\lambda}{\pi^2} e^{\alpha\bar{\lambda} - \bar{\alpha}\lambda} \mathrm{Tr}\,[e^{a^\dagger \partial_{\bar{\beta}}} |0\rangle\langle 0| e^{a\partial_\beta}\hat{\varrho}]\,|_{\beta = \bar{\beta} = 0}\, e^{|\beta|^2 + \bar{\beta}\lambda - \bar{\beta}\bar{\lambda} + \frac{1}{2}(s-1)|\lambda|^2} =$$

$$= \frac{2}{\pi(1-s)} \mathrm{Tr}\,[e^{a^\dagger \partial_{\bar{\beta}}} |0\rangle\langle 0| e^{a\partial_\beta}\hat{\varrho}]\,|_{\beta = \bar{\beta} = 0}\, e^{-\frac{1+s}{1-s}|\beta|^2 - \frac{2}{1-s}(|\alpha|^2 - \alpha\bar{\beta} - \bar{\alpha}\beta)} =$$

$$= \frac{2}{\pi(1-s)} e^{-\frac{2}{1-s}|\alpha|^2} \mathrm{Tr}\,\left[e^{\frac{2}{1-s}\alpha a^\dagger}\left(-\frac{1+s}{1-s}\right)^{a^\dagger a} e^{\frac{2}{1-s}\bar{\alpha}a}\hat{\varrho}\right],$$

one proves eq. (9). Continuing from the last result we have

$$W_s(\alpha, \bar{\alpha}) = \frac{2}{\pi(1-s)} e^{-\frac{2}{1-s}|\alpha|^2}.$$

$$\cdot \mathrm{Tr}\left[\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}a^\dagger a}\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}a^\dagger a} e^{\frac{2}{1-s}\alpha a^\dagger}\left(-\frac{1+s}{1-s}\right)^{a^\dagger a} e^{\frac{2}{1-s}\bar{\alpha}a}\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}a^\dagger a}\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}a^\dagger a}\hat{\varrho}\right] =$$

$$= \frac{2}{\pi(1-s)} e^{-\frac{2}{1-s}|\alpha|^2} \mathrm{Tr}\left[\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}a^\dagger a} e^{\frac{2}{\sqrt{1-s^2}}\alpha a^\dagger}(-)^{a^\dagger a} e^{\frac{2}{\sqrt{1-s^2}}\bar{\alpha}a}\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}a^\dagger a}\hat{\varrho}\right] =$$

$$= \frac{2}{\pi(1-s)} e^{-\frac{2s}{1-s^2}|\alpha|^2} \mathrm{Tr}\left[\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}a^\dagger a} D\left(\frac{2\alpha}{\sqrt{1-s^2}}\right)\left(\frac{1+s}{1-s}\right)^{\frac{1}{2}a^\dagger a}(-)^{a^\dagger a}\hat{\varrho}\right],$$

which proves eq. (11). Equation (10) is derived using the following identities:

$$e^{\frac{2\alpha}{1-s}a^\dagger}\left(\frac{S+1}{S-1}\right)^{a^\dagger a} e^{\frac{2\bar{\alpha}}{1-s}a} = \left(\frac{S+1}{S-1}\right)^{a^\dagger a} e^{-\frac{2\alpha}{1+s}a^\dagger} e^{\frac{2\bar{\alpha}}{1-s}a},$$

$$e^{\frac{4|\alpha|^2}{1-s^2}}\left(\frac{S+1}{S-1}\right)^{a^\dagger a} e^{\frac{2\bar{\alpha}}{1-s}a} e^{-\frac{2\alpha}{1+s}a^\dagger} = e^{\frac{4|\alpha|^2}{1-s^2}} e^{-\frac{2\bar{\alpha}}{1+s}a}\left(\frac{S+1}{S-1}\right)^{a^\dagger a} e^{-\frac{2\alpha}{1+s}a^\dagger}.$$

As a check, from eqs. (9)-(11) one can easily recover the usual definition of the Wigner function (1) for $s = 0$, and eq. (4) for the Q-function ($s = -1$), namely

$$W_{-1}(\alpha, \bar{\alpha}) = \frac{1}{\pi} e^{-|\alpha|^2} \mathrm{Tr}\,[(-O^+)^{a^\dagger a} e^{\bar{\alpha}a}\hat{\varrho}e^{\alpha a^\dagger}] =$$

$$= \frac{1}{\pi} e^{-|\alpha|^2} \mathrm{Tr}\,[|0\rangle\langle 0| e^{\bar{\alpha}a}\hat{\varrho}e^{\alpha a^\dagger}] = \frac{1}{\pi} Q(\alpha, \bar{\alpha}).$$

The inversion formula (14) is obtained using eq. (11) and the following formula [5]:

(16) $$\hat{O} = \int \frac{d^2\alpha}{\pi} \mathrm{Tr}\,[\hat{O}D(\alpha)]\, D^\dagger(\alpha),$$

that holds true for any Hilbert-Schmidt operator $\widehat{O}$, and hence for a (trace-class) density matrix. One has

$$(17) \qquad \left(\frac{s+1}{1-s}\right)^{\frac{1}{2}a^\dagger a}(-)^{a^\dagger a}\widehat{\varrho}\left(\frac{s+1}{1-s}\right)^{\frac{1}{2}a^\dagger a} =$$

$$= \int \frac{d^2\alpha}{\pi} \,\mathrm{Tr}\left[D(\alpha)\left(\frac{s+1}{1-s}\right)^{\frac{1}{2}a^\dagger a}(-)^{a^\dagger a}\widehat{\varrho}\left(\frac{s+1}{1-s}\right)^{\frac{1}{2}a^\dagger a}\right]D^\dagger(\alpha) =$$

$$= \frac{4}{1-s^2}\int \frac{d^2\alpha}{\pi} \, W_s(\alpha,\bar{\alpha}) \,\frac{\pi(1-s)}{2}\,e^{\frac{2s}{1-s^2}|\alpha|^2}\,D^\dagger\left(\frac{2\alpha}{\sqrt{1-s^2}}\right).$$

Hence,

$$\widehat{\varrho} = \frac{2}{1+s}\int d^2\alpha\, W_s(\alpha,\bar{\alpha})\,e^{\frac{2s}{1-s^2}|\alpha|^2}\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}a^\dagger a}(-)^{a^\dagger a}D^\dagger\left(\frac{2\alpha}{\sqrt{1-s^2}}\right)\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}a^\dagger a} =$$

$$= \frac{2}{1+s}\int d^2\alpha\, W_s(\alpha,\bar{\alpha})\,e^{-\frac{2}{1+s}|\alpha|^2}\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}a^\dagger a}e^{\frac{2\alpha}{\sqrt{1-s^2}}a^\dagger}(-)^{a^\dagger a}e^{\frac{2\bar{\alpha}}{\sqrt{1-s^2}}a}\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}a^\dagger a},$$

and then the result follows easily. In particular, for $s=0$ one has the inverse of the Glauber formula

$$(18) \qquad\qquad \widehat{\varrho} = 2\int d^2\alpha\, W(\alpha,\bar{\alpha})\, D(2\alpha)(-)^{a^\dagger a},$$

whereas for $s=1$ one recovers the relation (5) that defines the P-function.

The trace form of eqs. (9)-(11) can be used for an analysis of positivity of the Wigner function, usually a quite difficult task, as confirmed in ref. [10]. In particular, from eq. (9) one can immediately see that for $s<1$ (namely, with the only exception of the P-function) the s-Wigner function can become negative, because the operator $e^{\frac{2}{1-s}\bar{\alpha}a}\widehat{\varrho}\,e^{\frac{2}{1-s}\alpha a^\dagger}$ is positive-definite, whereas the preceding factor $\left(\frac{s+1}{s-1}\right)^{a^\dagger a}$ is negative for $s<1$, and positivity is guaranteed only for products of positive operators. On the other hand, from eq. (11) one can easily see that there is always a state (the eigenstate of $a^\dagger a$ with odd eigenvalue) that makes the s-Wigner function at $\alpha=0$ negative for $s<0$.

The representations (9), (10) for the generalized Wigner functions also provide the easiest way to derive differential representations for boson operators acting on a density matrix. By defining, analogously to eq. (1), the generalized *Wigner symbol* for any operator $\widehat{O}$,

$$(19) \qquad\qquad W_s(\alpha,\bar{\alpha}|\widehat{O}) \doteq \int \frac{d^2\lambda}{\pi^2}\,e^{\alpha\bar{\lambda}-\bar{\alpha}\lambda+\frac{s}{2}|\lambda|^2}\,\mathrm{Tr}\,[D(\lambda)\,\widehat{O}],$$

from eqs. (9), (10) one immediately derives the relations

$$(20) \qquad W_s(\alpha, \overline{\alpha} \,|\, a\widehat{\varrho}) = e^{-\frac{2}{1-s}|\alpha|^2} \frac{1-s}{2} \partial_{\overline{\alpha}} \, e^{\frac{2}{1-s}|\alpha|^2} W_s(\alpha, \overline{\alpha}) = \left(\alpha + \frac{1-s}{2}\partial_{\overline{\alpha}}\right) W_s(\alpha, \overline{\alpha}) \,,$$

$$(21) \qquad W_s(\alpha, \overline{\alpha} \,|\, a^\dagger \widehat{\varrho}) = e^{\frac{2}{1+s}|\alpha|^2} \left(-\frac{1+s}{2}\partial_{\alpha}\right) e^{-\frac{2}{1+s}|\alpha|^2} W_s(\alpha, \overline{\alpha}) =$$

$$= \left(\overline{\alpha} - \frac{1+s}{2}\partial_{\alpha}\right) W_s(\alpha, \overline{\alpha}) \,,$$

and analogous relations for right multiplication by the boson operator. More generally, one can write a differential representation for any *super-operator*—i.e. right or left multiplication by an operator $\widehat{O}$—namely

$$(22) \qquad W_s(\alpha, \overline{\alpha} \,|\, \widehat{O}\widehat{\varrho}) \doteq F_s[\widehat{O}\cdot] \, W_s(\alpha, \overline{\alpha}) \,, \qquad W_s(\alpha, \overline{\alpha} \,|\, \widehat{\varrho}\widehat{O}) \doteq F_s[\cdot\widehat{O}] \, W_s(\alpha, \overline{\alpha}) \,,$$

where $\widehat{O}\cdot$ and $\cdot\widehat{O}$ denote left and right multiplication by the operator $\widehat{O}$, respectively, and F_s are differential forms functions of α, $\overline{\alpha}$, ∂_α and $\partial_{\overline{\alpha}}$ with the following properties:

$$(23) \qquad F_s[\widehat{O}_1\widehat{O}_2\cdot] = F_s[\widehat{O}_1\cdot]\, F_s[\widehat{O}_2\cdot] \,,$$

$$(24) \qquad F_s[\cdot\widehat{O}_1\widehat{O}_2] = F_s[\cdot\widehat{O}_2]\, F_s[\cdot\widehat{O}_1] \,,$$

$$(25) \qquad [F_s[\cdot\widehat{O}_1],\, F_s[\widehat{O}_2\cdot]] = 0 \,,$$

$$(26) \qquad F_s[\cdot\widehat{O}] = \overline{F}_s[\widehat{O}^\dagger\cdot] \,.$$

TABLE I. – *Differential Wigner representation of some super-operators.*

Super-operator	F_s
$a\cdot$	$\alpha + \dfrac{1-s}{2}\partial_{\overline{\alpha}}$
$a^\dagger\cdot$	$\overline{\alpha} - \dfrac{1+s}{2}\partial_{\alpha}$
$\cdot a$	$\alpha - \dfrac{1+s}{2}\partial_{\overline{\alpha}}$
$\cdot a^\dagger$	$\overline{\alpha} + \dfrac{1-s}{2}\partial_{\alpha}$
$a\cdot a^\dagger$	$|\alpha|^2 + \dfrac{1-s}{2}(1 + \alpha\partial_{\alpha} + \overline{\alpha}\partial_{\overline{\alpha}}) + \left(\dfrac{1-s}{2}\right)^2 \partial_{\alpha\overline{\alpha}}$
$a^\dagger\cdot a$	$|\alpha|^2 - \dfrac{1+s}{2}(1 + \alpha\partial_{\alpha} + \overline{\alpha}\partial_{\overline{\alpha}}) + \left(\dfrac{1+s}{2}\right)^2 \partial_{\alpha\overline{\alpha}}$
$a^\dagger a\cdot$	$|\alpha|^2 + \dfrac{1}{2}[(1-s)\overline{\alpha}\partial_{\overline{\alpha}} - (1+s)\alpha\partial_{\alpha} - (1+s) - \dfrac{1}{2}(1-s^2)\partial_{\alpha\overline{\alpha}}]$
$\cdot a^\dagger a$	$|\alpha|^2 + \dfrac{1}{2}[(1-s)\alpha\partial_{\alpha} - (1+s)\overline{\alpha}\partial_{\overline{\alpha}} - (1+s) - \dfrac{1}{2}(1-s^2)\partial_{\alpha\overline{\alpha}}]$

The functional forms of the basic super-operators are summarized in table I. The representations of $\cdot a^\dagger$ and $\cdot a$ can be easily obtained from those of $a\cdot$ and $a^\dagger\cdot$ using identities (26). Equation (25) is just the obvious statement that «left multiplication commutes with right multiplication» (for a and $a^\dagger$ this corresponds to the identity $[\partial_\alpha + \kappa\bar\alpha, \partial_{\bar\alpha} + \kappa\alpha] = 0$). Then, the differential representation of higher-order super-operators is easily obtained from the composition rules (23) and (24). Using the differential representation for Bose super-operators, one can convert master equations into (possibly high order) Fokker-Planck equations. For example, the master equation of the damped harmonic oscillator (damping coefficient γ and thermal photons $\bar n$)

$$(27) \quad \partial_t\widehat\varrho = -\frac{\gamma}{2}(\bar n + 1)(a^\dagger a\widehat\varrho + \widehat\varrho a^\dagger a - 2\,a\widehat\varrho a^\dagger) - \frac{\gamma}{2}\,\bar n(aa^\dagger\widehat\varrho + \widehat\varrho aa^\dagger - 2\,a^\dagger\widehat\varrho a),$$

can be converted into the equivalent Fokker-Planck equation for the s-ordered Wigner function

$$(28) \quad \partial_t W_s(\alpha,\bar\alpha) = \frac{\gamma}{2}\left[\partial_\alpha\alpha + \partial_{\bar\alpha}\bar\alpha + (2\,\bar n + 1 - s)\partial_{\alpha\bar\alpha}\right]W_s(\alpha,\bar\alpha).$$

For solving Fokker-Planck equations, one can use very efficient Monte-Carlo Green-function simulation methods (see, for example, ref. [11]), choosing the parameter s such that both the Wigner function and the diffusion coefficient remain positive during the evolution. Then, from the inversion equation (14) one can recover the matrix elements $\langle n|\widehat\varrho|m\rangle$ of the operator $\widehat\varrho$ in form of Monte Carlo integrals of Laguerre polynomials.

In conclusion, we have presented simple trace formulas that connect the generalized Wigner functions with the density matrix, and vice versa, and we have shown how they can be practically used for: i) studying positivity of the generalized Wigner functions; ii) connecting master equations with Fokker-Planck equations; iii) evaluating the quantum state in Monte Carlo simulations of Fokker-Planck equations.

APPENDIX

Proof of identity (15).

From the relation

$$(A.1) \quad \partial_{\bar\alpha}^n\partial_\alpha^m\big|_{\alpha=\bar\alpha=0}\,e^{|\alpha|^2} = \delta_{nm}\,n!\,,$$

one has

$$(A.2) \quad e^{a^\dagger\partial_{\bar\alpha}}\,|0\rangle\langle 0|\,e^{a\partial_\alpha}\big|_{\alpha=\bar\alpha=0}\,e^{|\alpha|^2} = \sum_{n,\,m=0}^{\infty}(a^\dagger)^n\,|0\rangle\langle 0|\,a^m\,\frac{\partial_{\bar\alpha}^n\partial_\alpha^m}{n!m!}\,\bigg|_{\alpha=\bar\alpha=0}e^{|\alpha|^2} =$$

$$= \sum_{n=0}^{\infty}(a^\dagger)^n\,|0\rangle\langle 0|\,a^n\,\frac{1}{n!} = \sum_{n=0}^{\infty}|n\rangle\langle n| = \widehat 1\,.$$

Hence, using the identities

(A.3)
$$e^{a^\dagger \partial_{\bar{a}}} e^{\bar{a}\lambda} = e^{\lambda(a^\dagger + \bar{a})} e^{a^\dagger \partial_{\bar{a}}}, \qquad e^{a\partial_\alpha} e^{-\alpha\bar{\lambda}} = e^{-\bar{\lambda}(a+\alpha)} e^{a\partial_\alpha},$$

one obtains

(A.4)
$$e^{a^\dagger \partial_{\bar{a}}} |0\rangle\langle 0| e^{a\partial_\alpha} \big|_{\alpha=\bar{a}=0} e^{|\alpha|^2 + \bar{a}\lambda - \alpha\bar{\lambda} - \frac{1}{2}|\lambda|^2} =$$

$$= e^{-\frac{1}{2}|\lambda|^2} e^{\bar{a}\lambda - \alpha\bar{\lambda}} e^{a^\dagger \lambda} e^{a^\dagger \partial_{\bar{a}}} |0\rangle\langle 0| e^{a\partial_\alpha} \big|_{\alpha=\bar{a}=0} e^{|\alpha|^2} e^{-\bar{\lambda}a} = e^{-\frac{1}{2}|\lambda|^2} e^{a^\dagger \lambda} e^{-a\bar{\lambda}} = D(\lambda).$$

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