

## Feasible phase detection with ideal sensitivity

Giacomo M D'Ariano<sup>†</sup>||, Chiara Macchiavello<sup>‡</sup>|| and Matteo G A Paris<sup>§</sup>||

<sup>†</sup> Department of Electrical Engineering and Computer Science, Department of Physics and Astronomy, Northwestern University, Evanston, IL 60208, USA

<sup>‡</sup> Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, UK

<sup>§</sup> Arbeitsgruppe 'Nichtklassische Strahlung', Max-Planck-Gesellschaft an der Humboldt-Universität zu Berlin, Rudower Chaussee 5, 12489 Berlin, Germany

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**Abstract.** We present a first feasible scheme for detecting the phase of a single mode of radiation with ideal RMS sensitivity  $\delta\phi \sim \bar{n}^{-1}$  versus the average number of photons  $\bar{n}$ . It involves pairs of alternate independent homodyne measurements of two conjugated quadratures on a weakly squeezed state at the input. Nonunit quantum efficiency  $\eta$  of photodetectors degrades phase sensitivity leading to a power law  $\delta\phi \sim \bar{n}^{-\gamma(\eta)}$ , with  $\gamma$  increasing versus  $\eta$ .

In phase-sensing interferometers minute variations of environmental parameters are detected through changes in the phase shift of a light beam relative to a local oscillator (LO). The back-action effect of radiation pressure on the measured parameter poses the problem of optimizing phase sensitivity for a given average number of photons  $\bar{n}$  [1, 2]. In a shot-noise limited homodyne interferometer that uses coherent states, the root-mean-square (RMS) phase error  $\delta\phi$  is proportional to  $\bar{n}^{-1/2}$ . Sensitivity can be improved up to  $\delta\phi \sim \bar{n}^{-1}$  using squeezed states [1–3]. Such a power law, however, only holds in a small neighbourhood of a fixed working point that should be pursued by a feedback, whereas any sizeable phase shift would greatly degrade sensitivity. Moreover, nonideal quantum efficiency  $\eta < 1$  of photodetectors leads again to shot noise for  $\bar{n} \geq \eta/[8(1 - \eta)]$ .

Strictly speaking, homodyne based interferometers do not provide a proper phase detection, because the output photocurrent from the homodyne is proportional to a *single* quadrature of the field, say  $\hat{a}_\varphi = \frac{1}{2}(ae^{-i\varphi} + a^\dagger e^{i\varphi})$ , with  $a$  denoting the annihilator of the field mode and  $\varphi$  the tunable phase of the LO. Upon dividing  $\hat{a}_\varphi$  by the input field amplitude  $|\langle a \rangle|$  (which should be known in advance) one has a knowledge of the phase shift  $\phi$  only in an average sense, i.e.  $\langle \hat{x}_\varphi \rangle = |\langle a \rangle| \cos(\phi - \varphi)$ , but a single outcome  $x$  of  $\hat{a}_\varphi$  may still correspond to an unreal phase when  $x > |\langle a \rangle|$ . Moreover,  $\phi$  turns out to be defined in a  $\pi$ -window, instead of a  $2\pi$  one. For these reasons a scheme to detect the phase of one field mode should ultimately correspond to measuring the polar angle between *two* output (reduced) photocurrents, say  $I_1$  and  $I_2$  [4]. The outcomes of the detector are points distributed in the complex plane  $\alpha \equiv I_1 + iI_2 \equiv \rho e^{i\phi}$ , and the phase probability distribution  $p(\phi)$  is just the marginal one of the probability  $H(\alpha, \bar{\alpha})$  of the complex current  $\alpha$ , namely

$$p(\phi) = \int_0^\infty \rho d\rho H(\rho e^{i\phi}, \rho e^{-i\phi}). \quad (1)$$

|| Permanent address: Dipartimento di Fisica 'Alessandro Volta', via A Bassi 6, I-27100 Pavia, Italy.

We are now faced with the problem of optimizing the phase sensitivity  $\delta\phi$  for such *two-current* schemes. At the purely abstract level, the problem of optimizing  $\delta\phi$  versus  $\bar{n}$  has been addressed in general terms in the framework of quantum estimation theory [5]. Shapiro *et al* [6] adopted the reciprocal peak likelihood as a measure of sensitivity, and found that the ultimate quantum limit goes as  $\delta\phi \sim \bar{n}^{-2}$ . However, on the basis of numerical simulations Lane *et al* [7] have shown that the reciprocal peak likelihood is not an actual measure of phase sensitivity, and in [8] pathologies in this definition of  $\delta\phi$  have been found. In [9] the customary RMS error has been adopted for  $\delta\phi$ , and the ultimate quantum limit has been obtained†:

$$\delta\phi \simeq \frac{1.36}{\bar{n}}. \quad (2)$$

However, there is no known two-current scheme which achieves the ideal limit (2), and the optimal RMS sensitivity which ideally could be gained by an actual apparatus (double homodyne or heterodyne [10]) is  $\delta\phi \sim \bar{n}^{-2/3}$  [9], in between the shot noise level  $\delta\phi \sim \bar{n}^{-1/2}$  and the ideal bound (2). Thus, the current state of the art on phase sensitivity is represented by the ultimate limit (2), but with no available scheme for achieving it.

The aim of this paper is to provide a concrete detection scheme for reaching the ideal limit (2). We will show that, except for a constant factor, the power law (2) can actually be achieved by means of pairs of independent homodyne measurements of two conjugated quadratures on a (stable) weakly squeezed state at the input. We will also analyse the effect of nonunit quantum efficiency  $\eta < 1$  at photodetectors, and show that, in contrast to the single-measurement scheme, sensitivity is not unstable versus  $\eta$ , with slow degradation corresponding to a power law  $\delta\phi \sim \bar{n}^{-\gamma(\eta)}$ , with  $\gamma$  increasing as a function of  $\eta$ .

For heterodyne or double homodyne detectors the probability density  $H(\alpha, \bar{\alpha})$  in equation (1) is just the  $Q$ -function of the field density matrix  $\hat{\rho}$  [9, 11]

$$Q(\alpha, \bar{\alpha}) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle. \quad (3)$$

In this case the sensitivity  $\delta\phi \sim \bar{n}^{-2/3}$  can be obtained for optimal states which are almost indistinguishable from weakly squeezed states [9]. This phase noise is mainly related to the additional 3dB noise suffered by the distribution  $H(\alpha, \bar{\alpha})$ , which is due to the fact that these schemes achieve a *joint measurement* of two noncommuting quadratures [12]. For this reason one is led to consider a scheme of *independent* measurements of conjugated quadratures, and this will be contemplated in the following.

Generalizing the previous framework, one can think of the  $Q$ -function as just a particular case of the  $s$ -ordering Wigner function, namely

$$W_s(\alpha, \bar{\alpha}) = \int \frac{d^2\lambda}{\pi^2} e^{\alpha\bar{\lambda} - \bar{\alpha}\lambda} \text{Tr}(\hat{\rho} e^{\lambda a^\dagger - \bar{\lambda} a + \frac{1}{2}s|\lambda|^2}). \quad (4)$$

For  $s = -1$  one obtains the  $Q$ -function, which is the probability distribution for antinormally ordered fields;  $s = 0$  and  $s = 1$  correspond to symmetrical and normal orderings, respectively. From equation (4) one can see that lower negative  $s$  produce smoother functions  $W_s$ : thus, in order to improve sensitivity  $\delta\phi$ , a sharper Wigner function ( $s > -1$ ) should be considered instead of the  $Q$ -function. However, only for  $s \leq -1$  the Wigner

† The bound in (2) has been obtained numerically, and is actually given by

$$\delta\phi = \frac{1.36 \pm 0.01}{\bar{n}^{1.00 \pm 0.01}}.$$

function is non-negative for all states  $\hat{\rho}$ , whereas by definition the function  $H$  is a genuine experimental probability in all cases. Thus, for  $s > -1$  there remains only the possibility that  $H(\alpha, \bar{\alpha})$  coincides with  $W_s(\alpha, \bar{\alpha})$  for some special states: for  $s \leq 0$  this is true for squeezed states [13]. In fact, let us consider a squeezed state  $|\beta, r\rangle$ , with signal  $\beta$  and squeezing parameter  $r$  both real positive. For  $s \leq 0$  the Wigner function (4) is given by the double Gaussian

$$W_s(\alpha, \bar{\alpha}) = \frac{1}{2\pi\sigma_{1s}\sigma_{2s}} \exp\left[-\frac{(\text{Re}\alpha - \beta)^2}{2\sigma_{1s}^2} - \frac{(\text{Im}\alpha)^2}{2\sigma_{2s}^2}\right] \quad (5)$$

where the variances  $\sigma$  are the  $s$ -ordered second moments of the two conjugated quadratures  $\hat{a}_0$  and  $\hat{a}_{\pi/2}$ , namely

$$\sigma_{1s}^2 \equiv \langle : \Delta a_0^2 :_s \rangle = \frac{1}{4}(e^{2r} - s) \quad (6)$$

$$\sigma_{2s}^2 \equiv \langle : \Delta a_{\pi/2}^2 :_s \rangle = \frac{1}{4}(e^{-2r} - s). \quad (7)$$

The sharpest distribution clearly corresponds to  $s = 0$ .

The scheme for detecting the Gaussian Wigner function (5) for  $s = 0$  is based on homodyne detection, but with the highly excited LO alternately switching between phases  $\varphi = 0$  and  $\varphi = \pi/2$ . In this way each experimental event consists of a pair of independent measurements of the conjugated quadratures  $\hat{a}_0$  and  $\hat{a}_{\pi/2}$ , with measurements successively performed on the input field prepared again in the same state. This scheme is similar to the *optical homodyne tomography* [14], where the Wigner function is recovered by an ensemble of many repeated measurements of the quadratures  $\hat{a}_\varphi$  for different phases  $\varphi$  of the LO (in current experiments up to  $10^4$ – $10^5$  measurements can be performed within the stability time of the source). In our case only two measurements at different  $\varphi$  are needed, because we know in advance that the Wigner function is Gaussian. It is clear that the present scheme is used to detect time-dependent phase shifts due to some perturbing force or change of any environmental parameter (which is the actual motivation of any phase detection), so the pair of measurements should be performed within a time delay much shorter than the typical time scale of the perturbation. If we denote by  $x$  and  $y$  the outcomes of the measurements of  $\hat{a}_0$  and  $\hat{a}_{\pi/2}$ , respectively, each event corresponds to a point in the complex  $\alpha$ -plane defined by  $\alpha = x + iy$ . The probability distribution of the complex outcomes is given by

$$H(\alpha, \bar{\alpha}) = p_0(\text{Re}\alpha)p_{\pi/2}(\text{Im}\alpha), \quad (8)$$

where  $p_\phi$  represents the probability distribution of the quadrature  $\hat{a}_\phi$ . It is clear that  $H(\alpha, \bar{\alpha}) \equiv W_0(\alpha, \bar{\alpha})$  in equation (5) for squeezed states  $|\beta, r\rangle$ , because each Gaussian factor in the probability (5) is just a quadrature probability in equation (8). Hence, the proposed scheme detects the Wigner function  $W_0(\alpha, \bar{\alpha})$  for Gaussian states (squeezed or coherent).

Before addressing the problem of optimizing the phase sensitivity, let us consider the case of nonunit quantum efficiency at photodetectors. A photodetector with  $\eta < 1$  is equivalent to an ideal detector preceded by a beam splitter with transmissivity  $\eta$ . With this scheme in mind it is simple to check that the probability distribution  $p_\eta(I)$  of the output photocurrent of a homodyne detector becomes the following Gaussian convolution of the ideal probability  $p_1(I)$ :

$$p_\eta(I) = \int_{-\infty}^{\infty} dx p_1(x) \frac{\exp[-2\eta(I-x)^2/(1-\eta)]}{\sqrt{\pi(1-\eta)/2\eta}}. \quad (9)$$

The output probability distribution of the whole apparatus is thus given by

$$H_\eta(\alpha, \bar{\alpha}) = \int d^2\beta W_0(\beta, \bar{\beta}) \frac{\exp[-2\eta|\alpha - \beta|^2/(1-\eta)]}{\pi(1-\eta)/2\eta} \quad (10)$$

and coincides with the Wigner quasiprobability for negative  $s$ , namely

$$H_\eta(\alpha, \bar{\alpha}) = W_{1-\eta^{-1}}(\alpha, \bar{\alpha}). \quad (11)$$

Therefore, the Wigner function for negative  $s$  coincides with the probability distribution of the detector that has nonunit quantum efficiency at the photodetector  $\eta = 1/(1-s)$  [15]. Notice that  $\eta = \frac{1}{2}$  ( $s = -1$ ) leads to the  $Q$ -function, which, as already seen, can also be measured by means of heterodyne or double homodyne schemes with unit quantum efficiency: the effective quantum efficiency  $\eta = \frac{1}{2}$  corresponds to the additional 3 dB noise due to jointly measuring the pair of conjugated quadratures [12].

Now we address the problem of phase optimization for our scheme. For  $s \leq 0$  the marginal phase distribution of the Wigner function (5) is given by

$$p_s(\phi) = \frac{1}{4\pi\sigma_{1s}\sigma_{2s}\kappa(\phi)} \exp\left(-\frac{\bar{n} - \sinh^2 r}{2\sigma_{1s}^2}\right) \left\{ 1 + e^{\lambda(\phi)} \sqrt{\pi\lambda(\phi)} \left[ 1 + \operatorname{erf}\left(\sqrt{\lambda(\phi)}\right) \right] \right\} \quad (12)$$

where

$$\lambda(\phi) = \frac{(\bar{n} - \sinh^2 r)\sigma_{2s}^2}{2\sigma_{1s}^2(\sigma_{2s}^2 + \sigma_{1s}^2 \tan^2 \phi)} \quad (13)$$

$$\kappa(\phi) = \frac{1}{2} \left[ \frac{\cos^2 \phi}{\sigma_{1s}^2} + \frac{\sin^2 \phi}{\sigma_{2s}^2} \right] \quad (14)$$

and  $\operatorname{erf}(x)$  denotes the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}. \quad (15)$$

We evaluate the RMS phase sensitivity in the  $[-\pi, \pi]$  window, namely

$$\delta\phi_s = \left[ \int_{-\pi}^{\pi} d\phi \phi^2 p_s(\phi) \right]^{1/2}. \quad (16)$$

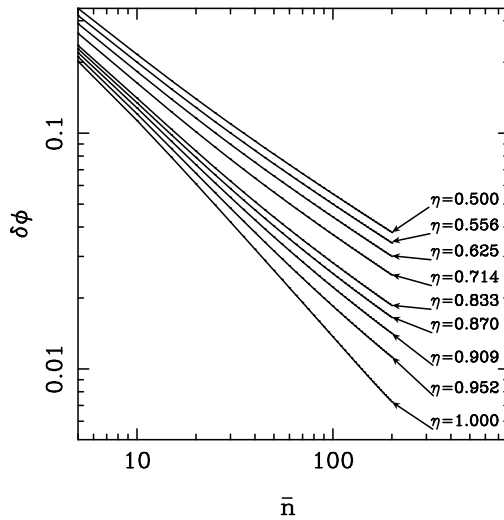
The phase sensitivity is optimized numerically by varying the fraction of squeezing photons  $\bar{n}_{sq} = \sinh^2 r$  at fixed total average photon number  $\bar{n}$ . The optimal  $\delta\phi$  versus  $\bar{n}$  is plotted in figure 1 for various  $\eta$ . In figure 2 the optimal fraction  $\bar{n}_{sq}/\bar{n}$  of squeezing photons is given. For all negative  $s$  the phase sensitivity obeys the power law

$$\delta\phi \sim \bar{n}^{-\gamma(\eta)} \quad (17)$$

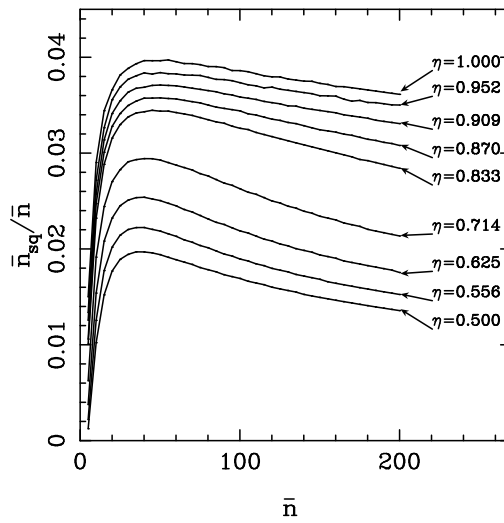
where the exponent  $\gamma$  versus  $\eta$  is plotted in figure 3. One can notice that for decreasing  $\eta$  there is a degradation of sensitivity, and the exponent  $\gamma$  increases as a function of  $\eta$  (roughly one has  $\gamma \simeq 1 - \eta\sqrt{1-\eta}$  in the considered range). Only a few per cent of squeezing photons is needed for optimal sensitivity, and fewer squeezing photons are required if the detectors are less efficient. (Physically a less efficient detector is more sensitive to the signal than to the squeezing photons.) The case  $s = -1$  corresponds to the result already obtained in [9] for a heterodyne or double homodyne detector. The best sensitivity is obviously attained for unit quantum efficiency ( $\eta = 1$ ,  $s = 0$ ). In this case the power law is explicitly given by

$$\delta\phi \simeq \frac{2.72}{\bar{n}}. \quad (18)$$

Equation (18) differs from the ultimate RMS sensitivity (2) by just a factor of two, and another factor of two must be accounted if  $\delta\phi$  is written in terms of the total number of photons  $n_T = 2\bar{n}$  per experimental point.

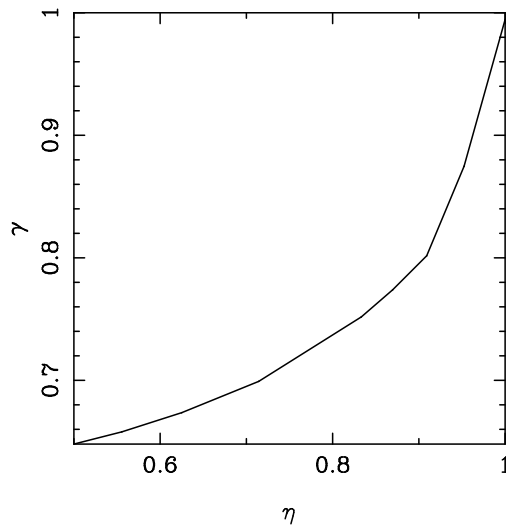


**Figure 1.** Optimized phase sensitivity versus  $\bar{n}$  for various quantum efficiencies  $\eta$ .



**Figure 2.** Fraction of squeezing photons  $\bar{n}_{sq}/\bar{n}$  corresponding to optimal sensitivities in figure 1.

In conclusion, we have presented a feasible two-current detection scheme that achieves ideal phase sensitivity  $\delta\phi \sim \bar{n}^{-1}$ . The scheme is based on pairs of independent homodyne measurements of two conjugated quadratures. Within the limits of our numerical analysis we have seen that a nonunit quantum efficiency slightly degrades the exponent of the power law: this is relevant from the experimental point of view, especially if one considers that the best sensitivity of conventional homodyne interferometric schemes is unstable versus  $\eta$  (any value of  $\eta < 1$  leads to shot noise for sufficiently large  $\bar{n}$ ). The present scheme is much more efficient than the heterodyne or double homodyne schemes considered in [9, 10], which have a halved effective quantum efficiency related to the additional 3dB noise due to the joint measurement of the two quadratures. With respect to the heterodyne/double



**Figure 3.** Exponent  $\gamma$  of phase sensitivity  $\delta\phi \sim \bar{n}^{-\gamma}$  versus quantum efficiency of photodetectors.

homodyne schemes, the one proposed here needs twice the number of measurements on the same state: however, for sufficiently high numbers of photons the improved sensitivity versus  $\bar{n}$  makes the present scheme the most convenient one.

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