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Clean positive operator valued measures

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In quantum mechanics the statistics of the outcomes of a measuring apparatus is described by a positive operator valued measure (POVM). A quantum channel transforms POVMs into POVMs, generally irreversibly, thus losing some of the information retrieved from the measurement. This poses the problem of *which POVMs are "undisturbed," i.e., they are not irreversibly connected to another POVM*. We will call such POVMs *clean*. In a sense, the clean POVMs would be "perfect," since they would not have any additional "extrinsical" noise. Quite unexpectedly, it turns out that such a "*cleanness*" property is largely unrelated to the convex structure of POVMs, and there are clean POVMs that are not extremal and vice versa. In this article we solve the cleannes classification problem for number n of outcomes $n \leqslant d$ (d dimension of the Hilbert space), and we provide a set of either necessary or sufficient conditions for $n > d$, along with an iff condition for the case of informationally complete POVMs for $n = d^2$. © *2005 American Institute of Physics.* [DOI: 10.1063/1.2008996]

I. INTRODUCTION

The new quantum information technology[1] has resurrected the interest in the theory of quantum measurements[2] and quantum open systems,[3,4] shifting the interest from applications to high-sensitivity and high-precision experiments[5] to its use in quantum information processing.[6] Depending on the particular kind of quantum processing—e.g., teleportation,[7,8] entanglement detection,[9] and distillation[10]—that are used in quantum computation[1,6] and quantum cryptography,[11] various new types of quantum measurements are now needed. The theory for engineering new quantum measurements optimized according to given criteria has been pioneered since the late 1960s by many authors[12] who concurred to the making of the quantum estimation theory,[13] the ancestor of the modern quantum information theory.

The general strategy of quantum estimation theory is to optimize the output statistics of the measuring apparatus according to a given criterion/fidelity, which depends on the specific use of

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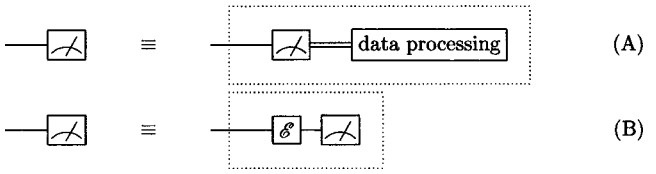

FIG. 1. There are two ways of processing POVMs: (a) the *postprocessing* of the output data and (b) *preprocessing* of the input state by a quantum channel. The postprocessing cannot generally achieve the same result of a *preprocessing:* the postprocessing is purely classical, whereas the preprocessing is quantum.

the measurement, the outcome statistics of the measurement for all possible input states being described by a positive operator valued measure (POVM).[13] POVMs form a convex set, where convex combinations correspond to random choices among different apparatuses. Most optimization problems actually resort to minimize a concave function on such a convex set, thereby optimization can be restricted to its extremal points, where concave functions attain their minimum. Coincidentally, due to the specific form of the optimization function, in many applications the optimal POVMs turn out to have unit rank—e.g., for phase estimation on pure states[13,14]—and this has led to the widespread belief that optimality is synonym of rank-one, whereas for sufficiently large dimension, and typically for optimization with input mixed states, the rank of extremal POVMs can be easily larger than one, as shown in Refs. 15–17.

In a specific application the optimal POVM does not necessarily attain the whole accessible information. At first sight, this assertion may appear contradictory: how a POVM can be optimal, if it wastes accessible information? However, once the measurement is performed, no other possibility for optimization is left apart from the processing of the outcome—*postprocessing* for short—and, being purely classical, the postprocessing cannot generally achieve the same result of a *preprocessing* by a quantum channel. The situation is depicted in Fig. 1. Clearly, the preprocessing can change the POVM irreversibly, reducing the information from the measurement. On the other hand, it is possible that a POVM optimal for a given criterion is obtainable from another *cleaner* one via an irreversible preprocessing as in Fig. 1(b). This means that in some cases we need to give up some *quantity* of information for the *quality* of the information.

The above-mentioned scenario poses the problem of *which POVMs are "undisturbed,"* namely *are not irreversibly connected to another POVM*. We will call such POVMs *clean*—in a sense a clean POVM would be "perfect," since it would not have any additional "extrinsical" noise, or it has lost no information irreversibly. Quite surprisingly, as announced, in this article we will see that the *cleanness* property of the POVM is largely unrelated to its extremality, and there are clean POVMs that are not extremal and vice versa. The problem of classifying clean POVMs turns out to be more difficult than that of classifying extremal ones, and in this article we will give a complete classification of clean POVMs only for a number n of outcomes $n \leqslant d$, whereas for $n > d$ we will give a set of interesting necessary conditions, and an iff condition for the case of informationally complete POVMs for $n = d^2$. Clearly, the need for a number of outcomes $n > d$ can be required by the particular optimization problem (see, e.g., Refs. 18 and 19), however, no more than $n = d^2$ elements are needed, which is the maximum number of outcomes for extremality.[15] Davies[20] proved d^2 to be an upper bound for the maximal cardinality of the POVM needed to attain the accessible information, and still it is debated if d^2 outcomes are actually needed (the cases of Refs. 18 and 19 proved that the lower bound is actually larger than d). This difficulties reflect those in classifying cleanness for $n > d$. In a sense it is clear that d^2 elements are needed to retrieve the accessible information, when the kind of information needs to be decided after the measurement has been performed. Indeed, an extremal POVM with d^2 outcomes is versatile to any kind of information encoding, as it is "informationally complete,"[21] namely it makes it possible to estimate any ensemble average by changing only the data processing of the outcomes (such an extremal measurement with d^2 elements can be proved to exist for any dimension d[15]). Clearly, for an extremal informationally complete measurement, a further optimization step can be achieved at the level of data processing,[22,23] once the kind of information of interest has been decided. Thus,

the postprocessing of the measurement is still a useful tool in retrieving the right information from a measurement.

The article is organized as follows. After introducing some notations and prerequisites in Sec. II, in Sec. III we state some general results about channels and POVMs which will be used throughout the article. In Sec. IV we analyze the convex set of channels connecting two POVMs. Section V is devoted to a complete analysis of the classical postprocessing, and give a complete characterization of "cleanness" under postprocessing. Section VI addresses the problem of the preprocessing ordering of POVMs, namely if a POVM is "cleaner" than another, and when they are "equivalent," which corresponds to the possibility of reversing the action of the channel connecting the two POVMs. Section VIII shows that for dimension $d=2$ equivalence under cleanness is the same as unitary equivalence. Section IX fully solves the case of number of outcomes $n \leqslant d$, and gives some interesting alternative theorems for the case of *effects*, namely the two-outcome POVMs. Section X analyzes the case of informationally complete POVMs, giving also an iff condition characterizing the clean POVMs. Section XI gives some conditions for rank-one measurements. Finally, we conclude the paper in Sec. XII with a list of most relevant results and of the main open problems.

II. NOTATION AND PREREQUISITES

Throughout this article we will consider a quantum system with Hilbert space H with finite dimension $d=\dim(\mathsf{H})$, and denote by S the set of states on H (corresponding to a positive unit-trace operator on H), and by $\mathsf{B}(\mathsf{H})$ the algebra of bounded operators on H. We will use capital script fonts e.g., $\mathcal{A},\mathcal{B},\ldots$, to denote operator algebras in $\mathsf{B}(\mathsf{H})$, and with the symbol $\mathcal{A}'$ we will denote the commutant of $\mathcal{A}$, namely the algebra defined as $\mathcal{A}' \doteq \{Y \in \mathsf{B}(\mathsf{H}) \,|\, [X,Y]=0, X \in \mathcal{A}\}$. Completely positive trace-preserving (CPT) and identity-preserving maps on S and $\mathsf{B}(\mathsf{H})$, respectively—all generally referred to as *channels*—will be denoted by capital calligraphic letters, e.g., $\mathcal{A},\mathcal{B},\ldots$, whereas we will always use capital Roman letters for operators. We will restrict attention to POVMs $\{P_e\}_{e\in\mathsf{E}}$ with finite sampling space E, namely

$$P_e \geqslant 0, \quad \forall\, e \in \mathsf{E}, \quad \sum_{e\in\mathsf{E}} P_e = I. \tag{1}$$

We will extensively use the vector notation $\mathbf{P} \equiv \{P_e\}$, $\mathsf{E}(\mathbf{P})$ denoting the sampling space of $\mathbf{P}$, and $|\mathbf{P}|$ the cardinality of $\mathsf{E}(\mathbf{P})$. The vector notation will be naturally extended to tensor products—e.g., $\mathbf{P}\otimes\mathbf{Q}$ for the POVM $\{P_e \otimes Q_f\}_{e\in\mathsf{E}(\mathbf{P}), f\in\mathsf{E}(\mathbf{Q})}$ on $\mathsf{H}\otimes\mathsf{H}$—and to functionals—e.g., $\mathrm{Tr}[\rho\mathbf{P}]$ for the vector of probabilities $\mathrm{Tr}[\rho P_e]$. By $\mathsf{Span}(\mathbf{P})$ we will denote the linear operator space spanned by the POVM elements $\{P_e\}_{e\in\mathsf{E}(\mathbf{P})}$, and by $\mathsf{Rng}(\mathbf{P})$ the range of the POVM $\mathbf{P}$, which is defined as the following convex subset of $\mathbb{R}_+^{|\mathbf{P}|}$

$$\mathsf{Rng}(\mathbf{P}) \doteq \{\mathbb{R}_+^{|\mathbf{P}|} \ni \boldsymbol{p} = \mathrm{Tr}[\rho\mathbf{P}], \quad \rho \in \mathsf{S}\}. \tag{2}$$

The convex set of POVMs with cardinality N will be denoted by $\mathcal{P}_N$.

Finally, we will use the symbol $|A\rangle\rangle$ to denote the following bipartite vector in $\mathsf{H}\otimes\mathsf{H}$

$$|A\rangle\rangle \doteq \sum_{m,n=1}^{d} A_{m,n}|m\rangle|n\rangle, \tag{3}$$

where $A \in \mathsf{B}(\mathsf{H})$ is the operator corresponding to the $d\times d$ matrix with elements $A_{m,n}$ on the basis $\{|n\rangle\}$. One can easily verify the following useful identities

$$A \otimes B^{\mathsf{T}}|C\rangle\rangle = |ACB\rangle\rangle,$$

$$\mathrm{Tr}_1[|A\rangle\rangle\langle\langle B|] = A^{\mathsf{T}}B^*, \tag{4}$$

$$\mathrm{Tr}_2[|A\rangle\rangle\langle\langle B|] = AB^\dagger,$$

where X^T denotes the transpose in the basis $\{|n\rangle\}$, while X^* is the complex conjugate in the same basis. Tr_i denotes the partial trace on the ith space.

III. USEFUL LEMMAS ABOUT CHANNELS AND POVMs

In the following we will name a map $\mathcal{E}$ *spectrum-width decreasing* when it reduces the "spectral width" of a real symmetric operator X, namely when

$$[\lambda_m(\mathcal{E}(X)), \lambda_M(\mathcal{E}(X))] \subseteq [\lambda_m(X), \lambda_M(X)], \tag{5}$$

$\lambda_M(X)$ and $\lambda_m(X)$ denoting the maximum and minimum eigenvalues of X, respectively.

Lemma III.1: Channels are spectrum-width decreasing.

Proof: Consider the eigenvector $|\psi_j\rangle$ of $\mathcal{E}(X)$ corresponding to the eigenvalue $\lambda_j(\mathcal{E}(X))$. One has

$$\lambda_j(\mathcal{E}(X)) = \mathrm{Tr}[\mathcal{E}(X)|\psi_j\rangle\langle\psi_j|] = \mathrm{Tr}[X\mathcal{E}^\mathsf{T}(|\psi_j\rangle\langle\psi_j|)] \in [\lambda_m(X), \lambda_M(X)], \tag{6}$$

since the dual map $\mathcal{E}^\mathsf{T}$ is CPT. ■

Notice that in the above-mentioned lemma the identity-preserving condition is crucial, since the lemma would not hold for a CPT map $\mathcal{E}$, e.g., $\mathcal{E}(\rho)=|\psi\rangle\langle\psi|$, and the spectral width increases from $[\lambda_m(\rho), \lambda_M(\rho)]$ to $[0, 1]$.

The inverse of a non-unitary invertible channel is necessarily not completely positive.

Theorem III. 2 (Wigner): *Any invertible channel has CP inverse iff it is unitary*.

Proof: Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two channels such that $\mathcal{E}_2^\mathsf{T} \circ \mathcal{E}_1^\mathsf{T}(\rho)=\rho$. Hence:

$$|\psi\rangle\langle\psi| = \sum_{ij} B_j A_i |\psi\rangle\langle\psi| A_i^\dagger B_j^\dagger, \quad \forall\, |\psi\rangle, \tag{7}$$

where A_i and B_j are canonical Kraus representations for $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Since all terms in the sum are positive, this means that $B_j A_i|\psi\rangle = \beta_{ij}^{\psi}|\psi\rangle$, for all $|\psi\rangle$ and all i,j. By linearity, it is clear that β_{ij} cannot depend on $|\psi\rangle$, implying that $B_j A_i = \beta_{ij} I$, for all i,j.

We can now prove that $\beta_{ij} \neq 0$, for all i,j. Otherwise, there exists a couple of operators B_k and A_l for which $B_k A_l = 0$. These two operators must both be noninvertible, since, if one is invertible, the other has to be null, and we can without loss of generality (w.l.o.g.) drop it from the Kraus representation (7). Let us fix the couple k,l for which $B_k A_l = 0$, namely both are not invertible. Now, the only possibility to have $B_j A_i = \beta_{ij} I$ for all i,j is that $B_k A_i = 0$ for all i (since B_k is not invertible, whence necessarily $B_k A_i$ cannot be full rank), and analogously $B_j A_l = 0$ for all j. In this case, all B_j's supports would be forced to be contained in the orthogonal complement to the range of A_l (which is strictly contained in the full Hilbert space), and this would be in contradiction with the normalization condition $\sum_j B_j^\dagger B_j = I$. Therefore, $\beta_{ij} \neq 0$ for all i,j, and the operators A_i and B_j are all invertible. This allows us to write

$$\begin{aligned} B_j &= \beta_{ij} A_i^{-1}, \quad \forall\, j, \\ A_i &= \beta_{ij} B_j^{-1}, \quad \forall\, i, \end{aligned} \tag{8}$$

whence all B_j's are proportional to each other, and analogously for the A_i. In other words, the Kraus representations of $\mathcal{E}_1$ and $\mathcal{E}_2$ are made of only one operator. This means that $\mathcal{E}_1$ and $\mathcal{E}_2$ are unitary, one the inverse of the other.

The converse direction is trivial. In Corollary X.4, we will prove that the inverse map of an invertible nonunitary channel is indeed nonpositive. ■

Theorem III.3 (Chefles, Jozsa, Winter): *Consider two sets of pure states on* H *with the same cardinality. There exist two channels mapping the elements of the first set to the corresponding elements of the second set and vice versa, iff the two sets of states are unitarily equivalent*.

Proof: See Ref. 24. ■

Lemma III.4. (Lindblad): A channel $\mathcal{E}$ stabilizes an algebra $\mathcal{S}\subseteq\mathsf{B}(\mathsf{H})$, namely

$$\mathcal{E}(X)=X, \quad \forall\, X\in\mathcal{S}, \tag{9}$$

iff the operators $\{E_k\}$ of any Kraus form $\mathcal{E}(X)=\Sigma_k E_k^\dagger X E_k$ belong to the commutant $\mathcal{S}'$ of the algebra $\mathcal{S}$.

Proof: See Ref. 25. ■

Finally let us state some results about extendiblity of completely positive maps (mostly taken from Ref. 26). To this end let us consider a linear subset $\mathcal{S}$ of $\mathsf{B}(\mathsf{H})$ which contains the identity and is closed under adjoints—each set $\mathcal{S}$ of this type will be called in the following an *operator system*. It is easy to see that $\mathcal{S}$ is generated (as a linear space) by its positive elements. It makes therefore sense to speak about positive maps $\mathcal{E}:\mathcal{S}\to\mathcal{A}$ into an algebra $\mathcal{A}$ and we can define also *complete positivity* in the usual way. Now the question arises whether such an $\mathcal{E}$ can be extended *as a completely positive map* to $\mathsf{B}(\mathsf{H})$. The following theorem gives a positive answer (Ref. 26, Theorems 6.2 and 7.5):

Theorem III.5. (Arveson's extension theorem): *Each completely positive map $\mathcal{E}:\mathcal{S}\to\mathsf{B}(\mathsf{H})$ defined on an operator system $\mathcal{S}\subset\mathsf{B}(\mathsf{H})$ can be extended to a completely positive map on $\mathsf{B}(\mathsf{H})$.*

If $\mathcal{E}$ is only positive (and not necessarily completely positive) a similar result is not available (cf. the corresponding discussion in Sec. VII). An important exception arises however, if the algebra $\mathcal{A}$ is abelian (Ref. 26, Theorem 3.9).

Theorem III.6: *If $\mathcal{E}:\mathcal{S}\to\mathcal{A}$ is positive, $\mathcal{S}$ an operator system and $\mathcal{A}$ an abelian algebra, the map $\mathcal{E}$ is completely positive.*

IV. THE CONVEX SET OF CHANNELS CONNECTING TWO POVMs

We now analyze the convex set of channels connecting two given POVMs **P** and **Q**, in equations

$$\mathcal{C}_{\mathbf{PQ}}=\{\mathcal{E}\ \text{channel}|\mathcal{E}(\mathbf{P})=\mathbf{Q}\}. \tag{10}$$

The extremal elements of $\mathcal{C}_{\mathbf{PQ}}$ can be characterized in terms of the operators $\{E_i\}$ of the canonical Krauss decomposition by the following theorem.

Theorem IV.1: *The map $\mathcal{E}\in\mathcal{C}_{\mathbf{PQ}}$ is extremal iff for some element P_k of the POVM* **P** *the operators $\{E_i^\dagger P_k E_j\}_{ij}$ made with the canonical Kraus operators $\{E_i\}$ of the map are linearly independent.*

Proof: First we show by contradiction that the condition is sufficient. In fact, suppose that $\mathcal{E}$, with $\{E_i^\dagger P_k E_j\}_{ij}$ linearly independent, is not extremal in $\mathcal{C}_{\mathbf{PQ}}$. Then there exist two different channels $\mathcal{E}_\pm\in\mathcal{C}_{\mathbf{PQ}}$ such that

$$\mathcal{E}=\tfrac{1}{2}(\mathcal{E}_+ +\mathcal{E}_-). \tag{11}$$

Upon defining $\mathcal{P}\equiv\mathcal{E}_+-\mathcal{E}$, clearly one has $\mathcal{P}(\mathbf{P})=0$ and $\mathcal{E}\pm\mathcal{P}=\mathcal{E}_\pm$, which are channels. Then $R_{\mathcal{E}_\pm}\equiv R_{\mathcal{E}}\pm R_{\mathcal{P}}\geqslant 0$, where for any channel $\mathcal{E}$ the positive operator $R_{\mathcal{E}}$ in linear correspondence with $\mathcal{E}$ is defined as $R_{\mathcal{E}}=\Sigma_j|E_j\rangle\rangle\langle\langle E_j|$ for $\{E_j\}$ Kraus operators of $\mathcal{E}$.[27] This implies that $\mathsf{Supp}(R_{\mathcal{P}})\subseteq\mathsf{Supp}(R_{\mathcal{E}})$, namely there exists a nonvanishing matrix p_{ij} such that $R_{\mathcal{P}}=\Sigma_{ij}p_{ij}|E_i\rangle\rangle\langle\langle E_j|$. As a consequence we have

$$\mathcal{P}(P_k)=\sum_{ij}p_{ij}E_i^\dagger P_k E_j=0, \quad \forall\, k. \tag{12}$$

This contradicts the hypothesis. The proof that it is also necessary is now straightforward. Suppose indeed that the operators $\{E_i^\dagger P_k E_j\}_{ij}$ are linearly dependent. Then there exists a nonvanishing matrix of coefficients a_{ij} such that $\Sigma_{ij}a_{ij}E_i^\dagger P_k E_j=0$ for all k. If we define $p_{ij}=\kappa(a_{ij}+a_{ij}^*)$, then the map $\mathcal{P}(X)=\Sigma_{ij}p_{ij}E_j^\dagger X E_i$ will annihilate all elements of the POVM **P**, namely $\mathcal{P}(\mathbf{P})=0$. Moreover,

for a sufficiently small $\kappa \neq 0$ both maps $\mathcal{E}_{\pm}=\mathcal{E}\pm\mathcal{P}$ will be channels and will belong to $\mathcal{C}_{\mathbf{PQ}}$. This implies that $\mathcal{E}=\frac{1}{2}(\mathcal{E}_{+}+\mathcal{E}_{-})$, namely $\mathcal{E}$ is not extremal. ■

One can prove that either any element of the border of $\mathcal{C}_{\mathbf{PQ}}$ is also an element of the border of the full convex set of channels, or $\mathcal{C}_{\mathbf{PQ}}\equiv\{\mathcal{E}\}$. This comes from the definition of the border of a convex set

Definition IV.2: For a convex set $\mathcal{C}$, an element $p\in\mathcal{C}$ belongs to its boundary $\partial\mathcal{C}$ if and only if there exists $q\in\mathcal{C}$ such that

$$p+\epsilon(q-p)\in\mathcal{C},\quad p-\epsilon(q-p)\notin\mathcal{C},\quad \forall\,\epsilon\in[0,1],. \tag{13}$$

or, equivalently iff there exists $q\in\mathcal{C}$ such that for all $\epsilon>0$ for which $p+\epsilon q\in\mathcal{C}$ then $p-\epsilon q\notin\mathcal{C}$.

We will now prove the following lemma.

Lemma IV.3: The border of the convex $\mathcal{C}_{\mathbf{PQ}}$ is a subset of the border of the convex of all channels.

Proof: Consider a channel $\mathcal{E}\in\mathcal{C}_{\mathbf{PQ}}$ and a "perturbation" $\mathcal{P}$ such that $\mathcal{E}+\epsilon\mathcal{P}\in\mathcal{C}_{\mathbf{PQ}}\forall\,\epsilon\in[0,1]$. By definition $\mathcal{P}(P_i)=0$ for all P_i, whence, if $\mathcal{E}-\epsilon\mathcal{P}$ is completely positive, then it necessarily belongs to $\mathcal{C}_{\mathbf{PQ}}$. Therefore, the only way to have $\mathcal{E}$ on the border of $\mathcal{C}_{\mathbf{PQ}}$ is to have $\mathcal{E}-\epsilon\mathcal{P}$ not CP, namely $\mathcal{E}$ lies on the border of the convex of all channels. ■

A "geometrical" proof is also the following. Since the constraint defining $\mathcal{C}_{\mathbf{PQ}}$ is linear, then $\mathcal{C}_{\mathbf{PQ}}$ is a linear section of the convex of all channels, whence its border belongs to the border of the convex of all channels.

Remark: Notice that the convex set $\mathcal{C}_{\mathbf{II}}$ will coincide with that of all channels, $\mathbf{I}=\{I\}$ denoting the trivial POVM.

Remark: From Lemma IV.3 it follows that when two POVMs are connected by a channel they can be always connected by a border channel, apart from the case in which the connecting channel is unique.

V. POSTPROCESSING

The most general postprocessing of a POVM, is a shuffling of the POVM elements with conditional probability $p(i|j)$, corresponding to the mapping

$$Q_i=\sum_j p(i|j)P_j. \tag{14}$$

When two POVMs $\mathbf{P}$ and $\mathbf{Q}$ are connected by a mapping of the form (14) for some conditional probability $p(i|j)$ we will write $\mathbf{P}>_p\mathbf{Q}$, and say that the POVM $\mathbf{P}$ is *cleaner under postprocessing*—for short *postprocessing cleaner*—than the POVM $\mathbf{Q}$. Notice that a relation of the form (14) is meaningful generally for $|\mathbf{P}|\neq|\mathbf{Q}|$, with the number of outcomes changing from input to output.

Relevant examples of post processing are:

(i) identification of two outcomes, e.g., j and k are identified with the same outcome l, corresponding to $p(n|j)=\delta_{ln}$, $p(n|k)=\delta_{ln}$ and
(ii) permutation π of outcomes, corresponding to $p(\pi(j)|k)=\delta_{jk}$.

The relation $>_p$ is a pseudo-ordering, since it is

(i) reflexive, corresponding to

$$\mathbf{P}>_p\mathbf{P},\quad p(i|j)=\delta_{ij}; \tag{15}$$

(ii) transitive, i.e., $\mathbf{P}>_p\mathbf{Q}>_p\mathbf{R}$, corresponding to

$$R_i=\sum_j p(i|k)Q_k,\ Q_k=\sum_j p'(k|j)P_j,\Rightarrow R_i=\sum_j p''(i|j)P_j,$$

$$p''(i|j) = \sum_k p(i|k)p'(k|j). \tag{16}$$

An equivalence relation under postprocessing can be defined as follows.

Definition V.1: The POVMs $\mathbf{P}$ *and* $\mathbf{Q}$ *are* postprocessing equivalent—*in symbols* $\mathbf{P} \simeq_p \mathbf{Q}$—*iff both relations* $\mathbf{P} \succ_p \mathbf{Q}$ *and* $\mathbf{Q} \succ_p \mathbf{P}$ *hold.*

We are now in position to define *cleanness under postprocessing*, namely

Definition V.2: A POVM $\mathbf{P}$ *is* postprocessing clean *if for any POVM* $\mathbf{Q}$ *such that* $\mathbf{Q} \succ_p \mathbf{P}$, *then also* $\mathbf{P} \succ_p \mathbf{Q}$ *holds, namely* $\mathbf{P} \simeq_p \mathbf{Q}$.

The characterization of cleanness under postprocessing (classical) is much easier than that of cleanness under preprocessing (quantum), and is given by the following theorem.

Theorem V.3: *A POVM* $\mathbf{P}$ *is postprocessing clean iff it is rank-one.*

Proof: First notice that a POVM $\mathbf{P}$ with elements having rank higher than one are not postprocessing clean. In fact, in this case one can diagonalize all the POVM elements and consider the POVM $\mathbf{P}'$ made of rank-one projections over all eigenvectors multiplied by the corresponding eigenvalue. Then, clearly $\mathbf{P}' \succ_p \mathbf{P}$ by identification of outcomes. In equations

$$P_i = \sum_{k_i} |\lambda_{k_i}^{(i)}\rangle\langle\lambda_{k_i}^{(i)}|, \quad P'_{i,k} = |\lambda_k^{(i)}\rangle\langle\lambda_k^{(i)}|, \Rightarrow \mathbf{P}' \succ_p \mathbf{P}, \tag{17}$$

corresponding to the identification of outcomes

$$p(i|j,k_j) = \delta_{ij} \,\forall\, k_j. \tag{18}$$

Reversely, all rank-one POVMs are postprocessing clean, namely if $\mathbf{Q} \succ_p \mathbf{P}$, then also $\mathbf{P} \simeq_p \mathbf{Q}$ must hold. In fact, suppose that $\mathbf{P}$ is rank one and that there exists a POVM $\mathbf{Q}$ such that $\mathbf{Q} \succ_p \mathbf{P}$, namely

$$P_i = \sum_j p(i|j)Q_j. \tag{19}$$

Now, since all elements P_i are rank one, the elements Q_j are necessarily proportional to P_i for all the indices j such that $p(i|j) \neq 0$, namely also $\mathbf{Q}$ is rank one, with

$$p(i|j)Q_j = \alpha_j P_i, \tag{20}$$

with $\Sigma_j \alpha_j = 1$, and $\alpha_j \geqslant 0$. For a fixed j, $p(i|j) = 0$ for at least one i, otherwise all the P_i's would be proportional. For the same reason, for a fixed i, $p(i|j) = 0$ for at least one j. We can then collect the indices i such that $p(i|j) \neq 0$ in the set $I(j)$, and write

$$Q_j = \sum_i p(i|j)Q_j = \sum_{i \in I(j)} p(i|j)Q_j = \sum_{i \in I(j)} \alpha_j P_i. \tag{21}$$

Finally, it is immediately verified that

$$q(j|i) = \begin{cases} \alpha_j, & i \in I(j) \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

is a conditional probability since for all i one has $\Sigma_j q(j|i) = \Sigma_j \alpha_j = 1$. Therefore, from Eq. (21) it follows that we have also $\mathbf{P} \succ_p \mathbf{Q}$, namely $\mathbf{P} \simeq_p \mathbf{Q}$. ■

VI. PREPROCESSING: ORDERING AND EQUIVALENCE OF POVMs

The action of channels allows to define the following pseudo-ordering.

Definition VI.1: Given the POVMs $\mathbf{P}$ *and* $\mathbf{Q}$ *with* $|\mathbf{P}| = |\mathbf{Q}|$ *we define* $\mathbf{P} \succ \mathbf{Q}$ *iff there exists a channel* $\mathcal{E}$ *such that*

$$\mathbf{Q} = \mathcal{E}(\mathbf{P}), \tag{23}$$

and we will say that the POVM $\mathbf{P}$ *is cleaner than the POVM* $\mathbf{Q}$.

Definition VI.2: We call a POVM $\mathbf{P}$ *"clean" iff for any POVM* $\mathbf{Q}$ *such that* $\mathbf{Q}\succ\mathbf{P}$ *one also has* $\mathbf{P}\succ\mathbf{Q}$.

It is easily proved that the relation $\succ$ is transitive and reflexive, namely it is a pseudo-ordering. Let us now define the following relation

Definition VI.3: We say that the two POVMs $\mathbf{P}$ *and* $\mathbf{Q}$ *are equivalent—denoted as* $\mathbf{P}\simeq\mathbf{Q}$*—when one has both* $\mathbf{P}\succ\mathbf{Q}$ *and* $\mathbf{Q}\succ\mathbf{P}$.

Clearly $\simeq$ is an equivalence relation. The pseudo-ordering $\succ$ now defines a partial ordering between equivalence classes. Indeed define the ordering between classes as follows:

$$[\mathbf{P}]\geqslant[\mathbf{Q}] \quad \text{iff} \quad \mathbf{P}'\succ\mathbf{Q}', \quad \forall\,\mathbf{P}'\in[\mathbf{P}], \quad \mathbf{Q}'\in[\mathbf{Q}]. \tag{24}$$

The above-mentioned ordering is consistently defined, since $\mathbf{P}',\mathbf{P}''\in[\mathbf{P}]$ means both $\mathbf{P}'\succ\mathbf{P}''$ and $\mathbf{P}''\succ\mathbf{P}'$, whence, by transitivity $\mathbf{P}''\succ\mathbf{P}'\succ\mathbf{Q}'\succ\mathbf{Q}''$, and the ordering does not depend on the chosen elements of the equivalence classes. This proves the consistency of the definition of $\geqslant$. Therefore, in the following we can consider a single element $\mathbf{P}$ instead of the class $[\mathbf{P}]$. In this way we can easily prove reflexivity $[\mathbf{P}]\geqslant[\mathbf{P}]$, since $\mathbf{P}\succ\mathbf{P}$, and transitivity

$$[\mathbf{P}]\geqslant[\mathbf{Q}], \quad [\mathbf{Q}]\geqslant[\mathbf{R}]\Rightarrow[\mathbf{P}]\geqslant[\mathbf{R}], \tag{25}$$

since $\mathbf{P}\succ\mathbf{Q}$, $\mathbf{Q}\succ\mathbf{R}$ implies $\mathbf{P}\succ\mathbf{R}$ by transitivity of $\succ$. Now let us consider the case when both $[\mathbf{P}]\geqslant[\mathbf{Q}]$ and $[\mathbf{Q}]\geqslant[\mathbf{P}]$. Then we have $\mathbf{P}\succ\mathbf{Q}$ and $\mathbf{Q}\succ\mathbf{P}$, namely $[\mathbf{P}]\equiv[\mathbf{Q}]$. ■

One would be tempted to conjecture that the relation $\simeq$ is equivalent to unitary equivalence, which is defined through

Definition VI.4: The POVMs $\mathbf{P}$ *and* $\mathbf{Q}$ *are unitarily equivalent, for short* $\mathbf{P}\simeq_U\mathbf{Q}$ *iff there exists a unitary operator* U *such that* $\mathbf{Q}=U\mathbf{P}U^\dagger$.

However, as we will see in the following, there exist equivalent POVMs which are not unitarily equivalent.

We have now the following necessary condition for equivalence under preprocessing

Theorem VI.5: *If* $\mathbf{P}\simeq\mathbf{Q}$ *then for each event* $e\in\mathsf{E}(\mathbf{P})$ *we have*

$$\lambda_M(P_e)=\lambda_M(Q_e)\equiv\lambda_M(e), \quad \lambda_m(P_e)=\lambda_m(Q_e)\equiv\lambda_m(e). \tag{26}$$

Proof: By Lemma III.1 we have both $\lambda_M(P_i)\geqslant\lambda_M(Q_i)$ and $\lambda_M(P_i)\leqslant\lambda_M(Q_i)$, and similarly for the minimum eigenvalues. ■

VII. PREPROCESSING: POSITIVE MAPS AND RELATED THEOREMS

There are two interesting variants of the relation $\succ$ just introduced, which help to get a more geometric insight into the structure. The first arises, if we replace the completely positive map $\mathcal{E}$ in Definition VI.1 by positive (but not necessarily *completely* positive) one. Hence we can define for two POVMs $\mathbf{P}$, $\mathbf{Q}$ with $|\mathbf{P}|=|\mathbf{Q}|$

$$\mathbf{P}\gg\mathbf{Q}\Leftrightarrow\mathbf{Q}=\mathcal{E}(\mathbf{P}), \quad \mathcal{E}\ \text{positive}. \tag{27}$$

It is obvious that $\mathbf{P}\succ\mathbf{Q}$ implies $\mathbf{P}\gg\mathbf{Q}$ but the other way round does not hold. This can be seen, if we consider an informationally complete POVM $\mathbf{P}$ and define $\mathbf{Q}=\Theta(\mathbf{P})$, where Θ denotes the transposition map (i.e. $\Theta(A)=A^\mathsf{T}$). Positivity of Θ implies $\mathbf{P}\gg\mathbf{Q}$. But Θ is only positive and not completely positive and it is the only map which connects $\mathbf{P}$ and $\mathbf{Q}$. The latter follows from informational completeness of $\mathbf{P}$, because this implies that the elements of $\mathbf{P}$ are a basis of $\mathsf{B}(\mathsf{H})$. Hence $\mathbf{P}\succ\mathbf{Q}$ *does not hold.*

Now consider the *ranges* $\mathsf{Rng}(\mathbf{P})$, $\mathsf{Rng}(\mathbf{Q})$ of $\mathbf{P}$ and $\mathbf{Q}$, defined in Eq. (2). If $\boldsymbol{p}\in\mathsf{Rng}(\mathbf{Q})$ there is by definition a density operator ρ with $\boldsymbol{p}=\mathrm{Tr}[\mathbf{Q}\rho]$. Hence, $\mathbf{P}\gg\mathbf{Q}$ implies

$$\boldsymbol{p}=\mathrm{Tr}[\mathbf{Q}\rho]=\mathrm{Tr}[\mathcal{E}(\mathbf{P})\rho]=\mathrm{Tr}[\mathbf{P}\mathcal{E}^\mathsf{T}(\rho)] \tag{28}$$

and therefore we get $\boldsymbol{p}\in\mathsf{Rng}(\mathbf{P})$. This observation motivates the definition:

$$\mathbf{P} \supset_r \mathbf{Q} \Leftrightarrow \mathsf{Rng}(\mathbf{Q}) \subset \mathsf{Rng}(\mathbf{P}). \tag{29}$$

According to our previous discussion we get in this way a hierarchy of relations

$$\mathbf{P} \succ \mathbf{Q} \Rightarrow \mathbf{P} \succcurlyeq \mathbf{Q} \Rightarrow \mathbf{P} \supset_r \mathbf{Q}. \tag{30}$$

We have already seen that the direction of the implication between $\succ$ and $\succcurlyeq$ cannot be reversed. For $\succcurlyeq$ and $\supset_r$ this is more difficult. To see that they are (very) closely related, consider the linear hull $\mathsf{Span}(\mathbf{P})$ of the elements of $\mathbf{P}$, which is obviously an *operator system* (cf. Sec. III). Hence we can speak about positive linear maps from $\mathsf{Span}(\mathbf{P})$ to $\mathsf{Span}(\mathbf{Q})$. This fact can be used to characterize the relation $\supset_r$ in the following way.

Proposition VII.1: Consider two POVMs $\mathbf{P}$, $\mathbf{Q}$ *with* $|\mathbf{P}|=|\mathbf{Q}|$. *Then the following statements are equivalent:*

(i) $\mathbf{P} \supset_r \mathbf{Q}$
(ii) *There is a (unique) positive map* $\mathcal{E}$: $\mathsf{Span}(\mathbf{P}) \to \mathsf{Span}(\mathbf{Q})$ *with* $\mathcal{E}(\mathbf{P})=\mathbf{Q}$.

Proof: The implication (ii)$\Rightarrow$(i) is trivial. Hence consider only the other direction. Here, the idea is to define the map $\mathcal{E}$ by

$$\mathcal{E}(P_e) = Q_e, \quad \forall\, e \in \mathsf{E}. \tag{31}$$

This map is well defined because we have (by assumption) for each density operator ρ a second density operator σ such that $\mathrm{Tr}[Q_e\rho]=\mathrm{Tr}[P_e\sigma]$ holds for all $e \in \mathsf{E}$. Hence if $\Sigma_e \lambda_e P_e = 0$ for some real λ_e we get

$$\sum_{e\in\mathsf{E}} \lambda_e \mathrm{Tr}[\rho Q_e] = \sum_{e\in\mathsf{E}} \lambda_e \mathrm{Tr}[\sigma P_e] = \mathrm{Tr}\left[\sigma \sum_{e\in\mathsf{E}} \lambda_e P_e\right] = 0. \tag{32}$$

Since ρ is arbitrary this implies $\Sigma_e \lambda_e Q_e = 0$. Therefore $\mathcal{E}$ defined in (31) is well defined, as stated. Using the same reasoning we can show that $\mathcal{E}$ is positive, which concludes the proof. ■

The difference between condition (ii) of this lemma and the definition of $\succcurlyeq$ in Eq. (27) is the *domain* of the the map $\mathcal{E}$. The following counter example which is taken (in a slightly modified form) from Ref. 26 shows that such a map is in general *not extendible* as a positive map to $\mathsf{B}(\mathsf{H})$.

Consider the diagonal 4×4 matrix $X=\mathrm{diag}(1,i,-1,-i)$ and the operator system $\mathcal{S}$ spanned by $I, X, X^\dagger$. It is easy to see that a general element $A=aI+bX+cX^\dagger$ is Hermitian iff $c=b^*$ and $a=a^*$ hold, and it is positive iff in addition $a \geqslant 2\max(|\Re b|, |\Im b|)$ hence,

$$A \geqslant 0 \Rightarrow c = b^*, \quad a \geqslant \sqrt{2}|b|. \tag{33}$$

Now consider the linear map

$$\mathcal{S} \ni A = aI + bX + cX^\dagger \mapsto \mathcal{E}(A) = \begin{pmatrix} a & \sqrt{2}b \\ \sqrt{2}c & a \end{pmatrix} \otimes I_2, \tag{34}$$

where I_2 denotes the 2×2 unit matrix. Since a 2×2 matrix is positive iff its diagonal elements and its determinant are positive, positivity of $\mathcal{E}$ follows directly from Eq. (33). On the other hand we have $\|\mathcal{E}(I)\|=1$ and $\|\mathcal{E}(X)\|=\sqrt{2}$. Since $\|X\|=1$ this implies $\|\mathcal{E}\| \geqslant \sqrt{2} > \|\mathcal{E}(I)\|$. But a positive map from a C^* algebra $\mathcal{A}$ into a a C^* algebra $\mathcal{B}$ always satisfies (Ref. 26, Corollary 2.9) $\|\mathcal{E}\|=\|\mathcal{E}(I)\|$. Hence the map defined in Eq. (34) can not be extended to $\mathsf{B}(\mathbb{C}^4)$—not even to the abelian algebra generated by $I, X, X^\dagger$. As a consequence of this reasoning we have shown that $\mathbf{P} \supset_r \mathbf{Q}$ *does not imply* $\mathbf{P} \succcurlyeq \mathbf{Q}$.

Hence positive maps can in general not be extended as a *positive* map to a bigger algebra. A very important special case arises, however, if the map $\mathcal{E}$ is *completely* positive. In this case a completely positive extension always exists (cf. Theorem III.5) This fact can be used along with Proposition VII.1 to get an interesting characterisation of $\succ$ in terms of ranges.

Theorem VII.2: *Consider two POVMs* $\mathbf{P}$, $\mathbf{Q}$ *with* $|\mathbf{P}|=|\mathbf{Q}|$. *Then the following statements are equivalent:*

(i) $\mathbf{P}\succ\mathbf{Q}$
(ii) *There is an informationally complete POVM* $\mathbf{M}$ *such that* $\mathbf{P}\otimes\mathbf{M}\supset_r\mathbf{Q}\otimes\mathbf{M}$
(iii) $\mathbf{P}\otimes\mathbf{M}\supset_r\mathbf{Q}\otimes\mathbf{M}$ *holds for all POVMs* $\mathbf{M}$.

Proof: The implication (i)$\Rightarrow$(iii) follows from the fact that (i) implies the existence of a map $\mathcal{E}$ such that $\mathbf{Q}=\mathcal{E}(\mathbf{P})$, and trivially the map $\mathcal{E}\otimes\mathcal{I}$ connects $\mathbf{P}\otimes\mathbf{M}$ with $\mathbf{Q}\otimes\mathbf{M}$, whence the statement via Eq. (29). Implication (i)$\Rightarrow$(ii) is just a special case of the previous one. Implication (iii)$\Rightarrow$(ii) is trivial. Hence only (ii)$\Rightarrow$(i) remains to be shown.

To this end note that informational completeness of $\mathbf{M}$ implies

$$\mathsf{Span}(\mathbf{Q}\otimes\mathbf{M})=\mathsf{Span}(\mathbf{Q})\otimes\mathsf{B}(\mathsf{H}), \tag{35}$$

and similarly for $\mathbf{P}\otimes\mathbf{M}$. Therefore we have [according to (ii) and Proposition VII.1] a unique positive map

$$\mathcal{F}:\mathsf{Span}(\mathbf{P})\otimes\mathsf{B}(\mathsf{H})\rightarrow\mathsf{Span}(\mathbf{Q})\otimes\mathsf{B}(\mathsf{H}) \tag{36}$$

with

$$\mathcal{F}(\mathbf{P}\otimes\mathbf{M})=\mathbf{Q}\otimes\mathbf{M}. \tag{37}$$

If we can show that $\mathcal{F}$ has the form

$$\mathcal{F}=\mathcal{E}\otimes\mathcal{I} \tag{38}$$

with a positive map $\mathcal{E}$: $\mathsf{Span}(\mathbf{P})\rightarrow\mathsf{Span}(\mathbf{Q})$ and the identity $\mathcal{I}$ on $\mathsf{B}(\mathsf{H})$, the theorem is proved because:

- Due to Eq. (38) and positivity of $\mathcal{F}$ the map $\mathcal{E}$ is completely positive as a map on the *operator system* $\mathsf{Span}(\mathbf{P})$. Hence by theorem III.5 it is extendible to a completely positive map on $\mathsf{B}(\mathsf{H})$.
- Eqs. (37) and (38) imply $\mathcal{E}(\mathbf{P})=\mathbf{Q}$ and therefore $\mathbf{P}\succ\mathbf{Q}$.

To prove Eq. (38) firstly note that (ii) implies $\mathbf{P}\supset_r\mathbf{Q}$. This follows from (with $e\in\mathsf{E}(\mathbf{Q})$ and a density matrix ρ on H):

$$\mathrm{Tr}[\rho Q_e]=\mathrm{Tr}\left[\frac{\rho\otimes I}{d}\left(Q_e\otimes\sum_{f\in\mathsf{E}(\mathbf{M})}M_f\right)\right] \tag{39}$$

$$=\sum_{f\in\mathsf{E}(\mathbf{M})}\mathrm{Tr}\left[(Q_e\otimes M_f)\left(\rho\otimes\frac{I}{d}\right)\right] \tag{40}$$

because we have by assumption a density matrix σ on $\mathsf{H}\otimes\mathsf{H}$ such that

$$\mathrm{Tr}\left[(\mathbf{Q}\otimes\mathbf{M})\left(\rho\otimes\frac{I}{d}\right)\right]=\mathrm{Tr}[(\mathbf{P}\otimes\mathbf{M})\sigma] \tag{41}$$

which in turn implies

$$\mathrm{Tr}[\rho Q_e]=\sum_{f\in\mathsf{E}(\mathbf{M})}\mathrm{Tr}[(P_e\otimes M_f)\sigma] \tag{42}$$

$$=\mathrm{Tr}\left[P_e\otimes\left(\sum_{f\in\mathsf{E}(\mathbf{M})}M_f\right)\sigma\right] \tag{43}$$

$$= \mathrm{Tr}[(P_e \otimes I)\sigma] = \mathrm{Tr}[P_e \, \mathrm{Tr}_2 \, \sigma], \tag{44}$$

where Tr_2 denotes the partial trace over the second tensor factor. Hence $\mathrm{Tr}[\rho \mathbf{Q}] = \mathrm{Tr}[(\mathrm{Tr}_2 \sigma)\mathbf{P}]$ which implies $\mathbf{P} \supset_r \mathbf{Q}$ as stated.

Now we can apply again Propostion VII.1 and get a positive map $\mathcal{E}$: $\mathsf{Span}(\mathbf{P}) \to \mathsf{Span}(\mathbf{Q})$ satisfying $\mathcal{E}(\mathbf{P}) = \mathbf{Q}$ and therefore $\mathcal{E} \otimes \mathcal{I}(\mathbf{P} \otimes \mathbf{M}) = \mathbf{Q} \otimes \mathbf{M}$. Since $\mathcal{F}$ is uniquely determined by Eq. (37) this implies $\mathcal{F} = \mathcal{E} \otimes \mathcal{I}$, which completes the proof. ■

This theorem gives a clear geometric picture for the relation $\succ$ and it helps to understand the difference between $\succ$ and $\gg$: if $\mathbf{P} \gg \mathbf{Q}$ holds we find for each *separable state* ρ on $\mathsf{H} \otimes \mathsf{H}$ a second separable state σ such that $\mathrm{Tr}[\mathbf{Q} \otimes \mathbf{M}\rho] = \mathrm{Tr}[\mathbf{P} \otimes \mathbf{M}\sigma]$. Hence, if $\mathbf{P} \succ \mathbf{Q}$ does not hold (but $\mathbf{P} \gg \mathbf{Q}$) there must be an *entangled* state ρ such that the probability vector $\mathrm{Tr}[\mathbf{Q} \otimes \mathbf{M}\rho]$ can not be reproduced by any expectation value of $\mathbf{P} \otimes \mathbf{M}$. This can be rephrased as follows: If $\mathbf{P} \gg \mathbf{Q}$ holds but not $\mathbf{P} \succ \mathbf{Q}$ we can reproduce the distribution of outcomes of $\mathbf{Q}$ measurements on *one system* by appropriate $\mathbf{P}$ measurements, but there is information about entangled states which can be gained only by $\mathbf{Q}$ and not by $\mathbf{P}$.

A second special case of Proposition VII.1 arises, when $\mathbf{Q}$ is abelian (i.e., all elements of the POVM commute). In this case the map $\mathcal{E}$ constructed in Proposition VII.1 is a map into an abelian algebra and therefore completely positive. Hence we get

Theorem VII.3: *Consider two POVMs* $\mathbf{P}, \mathbf{Q}$ *with* $|\mathbf{P}| = |\mathbf{Q}|$ *and assume that* $\mathbf{Q}$ *is abelian. Then* $\mathbf{P} \supset_r \mathbf{Q}$ *and* $\mathbf{P} \succ \mathbf{Q}$ *are equivalent*.

Proof: According to Proposition VII.1 there is a positive map $\mathcal{E}$ from $\mathsf{Span}(\mathbf{P})$ into the abelian C^* algebra $\mathcal{A}$ generated by the elements of $\mathbf{Q}$. According to Theorem III.6 this map is completely positive and by Theorem III.5 therefore extendible as a completely positive map to $\mathsf{B}(\mathsf{H})$. Hence $\mathbf{P} \supset_r \mathbf{Q}$ implies $\mathbf{P} \succ \mathbf{Q}$. Since the other implication is trivial the proof is completed. ■

Note that a similar result does not hold if $\mathbf{P}$ is abelian and $\mathbf{Q}$ is not. The counter example given after Proposition VII.1 applies even in this case.

The result from Theorem VII.3 is very useful, in particular because the range $\mathsf{Rng}(\mathbf{P})$ of an abelian POVM has a very simple structure, which is completely characterized by the joint eigenvalues of the elements of $\mathbf{P}$. To see this, consider a joint set of eigenvectors ψ_α, $\alpha = 1, \ldots, d$ and

$$P_e = \sum_{\alpha=1}^{d} \lambda_{e,\alpha} |\psi_\alpha\rangle\langle\psi_\alpha|, \quad \forall \, e \in \mathsf{E}. \tag{45}$$

The joint eigenvalues vectors

$$\boldsymbol{\lambda}_\alpha = (\lambda_{e,\alpha})_{e \in \mathsf{E}} \in \mathbb{R}^{|\mathbf{P}|} \tag{46}$$

form a set of probability vectors (in the case of joint degeneracies of the elements of $\mathbf{P}$ some of them may coincide) and for each convex linear combination

$$\boldsymbol{p} = \sum_{\alpha=1}^{d} p_\alpha \boldsymbol{\lambda}_\alpha, \quad p_\alpha \geqslant 0, \quad \sum_\alpha p_\alpha = 1 \tag{47}$$

we can find a density operator ($\rho = \Sigma_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ will do) such that $\boldsymbol{p} = \mathrm{Tr}[\rho \mathbf{P}]$ holds. Hence the *convex hull* of the $\boldsymbol{\lambda}_\alpha$ satisfies $\mathrm{conv}(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_d) \subset \mathsf{Rng}(\mathbf{P})$. On the other hand we have for each density operator ρ:

$$\mathrm{Tr}[\rho \mathbf{P}] = \sum_{\alpha=1}^{d} \langle \psi_\alpha, \rho \psi_\alpha \rangle \boldsymbol{\lambda}_\alpha \tag{48}$$

which implies $\mathrm{Tr}[\rho \mathbf{P}] \in \mathrm{conv}(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_d)$. Hence we have just shown the following proposition

Proposition VII.4: The range $\mathsf{Rng}(\mathbf{P})$ *of an abelian POVM* $\mathbf{P}$ *coincides with the convex hull of the* $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_d$.

The most simple example arises in the case of *effects*, i.e., measurements with two outcomes.

Obviously, each effect is abelian and has the form $\mathbf{P}=\{P, I-P\}$ with a positive operator $P \leqslant I$. If $\mu_1, \ldots, \mu_d$ are the eigenvalues of P given in decreasing order we get $\boldsymbol{\lambda}_\alpha=(\mu_\alpha, 1-\mu_\alpha)$. Hence all $\boldsymbol{\lambda}_\alpha \in \mathbb{R}^2$ are located on the graph of the function $\mathbb{R} \ni x \mapsto 1-x \in \mathbb{R}$, and $\boldsymbol{\lambda}_1$ respectively $\boldsymbol{\lambda}_d$ are the outermost points. This leads immediately to the following characterization of the relation $\succ$ for effects:

Theorem VII.5: *The effect* $\mathbf{P}$ *is "cleaner" than the effect* $\mathbf{Q}$, *i.e.* $\mathbf{P} \succ \mathbf{Q}$ *iff*

$$[\lambda_M(P), \lambda_m(P)] \supseteq [\lambda_M(Q), \lambda_m(Q)]. \tag{49}$$

Corollary VII.6: *Given two effects* $\mathbf{P}$ *and* $\mathbf{Q}$, *then* $\mathbf{P} \simeq \mathbf{Q}$ *iff* $\lambda_M(P)=\lambda_M(Q)$ *and* $\lambda_m(P)=\lambda_m(Q)$.

VIII. PREPROCESSING: EQUIVALENCE IN DIMENSION TWO

For dimension two the cleanness equivalence $\simeq$ and the unitary equivalence $\simeq_U$ coincide.

Theorem VIII.1: *For two-level systems* $\mathbf{P} \simeq \mathbf{Q}$ *iff* $\mathbf{P} \simeq_U \mathbf{Q}$.

Proof: If all the elements of both POVM are trivial, i.e., $P_e=\alpha_e I$ and $Q_e=\beta_e I$, $\forall e$, then the thesis follows easily. Therefore, we will focus on the nontrivial case, in which there exists at least one element P_i of $\mathbf{P}$ (or Q_i of $\mathbf{Q}$) that is nontrivial. Then, first, also Q_i (or P_i) is not proportional to the identity, since otherwise $P_i=\mathcal{F}(Q_i)=\alpha_i \mathcal{F}(I)=\alpha_i I$, which contradicts the hypothesis. Second, by Theorem VI.5 one has

$$P_i = \lambda_M(i) |\phi_M^{(i)}\rangle\langle\phi_M^{(i)}| + \lambda_m(i) |\phi_m^{(i)}\rangle\langle\phi_m^{(i)}|, \tag{50}$$

$$Q_i = \lambda_M(i) |\psi_M^{(i)}\rangle\langle\psi_M^{(i)}| + \lambda_m(i) |\psi_m^{(i)}\rangle\langle\psi_m^{(i)}|. \tag{51}$$

Now, by hypothesis, there exist channels $\mathcal{E}$ and $\mathcal{F}$ such that $Q_i=\mathcal{E}(P_i)$ and $P_i=\mathcal{F}(Q_i)$. Then, by linearity,

$$Q_i = \lambda_M(i) \mathcal{E}(|\phi_M^{(i)}\rangle\langle\phi_M^{(i)}|) + \lambda_m(i) \mathcal{E}(|\phi_m^{(i)}\rangle\langle\phi_m^{(i)}|). \tag{52}$$

We will now consider

$$\mathrm{Tr}[Q_i |\psi_M^{(i)}\rangle\langle\psi_M^{(i)}|] = \lambda_M(i) = \mathrm{Tr}[P_i \mathcal{E}^\mathsf{T}(|\psi_M^{(i)}\rangle\langle\psi_M^{(i)}|)], \tag{53}$$

and this clearly implies that $\mathcal{E}^\mathsf{T}(|\psi_M^{(i)}\rangle\langle\psi_M^{(i)}|)=|\phi_M^{(i)}\rangle\langle\phi_M^{(i)}|$. Analogous arguments lead to the conclusion that $\mathcal{E}^\mathsf{T}(|\psi_m^{(i)}\rangle\langle\psi_m^{(i)}|)=|\phi_m^{(i)}\rangle\langle\phi_m^{(i)}|$, and moreover $\mathcal{F}^\mathsf{T}(|\phi_M^{(i)}\rangle\langle\phi_M^{(i)}|)=|\psi_M^{(i)}\rangle\langle\psi_M^{(i)}|$ and $\mathcal{F}^\mathsf{T}(|\phi_m^{(i)}\rangle\langle\phi_m^{(i)}|)=|\psi_m^{(i)}\rangle\langle\psi_m^{(i)}|$. By collecting all the eigenstates of nondegenerate P_i's and Q_i's in two sets, namely,

$$\begin{aligned} \mathcal{E}^\mathsf{T}&:\{|\psi_M^{(i)}\rangle\langle\psi_M^{(i)}|, |\psi_m^{(i)}\rangle\langle\psi_m^{(i)}|\}_i \mapsto \{|\phi_M^{(i)}\rangle\langle\phi_M^{(i)}|, |\phi_m^{(i)}\rangle\langle\phi_m^{(i)}|\}_i \\ \mathcal{F}^\mathsf{T}&:\{|\phi_M^{(i)}\rangle\langle\phi_M^{(i)}|, |\phi_m^{(i)}\rangle\langle\phi_m^{(i)}|\}_i \mapsto \{|\psi_M^{(i)}\rangle\langle\psi_M^{(i)}|, |\psi_m^{(i)}\rangle\langle\psi_m^{(i)}|\}_i, \end{aligned} \tag{54}$$

and applying Theorem III.3 it follows that there exists a unitary U such that $Q_i=UP_iU^\dagger$ for all nontrivial Q_i's. Clearly, the same unitary transformation maps the trivial elements. ■

IX. PREPROCESSING: CLEANNESS FOR NUMBER OF OUTCOMES $n \leqslant d$

Lemma IX.1: For fixed number of elements $n \leqslant d$ *the POVM* $\mathbf{P}$ *is clean iff* $\lambda_M(P_i)=1$ *for all* i. *Such condition is also equivalent to* $\lambda_m(P_i)=0$ *with multiplicity at least* $n-1$, *and each vector which is eigenvector with unit eigenvalue for some element* P_j *must belong to the kernel of all other POVM elements*.

Proof: We first prove that the condition is also equivalent to $\lambda_m(P_i)=0$ for all i. Indeed, consider a normalized eigenvector $|u\rangle$ of P_j with eigenvalue 1, and suppose by absurd that some element P_i has $\lambda_m(P_i)>0$. Then

$$\langle u|u\rangle = \sum_k \langle u|P_k|u\rangle = \langle u|P_j|u\rangle + \langle u|P_i|u\rangle + \sum_{k\neq i,j} \langle u|P_k|u\rangle > 1, \tag{55}$$

and in order to have no contradiction one must have $\langle u|P_i|u\rangle = 0$, namely $\lambda_m(P_i) = 0$. Notice that Eq. (55) also implies that $\langle u|P_k|u\rangle = 0$ for all $k \neq j$, namely the same eigenvector $|u\rangle$ of P_j is eigenvector of all P_k for all $k \neq j$. Moreover, since there must be at least n vectors as $|u\rangle$, each being eigenvector of a different element P_j corresponding to unit eigenvalue, and since any two vectors must be orthogonal (since for some j they are eigenvectors corresponding to different eigenvalues of P_j), this means that the 0 eigenvalue for each POVM element must have multiplicity at least $n-1$, and all the eigenvectors of any element with eigenvalue 1 are in the kernel of the remaining elements.

We now prove that the condition is sufficient. Suppose that a POVM $\mathbf{Q}$ exists such that $\mathbf{Q} > \mathbf{P}$. Then by Lemma III.1 $\{0,1\} \subseteq \mathsf{Sp}(Q_i)$ for all i. We then need to prove that in this case $\mathbf{P} \simeq \mathbf{Q}$. From now on we will denote by $|u\rangle_i^P$ an eigenvector of P_i with eigenvalue 1 and by $|u\rangle_i^Q$ the same for Q_i. The proof is constructive: consider the map with Stinespring form $\mathcal{E}(X) = V^\dagger(I \otimes X)V$, where

$$V = \sum_i \sqrt{P_i} \otimes |u\rangle_i^Q, \tag{56}$$

and the notation $T = O \otimes |u\rangle$ denotes the operator defined as $T|\psi\rangle = O|\psi\rangle \otimes |u\rangle$ for all $|\psi\rangle \in \mathsf{H}$. It is clear that $\mathcal{E}(Q_i) = P_i$. Similarly, consider the map $\mathcal{F}(X) = W^\dagger(I \otimes X)W$, where

$$W = \sum_i \sqrt{Q_i} \otimes |u\rangle_i^P. \tag{57}$$

This is such that $\mathcal{F}(P_i) = Q_i$. We proved that POVMs $\mathbf{P}$ such that $\lambda_M(P_i) = 1$ for all i are clean. We will now prove that it is also a necessary condition. Consider indeed a generic POVM $\mathbf{Q}$ such that at least for one outcome j $\lambda_M(Q_j) < 1$. Then one can consider any POVM $\mathbf{P}$ with $\lambda_M(P_i) = 1$ for all i and construct the isometry

$$W = \sum_i \sqrt{Q_i} \otimes |u\rangle_i^P. \tag{58}$$

It is clear that the Stinespring form $W^\dagger(I \otimes X)W$ defines a channel $\mathcal{E}$ such that $\mathcal{E}(P_i) = Q_i$ for all i. Then $\mathbf{P} > \mathbf{Q}$. Moreover, by hypothesis $\lambda_M(P_j) > \lambda_M(Q_j)$ and then it is impossible that $\mathbf{P} \simeq \mathbf{Q}$. ■

An immediate corollary is the following

Corollary IX.2: The only clean elements with $n = d$ are the observables.

Proof: In Lemma IX.1 for $n = d$ the iff condition is equivalent to have eigenvalue 0 with multiplicity $d-1$ for each POVM element, namely each POVM element is rank one, and they are orthogonal. ■

Allowing mapping between POVMs with different number of outcomes, the situation simplifies:

Theorem IX.3: *For $n \leqslant d$ outcomes the set of clean POVMs coincides with the set of observables.*

Proof: Consider a generic POVM P_i with $i = 1, \ldots, n \leqslant d$. This can be always regarded as the preprocessing of any desired observable $\{|i\rangle\langle i|\}_{i=1,\ldots,d}$. In fact, using the isometry from H to $\mathsf{H}^{\otimes 2}$

$$V = \sum_{i=1}^{n} \sqrt{P_i} \otimes |i\rangle, \tag{59}$$

the following channel expressed in the Stinespring form

$$\mathcal{M}(X) = V^\dagger(I \otimes X)V \tag{60}$$

gives

$$\mathcal{M}(|i\rangle\langle i|) = P_i, \quad i = 1, \ldots d. \tag{61}$$

For a POVM with $n < d$ outcomes (strictly), notice that it is equivalent to a POVM with d outcomes and $d-n$ vanishing elements. On the other hand, for $n < d$ there is no channel that can increase the number of outcomes back to d, whence a POVM with $n < d$ outcomes cannot be clean. For $n = d$ Corollary IX.2 asserts that the only clean POVMs are the observables. ■

X. PREPROCESSING: ORDERING OF INFORMATIONALLY COMPLETE POVMs

Lemma X.1: If the POVM $\mathbf{Q}$ *is infocomplete then every* $\mathbf{P}$ *such that* $\mathbf{P} > \mathbf{Q}$ *is infocomplete, too.*

Proof: For d^2 outcomes POVMs, $\mathbf{P}$ and $\mathbf{Q}$ are infocomplete iff their elements are linearly independent. Suppose by absurd that there exists a nonnull vector of d^2 coefficients c_i such that $\Sigma_{i=1}^{d^2} c_i P_i = 0$, then also

$$\mathcal{E}\left(\sum_{i=1}^{d^2} c_i P_i\right) = 0 = \sum_{i=1}^{d^2} c_i Q_i = 0, \tag{62}$$

which contradicts the hypothesis.

If the number of outcomes is greater than d^2, suppose

$$\mathcal{E}(X) = 0, \tag{63}$$

for some $X \neq 0$, namely $\mathcal{E}$ would have non trivial kernel, in which case $\mathsf{Span}(\mathbf{Q}) \subseteq \mathsf{Rng}(\mathcal{E}) \subset \mathsf{B}(\mathsf{H})$, which contradicts the hypothesis that $\mathbf{Q} = \mathcal{E}(\mathbf{P})$ is infocomplete. Then $\mathcal{E}$ is invertible. Now, $\mathbf{P}$ must be infocomplete, otherwise the inverse of $\mathcal{E}$ would not have full rank, which is absurd. ■

The above theorem is immediately extended to any linearly independent POVM $\mathbf{Q}$. More interestingly, for any infocomplete POVM $\mathbf{P}$ one can prove the following lemma

Lemma X.2: If the POVM $\mathbf{P}$ *is infocomplete then every* $\mathbf{Q}$ *such that* $\mathbf{P} \simeq \mathbf{Q}$ *is infocomplete, too.*

Proof: It follows immediately from definition of $\simeq$ and Lemma X.1. ■

On the other hand, each POVM that is equivalent to an infocomplete one, is also unitarily equivalent to it, namely, more precisely

Theorem X.3: If $\mathbf{P}$ *is an infocomplete POVM, then* $\mathbf{P} \simeq \mathbf{Q}$ *iff* $\mathbf{P} \simeq_U \mathbf{Q}$.

Proof: Since the POVMs $\mathbf{P}$ and $\mathbf{Q}$ must be both infocomplete by the previous lemma, then the maps $\mathcal{E}$ and $\mathcal{F}$ are uniquely defined, and are the inverse of each other. Then, by Theorem III.2 $\mathcal{E}(X) = UXU^\dagger$ for some unitary U. ■

An alternative elegant proof of the above theorem would be the following.

Proof: By hypothesis, there exist $\mathcal{E}$ and $\mathcal{F}$ such that $\mathcal{E}(\mathbf{P}) = \mathbf{Q}$ and $\mathcal{F}(\mathbf{Q}) = \mathbf{P}$. This means that $\mathcal{F} \circ \mathcal{E}$ stabilizes the algebra generated by $\mathbf{P}$, that is $\mathsf{Span}(\mathbf{P}) \equiv \mathsf{B}(\mathsf{H})$. On the other hand, the commutant of an infocomplete POVM is only the identity, since $[P_i, X] = 0$ for all i implies $[A, X] = \Sigma_i a_i [P_i, X] = 0$ for all $A \in \mathsf{B}(\mathsf{H})$. This fact along with Lemma III.4 implies that $\mathcal{F} \circ \mathcal{E}$ is the identical map. The thesis is then a straightforward consequence of Theorem III.2. ■

Corollary X.4: For each non unitary invertible channel $\mathcal{E}$ *on* $\mathsf{B}(\mathsf{H})$ *there exists at least a pure state* $\psi \in \mathsf{H}$ *such that* $\mathcal{E}^{\mathsf{T}-1}(|\psi\rangle\langle\psi|) \ngeq 0$.

Proof: Let us consider an extremal POVM with d^2 rank-one elements $\{|\alpha_i\rangle\langle\alpha_i|\}$ $i = 1, \ldots, d^2$ (according to Ref. 15 such a POVM always exists for any dimension d, and it is necessarily informationally complete). Assuming $\mathcal{E}$ invertible, let's consider $Q_i = \mathcal{E}^{-1}(|\alpha_i\rangle\langle\alpha_i|)$. The POVM $|\alpha_i\rangle\langle\alpha_i|$ is clean since it is rank-one. However, since it is also infocomplete, then Q_i cannot be itself a POVM, otherwise according to Theorem X.3 it would be unitarily equivalent to $|\alpha_i\rangle\langle\alpha_i|$. Moreover, being both $|\alpha_i\rangle\langle\alpha_i|$ and Q_i infocomplete, the map $\mathcal{E}$ would be univocally defined, whence itself unitary, contrarily to the hypothesis. Then, $\{Q_i\}$ is not a POVM. However, since the map $\mathcal{E}$ is a channel, whence $\mathcal{E}^{-1}$ must be identity preserving, one has $\Sigma_i Q_i = I$, then necessarily at least one element Q_j cannot be positive, namely there exists a vector $\psi \in \mathsf{H}$ for which

$$\langle\psi|Q_j|\psi\rangle < 0. \tag{64}$$

This inequality can be rewritten as follows:

$$\mathrm{Tr}[|\psi\rangle\langle\psi|\mathcal{E}^{-1}(|\alpha_j\rangle\langle\alpha_j|)] = \mathrm{Tr}[\mathcal{E}^{\mathsf{T}-1}(|\psi\rangle\langle\psi|)|\alpha_j\rangle\langle\alpha_j|] < 0, \tag{65}$$

namely $\mathcal{E}^{\mathsf{T}-1}(|\psi\rangle\langle\psi|)$ is not positive. ■

We have also the following interesting theorem.

Theorem X.5: *Every channel $\mathcal{F}$ which maps the set of states S surjectively on itself, i. e. such that $\mathcal{F}(\mathsf{S})\equiv\mathsf{S}$, is necessarily unitary.*

Proof: First, suppose that $\mathcal{F}$ is invertible, then $\mathcal{F}$ must be unitary, otherwise $\mathcal{F}^{-1}(\mathsf{S})=\mathsf{S}$ would not be possible by Lemma X.4. On the other hand, if $\mathcal{F}$ is not invertible, then its range must have dimension strictly smaller than d^2. Now, consider a rank-one infocomplete POVM $\mathbf{P}$ with $|\mathbf{P}|=d^2$. Clearly, some POVM element cannot belong to $\mathcal{F}(\mathsf{S})$, and this proves that $\mathcal{F}(\mathsf{S})\subset\mathsf{S}$ strictly, since such normalized POVM elements are just pure states. ■

For qubits this theorem has the simple geometric interpretation that the Bloch sphere transformed under $\mathcal{F}^{-1}$ for any invertible non unitary $\mathcal{F}$ becomes an ellipsoid which contains elements outside the Bloch sphere.

By definition, and according to Theorem X.3 an infocomplete POVM $\mathbf{P}$ is clean iff $\mathcal{E}^{-1}(\mathbf{P})$ is not a POVM for all invertible non unitary maps $\mathcal{E}$. This means that as soon as the set S of states is transformed by $\mathcal{E}^{-1}$, the POVM is able to detect at least one of the points in $\mathcal{E}^{-1}(\mathsf{S})-\mathsf{S}$, say $\mathcal{E}^{-1}(|\psi\rangle\langle\psi|)$, since the "probability distribution" corresponding to $\mathcal{E}^{-1}(|\psi\rangle\langle\psi|)$ is no longer positive.

XI. PREPROCESSING: ORDERING OF RANK-ONE POVMs

Intuitively one thinks that a rank-one POVM is clean. This is actually true, and it is more precisely stated by theorem XI.2 in this section. In order to prove it, we first need the following

Lemma XI.1: If the POVM $\mathbf{Q}$ is rank-one (i.e. each element Q_i can be written as $Q_i=|w_i\rangle\langle w_i|$), then for any POVM $\mathbf{P}$ such that $\mathbf{P}\succ\mathbf{Q}$, also $\mathbf{P}$ is rank one, and $\mathrm{Tr}[P_i]=\mathrm{Tr}[Q_i]$, $\forall i$.

Proof: Consider the following normalized vectors

$$|\widetilde{w}_i\rangle = \frac{1}{\sqrt{N_i}}|w_i\rangle, \quad Q_i = N_i|\widetilde{w}_i\rangle\langle\widetilde{w}_i|, \tag{66}$$

where $N_i=\mathrm{Tr}[Q_i]=\|w_i\|^2$, whence $\Sigma_i N_i=d$. Suppose $\mathbf{P}\succ\mathbf{Q}$, and $\mathcal{E}(\mathbf{P})=\mathbf{Q}$. Then one can easily verify the following identity:

$$N_i = \mathrm{Tr}[Q_i|\widetilde{w}_i\rangle\langle\widetilde{w}_i|] = \mathrm{Tr}[\mathcal{E}(P_i)|\widetilde{w}_i\rangle\langle\widetilde{w}_i|] = \mathrm{Tr}[P_i\mathcal{E}^{\mathsf{T}}(|\widetilde{w}_i\rangle\langle\widetilde{w}_i|)]. \tag{67}$$

Now, by the CPT property of $\mathcal{E}^{\mathsf{T}}$, $\mathcal{E}^{\mathsf{T}}(|\widetilde{w}_i\rangle\langle\widetilde{w}_i|)$ is a state and clearly the last expression in Eq. (67) is less than or equal to the maximum eigenvalue $\lambda_M(P_i)$ of P_i. We have than the following situation:

$$N_i \leqslant \lambda_M(P_i) \leqslant \mathrm{Tr}[P_i]. \tag{68}$$

By the normalization and positivity of POVMs, we have that $d=\Sigma_i N_i=\Sigma_i\mathrm{Tr}[P_i]$ and $N_i\geqslant 0$, $\mathrm{Tr}[P_i]\geqslant 0$. These conditions along with Eq. (68) imply

$$N_i \equiv \mathrm{Tr}[P_i] \ \ \forall\, i, \tag{69}$$

and this in turn implies $\lambda_M(P_i)=\mathrm{Tr}[P_i]$, namely P_i is rank one. ■

We will now prove the following theorem.

Theorem XI.2: *If $\mathbf{Q}$ is rank one, then $\mathbf{P}\succ\mathbf{Q}$ iff $\mathbf{P}\simeq_U\mathbf{Q}$. Namely, rank-one POVMs are clean.*

Proof: First, notice that by Lemma XI.1, $\mathbf{P}\succ\mathbf{Q}$ implies that $\mathbf{P}$ is rank one with $\mathrm{Tr}[P_i]=\mathrm{Tr}[Q_i]$, for all i. We have then

$$P_i = |v_i\rangle\langle v_i| = M_i|\widetilde{v}_i\rangle\langle\widetilde{v}_i|, \quad \|\widetilde{v}_i\| = 1, \tag{70}$$

$$Q_i = |w_i\rangle\langle w_i| = M_i|\tilde{w}_i\rangle\langle\tilde{w}_i|, \quad \|\tilde{w}_i\| = 1, \tag{71}$$

where $M_i \equiv \mathrm{Tr}[P_i] = \mathrm{Tr}[Q_i]$, consistently with Lemma XI.1. Now, by hypothesis we have

$$M_i = \mathrm{Tr}[\mathcal{E}(P_i)|\tilde{w}_i\rangle\langle\tilde{w}_i|] = \mathrm{Tr}[P_i\mathcal{E}^{\mathsf{T}}(|\tilde{w}_i\rangle\langle\tilde{w}_i|)] = M_i\,\mathrm{Tr}[|\tilde{v}_i\rangle\langle\tilde{v}_i|\mathcal{E}^{\mathsf{T}}(|\tilde{w}_i\rangle\langle\tilde{w}_i|)]. \tag{72}$$

As a consequence, necessarily $\mathrm{Tr}[|\tilde{v}_i\rangle\langle\tilde{v}_i|\mathcal{E}^{\mathsf{T}}(|\tilde{w}_i\rangle\langle\tilde{w}_i|)] = 1$, and by CPT property of $\mathcal{E}^{\mathsf{T}}$ this implies $\mathcal{E}^{\mathsf{T}}(|\tilde{w}_i\rangle\langle\tilde{w}_i|) \equiv |\tilde{v}_i\rangle\langle\tilde{v}_i|$. Notice that since $\mathcal{E}^{\mathsf{T}}(I) = \Sigma_i M_i \mathcal{E}^{\mathsf{T}}(|\tilde{w}_i\rangle\langle\tilde{w}_i|) = \Sigma_i M_i|\tilde{v}_i\rangle\langle\tilde{v}_i| = I$, then $\mathcal{E}^{\mathsf{T}}$ and $\mathcal{E}$ are unital, namely both CPT and CPI. Then, by applying Theorem III.3 one has $\mathbf{P} \simeq_U \mathbf{Q}$. The converse is trivial. ■

XII. CONCLUSIONS

In this paper we have introduced the notion of *clean* POVMs, namely which are not irreversibly connected to another POVM via a quantum channel. We used the adjective clean for such POVMs in the sense that they are not affected by extrinsical quantum noise from the action of a channel which is in principle avoidable. We have seen that, quite unexpectedly, the *cleanness* property is largely unrelated to the convex structure of POVMs, and there are clean POVMs that are not extremal and extremal POVMs that are not clean.

The classification problem of POVMs cleanness turned out to be much harder than that of their extremality, and in this paper we gave a complete classification of clean POVMs only for number n of outcomes $n \leqslant d$ (d dimension of the Hilbert space), whereas for $n > d$ we gave a set of either necessary or sufficient conditions, and an iff condition for the case of informationally complete POVMs for $n = d^2$. The difficulty for classifying the case $n > d$ reflects analogous difficulties in the theory of quantum measurements in assessing the maximal POVM cardinality needed to attain the accessible information, cardinality whose lower bound has been shown to be actually larger than d.[18,19]

The novel issue of clean POVMs naturally opens new problems in the theory of quantum information and quantum measurements. Besides the problem of the general classification of cleanness, it raises the problem of characterizing all POVMs achievable from a given one via a quantum channel, or, reversely, of all POVMs which can be evolved toward a given one via a quantum channel. These are only initial steps toward a thorough analysis of the general problem of the partial ordering induced by channels on the convex set of measurements, an issue which is not an academic mathematical problem, but which is relevant for engineering new quantum measurements with minimal available resources.

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