

Lower bounds on phase sensitivity in ideal and feasible measurements

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The detection of the phase shift between a single mode of the em field and a local reference oscillator is analyzed in the proper framework of quantum estimation theory. Such a fully quantum treatment clarifies the meaning of the "operational approach" suggested by Noh, Fougères, and Mandel [Phys. Rev. A **45**, 424 (1992)], in which different measurement schemes correspond to different phase operators. We show that the phase shift is actually measured in the form of the polar angle between two real output photocurrents, namely through a joint measurement of two conjugated quadratures of the field. This scheme is the only feasible one for detecting the quantum phase, and it is equivalently performed by either heterodyne detection or double-homodyne detection of the field. On the contrary, the customary homodyne detection (and, generally, any kind of measurement of a single phase-dependent variable) is more properly a zero-point phase measurement. As a definition of sensitivity we consider the output rms noise of the detection scheme, the only relevant one for actual experiments, in contrast with many other different notions currently adopted in the literature. We show that the r.m.s. phase sensitivity versus the average photon number \bar{n} is bounded by the ideal limit $\Delta\phi \sim \bar{n}^{-1}$, whereas for the feasible schemes the bound is $\Delta\phi \sim \bar{n}^{-2/3}$, in between the shot-noise level $\Delta\phi \sim \bar{n}^{-1/2}$ and the ideal bound. The latter can actually be achieved by single homodyne detection of suitable squeezed states, but only in the neighborhood of a fixed zero-phase working point. The phase sensitivity bound $\Delta\phi \sim \bar{n}^{-2/3}$ can be reached using coherent states with only 2% of squeezing photons, in contrast with the homodyne-detection bound $\Delta\phi \sim \bar{n}^{-1}$ which is reached with 50%. The uncertainty product of two conjugated phase quadratures largely exceeds the Heisenberg limit, even for the ideal joint measurement. We also show that no sizeable improvement in sensitivity is found for detection schemes which involve wideband states, at least for the case of nonentangled states. At the end of the paper we give an extensive table of asymptotic sensitivities at large \bar{n} for both ideal and feasible schemes.

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I. INTRODUCTION

The principle of any interferometric detection lies in the possibility of monitoring very small changes in some environmental parameter through an induced phase shift of the em field. The back-action effect on the measured parameter—which is due to the unavoidable radiation pressure—poses the problem of optimizing the phase sensitivity $\Delta\phi$ as a function of the radiation intensity \bar{n} . The problem of optimizing $\Delta\phi$ versus \bar{n} has stimulated many discussions in the literature [1–9], and the debate has recently risen again with renewed interest [10–13] after the seminal papers of Noh, Fougères, and Mandel [14].

In the quantum limit of very weak em fields the problem of detecting the phase shifts becomes threefold. On one hand, there is no Hermitian operator representing the phase, nor is the phase itself conjugated to the photon number, as in the original heuristic approach of Dirac [15]. On the other hand, one can define many Hermitian operators which are functions of the phase itself, but which apparently violate the usual trigonometric calculus [16–18]. And finally, as pointed out in Ref. [14], the detection scheme itself is involved in the definition of the actually measured phase operator.

Among the numerous attempts [19–27] which have been made after Dirac in order to introduce a dynamical variable for the phase, the limiting procedure of Pegg and Barnett [20] has become the most popular technique,

because it allows the evaluation of expected values with very simple and reliable rules. However, despite its simplicity and effectiveness as a mathematical tool, this approach has no obvious physical interpretation, and leaves most of the conceptual problems of phase detection still open.

It has been realized by some authors [4, 26, 28, 29] that the most appropriate approach to the phase of the field is the "quantum estimation theory" of Helstrom [30]. Even though it is easy to show that this method is equivalent in the end to the Pegg and Barnett procedure (see Ref. [31] for more details), nevertheless it provides a physically meaningful scheme for the phase measurement where all conceptual problems disappear.

Despite the fact that it has long been recognized as the most natural framework for analyzing any kind of quantum detection, the quantum estimation theory has not gained the necessary popularity yet, perhaps due to the fact that its main ingredient—the probability operator measure (POM)—is generally a nonorthogonal spectral decomposition, and thus appears to be in conflict with the conventional dictum of quantum mechanics that only "observables"—i.e., orthogonal POM's—can be measured. This point has been well clarified in some papers [31, 32], where it is shown that nonorthogonal POM's correspond to actual observables on a larger Hilbert space which includes also modes pertaining to the measuring apparatus (all together referred to as the "probe"). It

is clear that this assertion provides the proper quantum setting for the operational point of view of Ref. [14], where the dependence of the measured operator on the detection scheme just corresponds to the involvement of the probe variables in the measuring process, an involvement which becomes unavoidable when the phase of the field is detected.

In this paper we again analyze the problem of optimizing the phase measurement using the quantum estimation theory approach of Ref. [4]. However, differently from Refs. [4, 6, 8, 29], we adopt the usual output rms noise as a measure of the phase sensitivity, because the “reciprocal peak likelihood” of Ref. [4] has been proven to be not a good measure of phase sensitivity [33]. We analyze both ideal and feasible measurements of the phase. The ideal measurement is just one of the main results of the quantum estimation theory itself: here the quantum states optimizing the rms sensitivity are given. No viable method for experimentally implementing such ideal measurement has, however, been envisaged yet. As regards the feasible schemes of phase detection, here we distinguish between two main different classes: the genuine phase-detection schemes and the measurements of a single phase-dependent observable. In the former class, the phase shift of the field is related to the polar angle between two measured photocurrents which, in turn, correspond to two conjugated quadratures of the field. Such a scheme is the only viable one for phase detection, and corresponds equivalently to either heterodyne detection or double-homodyne detection of the field. This also clarifies the subtle nature of the phase itself which, despite being a single real parameter, nonetheless requires a joint measure of two conjugated operators. In contrast, in the second class of measurements, only a single observable is detected—typically, during homodyne detection of a single quadrature of the field. Here we want to emphasize that a single-quadrature measurement cannot be used to infer the value of the phase, because the knowledge of a quadrature would require an additional measurement on the field—essentially its intensity—which unavoidably would destroy the information on the phase. Thus, the single-homodyne-detection scheme can be used only as a zero-phase monitoring technique, which, however, is the essential of a typical interferometric measurement. As is shown in this paper, the requirement of detecting the whole phase probability distribution makes the measurement less sensitive than in the case of a zero-point detection. In fact, the phase sensitivity for a double-quadrature detector turns out to be bounded by $\Delta\phi \sim \bar{n}^{-2/3}$, but for the single-homodyne-detection scheme by $\Delta\phi \sim \bar{n}^{-1}$. Moreover the states achieving the two bounds are dramatically different: they are weakly squeezed (about 2% of squeezing photons) for the double-quadrature measurement, whereas they become strongly squeezed (50%) for one-quadrature detection.

In Sec. II we briefly recall the quantum estimation theory of the phase. We show that such an approach naturally comes from a conventional quantum treatment of the apparatus, where the output-photocurrent probability distributions—and hence their relative polar angle distribution—are evaluated in terms of the input quan-

tum state of radiation which carries the phase information (the *system*), along with the states of the other modes involved in the apparatus (the *probe*). The main ingredient of this approach is the above-mentioned POM which is a suitable probability operator describing the detection scheme in a way independent of the system state (i.e., it gives the output phase probability for any given input state). The optimization of such POM over all possible apparatus leads to the ideal measurement of the phase.

Section III is devoted to a critical discussion of the problem of defining a proper phase sensitivity, comparing the different definitions introduced in the literature. The circular topology of the phase has led some authors to define the sensitivity in a way different from the customary rms, but we stress that only the rms noise represents the actual sensitivity of a real measurement.

In Sec. IV the double-quadrature detection schemes are analyzed. We show the equivalence between heterodyne and double-homodyne detection, and we give a detailed quantum description of the latter, also providing a computer simulation of the experimental procedure. It is shown that an analysis for wideband states does not lead to substantial differences with respect to the narrowband case, at least for nonentangled states.

In Sec. V both ideal and feasible measurements of a single phase-dependent variable are analyzed, including the case of *zero-point* measurements. We compare these single-quadrature schemes with the double-quadrature ones and we conclude that the latter lead to more reliable results than the former, as unphysical quantum statistics are avoided and no violation of the trigonometric calculus occurs for expectation values.

In Sec. VI lower bounds to phase sensitivity of both ideal and feasible detection are evaluated. A table of asymptotical bounds in the limit of large photon numbers \bar{n} is given, for various schemes and states.

Section VII closes the paper with some concluding remarks.

II. IDEAL QUANTUM PHASE DETECTION

The quantum-classical correspondence in the harmonic oscillator description of a single field mode fails when a quantum analog of the phase is considered. Despite the fact that the classical Poisson brackets seem to suggest an operator $\hat{\phi}$ which is canonically conjugated to the number \hat{n} [15], the resulting commutation relation turns out to be ill defined, due to discreteness of the spectrum of \hat{n} . Also a definition of the phase $\hat{\phi}$ via a polar decomposition of the annihilator $e^{i\hat{\phi}} = (\hat{n} + 1)^{-1/2} a$ does not lead to a unitary phase factor [16], due to the lower bound of the \hat{n} spectrum [16,34]. However, the absence of a phase observable does not prevent a quantum description of the phase statistics, as it is just sufficient to give a rule connecting the density matrix of radiation with the probability distribution of the phase: this is exactly the point of view adopted in the quantum estimation theory of Helstrom [30].

The main ingredient of the quantum estimation theory is represented by the mathematical concept of *probability-*

operator-valued measure (POM) on the Hilbert space \mathcal{H}_S of the system, which extends the conventional description by observables. Using a notation which is familiar to physicists—even though not strictly legitimate from the mathematical point of view [31]—given a vector \mathbf{z} of real parameters to be measured, a POM $d\hat{\lambda}(\mathbf{z})$ is a self-adjoint measure with the following properties:

$$d\hat{\lambda}(\mathbf{z}) \geq 0, \quad \int_Z d\hat{\lambda}(\mathbf{z}) = \hat{1}, \quad (1)$$

where Z denotes the space of \mathbf{z} . From Eqs. (1) it follows that the density-matrix state $\hat{\rho}$ of the system can be connected to the probability distribution of \mathbf{z} according to the following rule:

$$dP(\mathbf{z}) = \text{tr}\{\hat{\rho}d\hat{\lambda}(\mathbf{z})\}. \quad (2)$$

The POM $d\hat{\lambda}(\mathbf{z})$ itself depends on the particular experimental setup, and different probability distributions $dP(\mathbf{z})$ result from different detectors with the system state $\hat{\rho}$ being fixed. Using the POM in Eqs. (1) a set of self-adjoint (generally noncommuting) operators $\hat{\Lambda}$ can be defined as follows

$$\hat{\Lambda} = \int_Z \mathbf{z} d\hat{\lambda}(\mathbf{z}). \quad (3)$$

More generally, one can define operator functions $\widehat{f(\Lambda)}$ of the form

$$\widehat{f(\Lambda)} = \int_Z f(\mathbf{z}) d\hat{\lambda}(\mathbf{z}). \quad (4)$$

For orthogonal $d\hat{\lambda}(\mathbf{z})$ the present description corresponds to a measure of the observables $\hat{\Lambda}$ in the usual sense, and the function calculus holds

$$\widehat{f(\Lambda)} = f(\hat{\Lambda}). \quad (5)$$

Relation (5), however, no longer holds true for nonorthogonal POM's. In this case the self-adjoint operators $\hat{\Lambda}$ only provide the correct expectation value of \mathbf{z} , whereas the higher moments of the probability distribution are not given by $\langle \hat{\Lambda}^n \rangle$, but must be evaluated through the original recipe (2).

The present scheme is not in conflict with the basic assertion of quantum mechanics that “only observables can be measured.” Despite the fact that the POM's generally describe measurements that do not correspond to observables in the usual sense, nonetheless the Naimark theorem [31, 32] assures that every POM can be also obtained as a partial trace of a customary orthogonal projection-valued measure on a larger Hilbert space which, itself, represents the original system interacting with an appropriate apparatus. Thus, a nonorthogonal POM essentially describes generalized observables, whose definition in the usual sense would unavoidably involve the measuring apparatus itself. Upon denoting by $|\psi(\mathbf{z})\rangle \in \mathcal{H}_S \otimes \mathcal{H}_P$ a complete orthonormal set of eigenvectors for commuting self-adjoint operators acting on the enlarged system-probe space, the probability distribution

$$dP(\mathbf{z}) = \text{tr}\{\hat{\rho}_S \otimes \hat{\rho}_P |\psi(\mathbf{z})\rangle \langle \psi(\mathbf{z})|\} \quad (6)$$

$$= \text{tr}_S \{\hat{\rho}_S \text{tr}_P [\hat{\rho}_P |\psi(\mathbf{z})\rangle \langle \psi(\mathbf{z})|]\}, \quad (7)$$

corresponds to the following POM on the system space \mathcal{H}_S ,

$$d\hat{\lambda}(\mathbf{z}) = \text{tr}_P \{\hat{\rho}_P |\psi(\mathbf{z})\rangle \langle \psi(\mathbf{z})|\} \equiv \langle \psi(\mathbf{z}) | \hat{\rho}_P | \psi(\mathbf{z}) \rangle. \quad (8)$$

For the case of the oscillator phase, the above analysis starts with the following definition of the measured random parameter ϕ : given a reference state $|\psi\rangle \in \mathcal{H}_S$, which is modulated by a phase shifter with unknown shift $\phi \in [-\pi, \pi)$, the outgoing state is given by $|\psi_\phi\rangle = e^{i\phi\hat{n}}|\psi\rangle$. From this definition the quantum estimation theory optimizes the POM at a purely abstract level in order to obtain the ideal—i.e., the least noisy—measurement. For a generic POM $d\hat{\mu}(\phi)$ which represents a particular apparatus, the following quantity,

$$dP(\phi, \bar{\phi}) = p(\phi, \bar{\phi}) d\bar{\phi} = \frac{1}{2\pi} \langle \psi_\phi | d\hat{\mu}(\bar{\phi}) | \psi_\phi \rangle, \quad (9)$$

gives the conditional probability distribution of obtaining the outcome $\bar{\phi}$ when the true value is ϕ . The optimization procedure depends on the choice of an appropriate error-cost function $W(\phi - \bar{\phi})$ —which weights the penalty for the error $\phi \neq \bar{\phi}$ —and the optimum apparatus corresponds to the absolute minimum of the average cost

$$\bar{C} = \int dP(\phi, \bar{\phi}) W(\phi - \bar{\phi}). \quad (10)$$

The optimization problem (10) has been solved for a large class of error-cost functions [30], as, for example

$$W(\phi) = 4 \sin^2 \frac{\phi}{2} \quad \text{or} \quad W(\phi) = -\delta_{2\pi}(\phi). \quad (11)$$

The first function in Eq. (11) approximates a Gaussian distribution of estimates for a 2π -periodic variable; the second corresponds to the *maximum likelihood strategy*, and $\delta_{2\pi}(\phi)$ is the periodic δ function. The optimum POM for both error-cost functions is

$$d\hat{\mu}(\phi) = \frac{d\phi}{2\pi} |e^{i\phi}\rangle \langle e^{i\phi}|, \quad (12)$$

where $|e^{i\phi}\rangle$ are the Susskind-Glogower phase states [16]

$$|e^{i\phi}\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle. \quad (13)$$

Equation (12) provides the POM of an ideal phase measurement. Some examples of commuting pairs of self-adjoint operators have been proposed in Refs. [27] and [35], achieving the optimum POM (12) on a system-probe Hilbert space; however, no viable method for experimentally implementing a corresponding setup has been devised yet, and hence the POM (12) only represents an ideal limit.

Corresponding to the optimum POM (12) one can define a self-adjoint phase operator

$$\hat{\phi} = \int_{-\pi}^{\pi} \phi d\hat{\mu}(\phi) = -i \sum_{n \neq m} (-1)^{n-m} \frac{1}{n-m} |n\rangle \langle m| \quad (14)$$

along with the *squared phase operator*

$$\begin{aligned}\widehat{\phi^2} &= \int_{-\pi}^{\pi} \phi^2 d\hat{\mu}(\phi) \\ &= \frac{\pi^2}{3} + 2 \sum_{n \neq m} (-1)^{n-m} \frac{1}{(n-m)^2} |n\rangle\langle m|, \quad (15)\end{aligned}$$

which produces the correct rms noise. As previously noted, one has that

$$\widehat{\phi^2} \neq \widehat{\phi}^2, \quad (16)$$

and more generally $f(\widehat{\phi}) \neq f(\widehat{\phi})$. The failure of operator function calculus is a consequence of the nonorthogonality of the Susskind-Glogower states $|e^{i\phi}\rangle$, and physically corresponds to the impossibility of defining the phase as a conventional observable without any reference to the measuring apparatus.

The optimum POM (12) describes the ideal phase detection: for actual detection schemes one has different (suboptimal) POM's and corresponding different definitions for the operators in Eqs. (14) and (15), depending on the scheme itself. This assertion clarifies and formalizes the operational approach of Noh, Fougères, and Mandel in Refs. [14] and [36] where it has been clearly recognized that the definition of the phase itself cannot be divorced from the measurement process that is used to determine it. In Sec. IV we examine a feasible detection scheme, giving the explicit form of both the POM (1) and the operators (14) and (15).

III. SENSITIVITY OF QUANTUM PHASE DETECTION

The sensitivity of a measurement of a parameter $x \in \mathbf{R}$ is usually defined as the rms of the experimental probability distribution $dP(x)$, namely

$$\overline{\Delta x^2} = \int_{\mathbf{R}} dP(x) x^2 - \left(\int_{\mathbf{R}} dP(x) x \right)^2. \quad (17)$$

The phase variable ϕ is defined in the bounded domain $[-\pi, \pi)$ with 2π periodicity, and this peculiarity has led many authors to the conclusion that the rms of the phase is not the appropriate quantity to be considered as sensitivity, claiming that it is not invariant under phase translations $\phi \rightarrow \phi + \chi$. As a consequence, different definitions of phase sensitivity have been adopted, which became essentially the same for nearly Gaussian distributions. The relevance of this subject needs a critical revision of the most commonly adopted definitions of phase sensitivity: we will conclude that the customary rms noise itself is the correct sensitivity to be considered.

A. Phase dispersion D

Dispersion D [9] is defined as follows

$$D \equiv (1 - \langle \widehat{\cos \phi} \rangle^2 - \langle \widehat{\sin \phi} \rangle^2)^{\frac{1}{2}} = 1 - \left| \sum_{n=0}^{\infty} c_{n+1}^* c_n \right|^2, \quad (18)$$

where c_n are the coefficients of the number representation of the state and the sine and cosine operators are defined according to Eq. (4) as follows

$$\widehat{\cos \phi} = \int_{-\pi}^{\pi} d\hat{\mu}(\phi) \cos \phi, \quad (19)$$

$$\widehat{\sin \phi} = \int_{-\pi}^{\pi} d\hat{\mu}(\phi) \sin \phi, \quad (20)$$

and coincide with the sine and cosine operators of Susskind and Glogower [16]. The definition (18) follows from elementary error-propagation calculus, the phase ϕ being regarded as a function of the two “independent variables” $\sin \phi$ and $\cos \phi$ as follows

$$\phi = -i \ln(\cos \phi + i \sin \phi). \quad (21)$$

In Eq. (21) the correct logarithm branch is selected in order to obtain the desired domain for ϕ . Apart from the minor point that Eq. (18) would lead to dispersion $D = 1$ for constant distributions—instead of $\overline{\Delta \phi^2} = \pi^2/3$ —the main criticism is that $\sin \phi$ and $\cos \phi$ cannot be considered as independent variables, because they correspond to a noncommuting pair of operators which are jointly measured when detecting ϕ .

B. Reciprocal peak likelihood $\delta\phi$

The peak likelihood $p(\phi|\phi)$ [see Eq. (9)] is the maximum height of the probability distribution. Its inverse, namely

$$\delta\phi = \frac{1}{p(\phi|\phi)} = 2\pi \left(\sum_{n=0}^{\infty} |c_n|^2 \right)^{-2}, \quad (22)$$

has been introduced in Ref. [4] as a measure of the width of

the distribution, coherently with the maximum-likelihood strategy used in the quantum estimation theory. Here, the following criticisms are in order: (i) $\delta\phi$ is a *local* criterion, namely it checks only one point of the distribution, whereas there is no control on the global behavior as, for example, on the eventual occurrence of high tails. The most degenerate situation occurs when the tails are so high that the distribution itself converges to $P(\phi) = 1/2\pi$, apart from one point with infinite probability density and zero integral [37, 38], thus leading to vanishing $\delta\phi$ instead of $\delta\phi = \pi^2/3$; (ii) the coherence of this sensitivity definition with the maximum-likelihood strategy [4] cannot be considered as a valid argument, in view of the aforementioned equivalence between the likelihood strategy and the (quasi)-Gaussian one; (iii) recent numerical results [33] have shown that the simulated sensitivity does not actually correspond to $\delta\phi$.

C. POM root mean square $\langle \widehat{\Delta \phi^2} \rangle$

Given a physical apparatus (or an ideal detector) one has a corresponding POM and, in turn, a probability distribution $dP(\phi)$ according to Eq. (2). Such a probability has a rms error (17) given by

$$\langle \widehat{\Delta\phi^2} \rangle = \langle \widehat{\phi^2} \rangle - \langle \widehat{\phi} \rangle^2. \quad (23)$$

Here $\langle \dots \rangle$ denotes the ensemble quantum average on the system space \mathcal{H}_S , and the operators $\widehat{\phi}$ and $\widehat{\phi^2}$ depend on the considered POM [for the optimum POM they are given in Eqs. (14) and (15)]. Notice that there is no ambiguity in choosing between the two operators $\widehat{\phi^2} \neq \widehat{\phi}^2$, because Eq. (23) directly follows from the probability (2). For a random-phase state—namely a constant probability distribution—one correctly has $\langle \widehat{\Delta\phi^2} \rangle = \pi^2/3$.

As regards the problem of invariance under phase shifts, here we stress that this actually is not a problem. In fact, the only concern is the correspondence between experimental and theoretical quantities, and the circular topology of the phase arises at both experimental and theoretical levels in the same way. Whatever procedure is considered for measuring the phase, the information on it has always to be inferred from a joint sine-cosine measurement, and hence the experimental equipment itself has to be tuned on a selected 2π window. Once the domain is fixed, the experimental noise is, by definition, the rms noise on such a domain. Therefore, different choices of the 2π window actually lead to different experimented amounts of noise, and also theoretically the rms noise has to be evaluated on the chosen domain (hereafter we will always use the $[-\pi, \pi)$ window).

IV. GENUINE PHASE DETECTION: THE DOUBLE-QUADRATURE SCHEMES

In this section we analyze in detail the class of feasible phase-detection schemes based on joint measurement of two conjugated quadratures of the field. After evaluating the POM of the double-homodyne-detection scheme suggested in Refs. [14,39] we show the equivalence with heterodyne detection. We also give an analysis of the double-homodyne detection for wideband states, showing that no substantial difference is found in the end with respect to the one-mode case, at least for nonentangled states.

A. Double-homodyne detection

Double-balanced-homodyne detection provides a way for simultaneously measuring a pair of conjugated field quadratures for one mode of the em field. The schematic diagram of the detector setup is reported in Fig. 1. There are four 50-50 beam splitters and four photodetectors, whereas a $\pi/2$ phase shifter is inserted in one arm. The mode carrying the measured phase is a , and a stable reference for the phase is provided by the local oscillator (LO) which is synchronous with a and is prepared in a highly excited coherent state $|z\rangle$.

The double-homodyne scheme can be used to perform a phase measurement: however, as we will see, it leads to a probability distribution which is more noisy than the ideal one, as a consequence of a sort of added instrumental noise. The phase distribution is obtained through the following procedure. Each experimental event consists of a simultaneous detection of the two commuting dif-

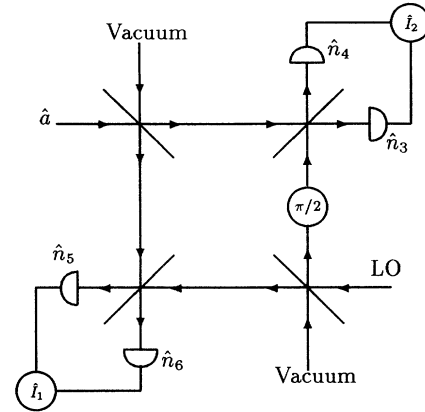


FIG. 1. Outline of the double-balanced-homodyne detector.

ference photocurrents $\hat{I}_1 = \hat{n}_6 - \hat{n}_5$ and $\hat{I}_2 = \hat{n}_4 - \hat{n}_3$, which trace a pair of field quadratures. Each event corresponds to a plotted point in the complex plane representing the field amplitude, and the phase value inferred from the event is the polar angle of the point itself. The experimental histogram of the phase distributions is obtained upon dividing the plane into “infinitesimal” angular bins of equal width $\delta\phi$, from $-\pi$ to π , and then counting the number of points which fall into each bin. In formulas, the statistical frequency P_n for the n th bin $\theta_n \equiv [-\pi + n\delta\phi, -\pi + (n+1)\delta\phi)$ is defined as follows

$$P_n = \frac{1}{N} \left\{ \begin{array}{l} \text{number of events with } I_1 = \rho \cos \phi, \\ I_2 = \rho \sin \phi, \phi \in \theta_n \end{array} \right\}, \quad (24)$$

where $\rho = \sqrt{I_1^2 + I_2^2}$ and N is the total number of events.

In Fig. 2, as an example, a computer simulation of the above experimental procedure is illustrated for a squeezed state with equal number $\bar{n}_s = \bar{n}_{sq} = 10$ of signal and squeezing photons. The simulated experimental histogram (10^4 events) is compared with the theoretical probability obtained from the POM of the detector. The POM of the detector is derived in the following.

The difference photocurrents \hat{I}_1 and \hat{I}_2 are commut-

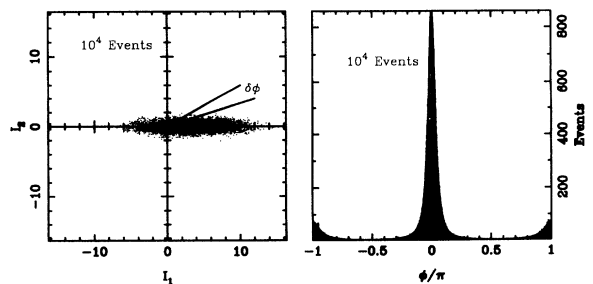


FIG. 2. Computer simulation of a double-quadrature experiment for a squeezed state with equal number $\bar{n} = 10$ of signal and squeezing photons. The histogram (10^4 events) is compared with the theoretical result from the POM (39).

ing operators with factorized probability $P(I_1, I_2) = P(I_1)P(I_2)$. The probability distribution for each photocurrent can be evaluated as the Fourier transform of the moment-generating function. Introducing the *reduced current* $\hat{\mathcal{I}} = \hat{I}/|z|$ (we recall that $|z\rangle$ is the coherent state of the LO) one has

$$P(\mathcal{I}) = \int_{-\pi|z|}^{\pi|z|} \frac{d\lambda}{2\pi} \text{tr}\{\hat{\rho} e^{i\lambda(\hat{\mathcal{I}} - \mathcal{I})}\}, \quad (25)$$

where $\hat{\rho}$ denotes the density-matrix state of the photocurrent modes. The phase distribution is the marginal probability integrated over the modulus ρ defined in Eq. (24), namely

$$P(\phi) = \int_0^\infty \rho d\rho P_1(\rho \cos \phi) P_2(\rho \sin \phi). \quad (26)$$

Using Eq. (25) one has

$$P(\phi) = \int_0^\infty \rho d\rho \int_{-\pi|z|}^{\pi|z|} \frac{d\mu}{2\pi} \int_{-\pi|z|}^{\pi|z|} \frac{d\nu}{2\pi} \text{tr}\{\hat{\rho}_P \otimes \hat{\rho}_S e^{i\mu(\hat{\mathcal{I}}_1 - \rho \cos \phi) + i\nu(\hat{\mathcal{I}}_2 - \rho \sin \phi)}\}, \quad (27)$$

$\hat{\rho}_S$ being the density matrix of the mode a (the system) and

$$\hat{\rho}_P = |0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |z\rangle\langle z| \quad (28)$$

the density matrix of the probe. From Eqs. (7) and (8) one can see that the POM is obtained upon tracing over the probe Hilbert space \mathcal{H}_P , thus obtaining the operator which acts on the system space \mathcal{H}_S only,

$$d\hat{\mu}_D(\phi) = d\phi \int_0^\infty \rho d\rho \int_{-\pi|z|}^{\pi|z|} \frac{d\mu}{2\pi} \int_{-\pi|z|}^{\pi|z|} \frac{d\nu}{2\pi} \text{tr}_P\{\hat{\rho}_P \otimes \hat{1}_S e^{i\mu(\hat{\mathcal{I}}_1 - \rho \cos \phi) + i\nu(\hat{\mathcal{I}}_2 - \rho \sin \phi)}\}, \quad (29)$$

(hereafter the subscript D is used to denote all quantities related to the double-homodyne scheme, in order to distinguish them from the same ideal quantities). Using the coherent-state resolution of identity one has

$$d\hat{\mu}_D(\phi) = d\phi \int_0^\infty \rho d\rho \int_{-\pi|z|}^{\pi|z|} \frac{d\mu}{2\pi} \int_{-\pi|z|}^{\pi|z|} \frac{d\nu}{2\pi} \int_{\mathcal{C}} \frac{d^2u}{\pi} \int_{\mathcal{C}} \frac{d^2w}{\pi} e^{-i\rho(\mu \cos \phi + \nu \sin \phi)} |u\rangle R \langle w|, \quad (30)$$

where R is the matrix element

$$R = \langle u, z, 0, 0 | \hat{U}^\dagger e^{i(\mu \hat{\mathcal{I}}_1 + \nu \hat{\mathcal{I}}_2)} \hat{U} | 0, 0, z, w \rangle. \quad (31)$$

In Eq. (31) \hat{U} denotes the unitary evolution operator of the detector, which acts on the state (31) as follows

$$\hat{U} | 0, 0, z, w \rangle = \frac{1}{2} (z + w), \frac{1}{2} (z - w), \frac{1}{2} (iz + w), \frac{1}{2} (iz - w) \rangle. \quad (32)$$

The explicit expression of the matrix element R is given by

$$\begin{aligned} \ln R = & -|z|^2 - \frac{1}{2}|u|^2 - \frac{1}{2}|w|^2 + \frac{1}{2}|z|^2 \left(\cos \frac{\mu}{|z|} + \cos \frac{\nu}{|z|} \right) + \frac{1}{2}z\bar{u} \left(i \sin \frac{\mu}{|z|} + i \sin \frac{\nu}{|z|} \right) \\ & + \frac{1}{2}w\bar{z} \left(i \sin \frac{\mu}{|z|} - i \sin \frac{\nu}{|z|} \right) + \frac{1}{2}w\bar{u} \left(\cos \frac{\mu}{|z|} + \cos \frac{\nu}{|z|} \right). \end{aligned} \quad (33)$$

Taking the strong LO limit $|z| \rightarrow \infty$ and introducing the complex variable $\alpha = \frac{1}{2}(\nu + i\mu)e^{i\arg(z)}$ one gets

$$R = \exp \left[-\frac{1}{2}|u|^2 - \frac{1}{2}|w|^2 - |\alpha|^2 + \alpha\bar{u} - w\bar{\alpha} + w\bar{u} \right] = \sum_{p=0}^{\infty} \frac{1}{p!} \langle u | \hat{a}^\dagger | \alpha \rangle \langle -\alpha | \hat{a} | w \rangle. \quad (34)$$

Substituting Eq. (34) into Eq. (30) leads to

$$\begin{aligned} d\hat{\mu}_D(\phi) = & d\phi \int_0^\infty \rho d\rho \int_{-\infty}^\infty \frac{d\mu}{2\pi} \int_{-\infty}^\infty \frac{d\nu}{2\pi} e^{-i\rho(\mu \cos \phi + \nu \sin \phi)} \sum_{p=0}^{\infty} \frac{1}{p!} \int_{\mathcal{C}} \frac{d^2u}{\pi} |u\rangle \langle u | \hat{a}^{p\dagger} | \alpha \rangle \langle -\alpha | \hat{a}^p \int_{\mathcal{C}} \frac{d^2w}{\pi} |w\rangle \langle w| \\ = & d\phi \sum_{p=0}^{\infty} \frac{1}{p!} \frac{1}{2\pi} \hat{a}^{p\dagger} \left[2 \int_0^\infty \rho d\rho \int_{\mathcal{C}} \frac{d^2\alpha}{\pi} \exp(\rho \hat{a} e^{i\phi}) | \alpha \rangle \langle \alpha | e^{i\pi \hat{n}} \exp(\rho \hat{a}^\dagger e^{-i\phi}) \right] \hat{a}^p. \end{aligned} \quad (35)$$

Using the coherent resolution of the identity and integrating over ρ one obtains

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{2\pi} \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{m,n=0}^{\infty} (-i)^{n-m} e^{i(n-m)\phi} \times \frac{\Gamma(\frac{n+m}{2} + 1)}{n!m!} \hat{a}^p \dagger e^{i\frac{\pi}{2}\hat{n}} \hat{a}^{m\dagger} \hat{a}^n e^{i\frac{\pi}{2}\hat{n}} \hat{a}^p, \quad (36)$$

where $\Gamma(x)$ is Euler's Gamma function. The normal ordered representation of the vacuum state

$$\lim_{\varepsilon \rightarrow 1} \sum_{p=0}^{\infty} \frac{(-\varepsilon)^p}{p!} \hat{a}^p \dagger \hat{a}^p = |0\rangle\langle 0|, \quad (37)$$

leads to

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{2\pi} \sum_{m,n=0}^{\infty} (-i)^{n-m} e^{i(n-m)\phi} \times \frac{\Gamma(\frac{n+m}{2} + 1)}{n!m!} e^{i\frac{\pi}{2}\hat{n}} \hat{a}^{m\dagger} |0\rangle\langle 0| \hat{a}^n e^{i\frac{\pi}{2}\hat{n}}. \quad (38)$$

From Eq. (38) one obtains the POM of the detector in the form of a double series

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{2\pi} \sum_{n,m} e^{i(n-m)\phi} \frac{\Gamma(\frac{n+m}{2} + 1)}{\sqrt{n!m!}} |n\rangle\langle m|. \quad (39)$$

Alternatively, using the Γ -function integral representation one can write

$$\begin{aligned} d\hat{\mu}_D(\phi) &= \frac{d\phi}{\pi} \int_0^{\infty} \rho d\rho e^{-\rho^2} e^{\rho e^{i\phi} a^\dagger} |0\rangle\langle 0| e^{\rho e^{-i\phi} a} \\ &= \frac{d\phi}{\pi} \int_0^{\infty} \rho d\rho |\rho e^{i\phi}\rangle \langle \rho e^{i\phi}|. \end{aligned} \quad (40)$$

The POM for the detector in Eq. (39) corresponds to the *effectively measured phase operator*

$$\begin{aligned} \hat{\phi}_D &= \int \phi d\hat{\mu}_D(\phi) \\ &= -i \sum_{n \neq m} (-1)^{n-m} \frac{1}{n-m} \frac{\Gamma(\frac{n+m}{2} + 1)}{\sqrt{n!m!}} |n\rangle\langle m|, \end{aligned} \quad (41)$$

and to the *squared phase operator*

$$\begin{aligned} \hat{\phi}_D^2 &= \int \phi^2 d\hat{\mu}_D(\phi) \\ &= \frac{\pi^2}{3} + 2 \sum_{n \neq m} (-1)^{n-m} \frac{1}{(n-m)^2} \frac{\Gamma(\frac{n+m}{2} + 1)}{\sqrt{n!m!}} \\ &\quad \times |n\rangle\langle m|, \end{aligned} \quad (42)$$

which is needed for evaluating the instrumental sensitivity $\langle \widehat{\Delta\phi_D^2} \rangle$. Notice that

$$\frac{\Gamma(\frac{n+m}{2} + 1)}{\sqrt{n!m!}} \leq 1 \quad \forall n, m \geq 0, \quad (43)$$

and thus, for any state of the mode a one has the inequality

$$\langle \widehat{\Delta\phi_D^2} \rangle \geq \langle \widehat{\Delta\phi^2} \rangle, \quad (44)$$

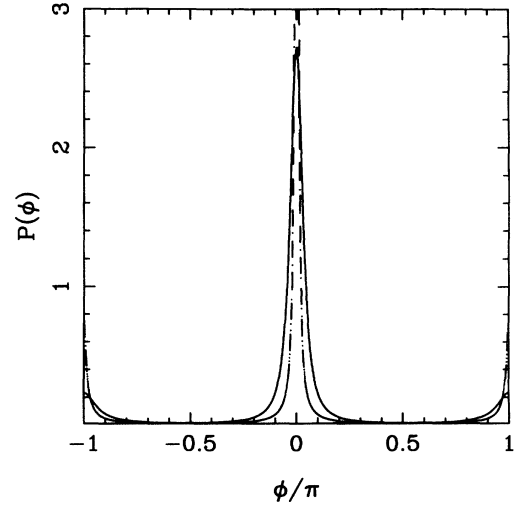


FIG. 3. Comparison between the ideal and double-quadrature phase probability distributions for the same squeezed state of Fig. 2.

namely the double-homodyne detector adds some extrinsic instrumental noise with respect to the ideal measurement. In Fig. 3 a comparison between the ideal and the double-homodyne phase probability distributions is given for the same state of Fig. 2: the added noise is evident from the fact that the ideal probability is sharper than that of the double-homodyne detection. However, we stress again that double-homodyne detection is the best available method for detecting the phase.

B. Equivalence with heterodyne detection

The first proposed method to perform simultaneous measurements of two field quadratures was heterodyne detection [40]. Here we synthetically analyze this scheme, only in order to make a connection with the double-homodyne detector and show that the two apparatuses are completely equivalent from the point of view of the measured physical quantities.

The detector along with the relevant field modes is outlined in Fig. 4. The input field E_{in} impinges into a beam splitter and has nonzero photon number only at the frequency $\omega_0 + \omega_{\text{IF}}$. The local oscillator works at the different frequency ω_0 , and the output photocurrent \hat{I}_{out} is measured at the *intermediate frequency* ω_{IF} . The

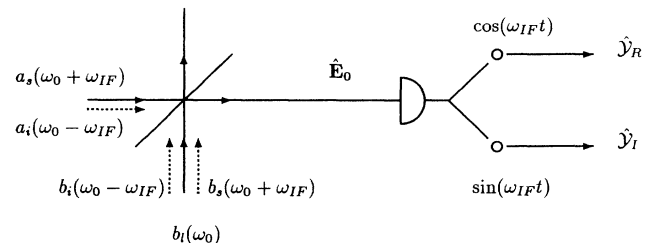


FIG. 4. Scheme of the heterodyne detector and the relevant field modes involved in the measurement. Dashed lines denote vacuum states.

measured photocurrent is given by

$$\hat{I}_{\text{out}}(t) = \hat{E}_{\text{out}}^-(t)\hat{E}_{\text{out}}^+(t), \quad (45)$$

where E^\pm denote the usual positive and negative frequency components of the field. The component at frequency ω_{IF} is given by

$$\begin{aligned} \hat{I}_{\text{out}}(\omega_{\text{IF}}) &= \int dt \hat{I}_{\text{out}}(t)e^{i\omega_{\text{IF}}t} \\ &= \int d\omega \hat{E}_{\text{out}}^-(\omega + \omega_{\text{IF}})\hat{E}_{\text{out}}^+(\omega). \end{aligned} \quad (46)$$

For a nearly transparent beam splitter, and in the limit of strong LO in the coherent state $|z\rangle$, one can define the reduced complex current \hat{Y}

$$\hat{Y} = \lim_{\substack{\eta \rightarrow 1, |z| \rightarrow \infty \\ \gamma = \text{const}}} \gamma^{-1} \hat{I}_{\text{out}}(\omega_{\text{IF}}), \quad (47)$$

where $\gamma = |z|\sqrt{\eta(1-\eta)}$. In this limit the expression for \hat{Y} is given by

$$\hat{Y} = |z|^{-1}(a_s^\dagger b_l + a_i b_l^\dagger) + (\text{vanishing terms}), \quad (48)$$

where the subscript s, l , and i refer to the signal, LO, and image component of the field, respectively, a are signal modes, b the LO modes, and the vanishing terms denote operators which do not give contributions in the strong LO limit. In double-homodyne detection in the same limit the role of the complex current (48) is played by

$$\hat{I} = \hat{I}_1 + i\hat{I}_2 = |z|^{-1} \left(a_2 a_1^\dagger + b_0 a_2^\dagger \right), \quad (49)$$

where subscript 1 refers to the input signal and subscript 2 to the local oscillator, whereas b_0 is the vacuum mode at the unused port of the beam splitter which contains the input signal. The full equivalence between heterodyne and double-homodyne detection is apparent when comparing Eq. (48) and Eq. (49). As in the double-homodyne case, now the real and imaginary parts of the current trace the two conjugated quadratures $a_{s\phi}$ and $a_{s\phi+\pi/2}$ of the signal mode. In Ref. [5] the POM of the heterodyne detector has been derived in a different context, leading to the same result obtained for the double-homodyne detector in Sec. IV A.

At the end of this section we notice that the actual sources of extrinsic added noise are the vacuum modes a_i for the heterodyne detector and b_0 for the double-homodyne detector: the other vacuum modes are totally irrelevant in the limit of strong LO.

C. Double-homodyne detector: a wideband analysis

In this subsection we give a wideband analysis of the double-homodyne detector, in order to show that no phase-sensitivity improvement is possible, at least for nonentangled multimode states. This result is in contrast with that of Ref. [42], where from information-theory arguments it is claimed that the phase sensitivity can be exponentially improved as a function of the total average number of photons, also when using coherent states.

Let us consider a double-homodyne detector with co-

herent LO and signal field both excited in the wide frequency band $\Delta\omega$. For the optical arrangement here considered it is sufficient to adopt a paraxial description of the light beams. A light beam propagating along a fixed direction (and with a fixed polarization) is described by the operator [43]

$$\hat{E}(x, t) = \hat{E}^+(x, t) + \hat{E}^-(x, t), \quad (50)$$

where

$$\hat{E}^\pm(x, t) = i\sqrt{\frac{\hbar}{4\pi\epsilon_0 \mathcal{A}c}} \int_0^\infty d\omega \sqrt{\omega} a(\omega) e^{i\omega(t \mp \frac{x}{c})}, \quad (51)$$

\mathcal{A} is the section of the beam, c is the light speed in the vacuum, and the continuum-mode operators commute as follows

$$[a(\omega), a^\dagger(\omega')] = \delta(\omega - \omega'). \quad (52)$$

If the field is populated in a not too broad band $\Delta\omega$ it is possible to extend the integral in Eq. (51) to the whole real axis, so that one can define δ -like commutations for the field also in the time domain. The Poynting vector describing the energy by the propagating light beam is given by

$$\hat{S}(x, t) = 2\epsilon_0 c \hat{E}^-(x, t)\hat{E}^+(x, t), \quad (53)$$

and in the frequency domain it becomes

$$\begin{aligned} \hat{S}(x, \omega_0) &= \int dt \hat{S}(x, t)e^{i\omega_0 t} = e^{i\omega_0 \frac{x}{c}} \frac{\hbar}{\mathcal{A}} \\ &\quad \times \int d\omega \sqrt{\omega(\omega + \omega_0)} a^\dagger(\omega + \omega_0)a(\omega). \end{aligned} \quad (54)$$

The response of the photodetector can be conveniently described by means of a spectral sensitivity $R(\omega)$, which is dimensionally homogeneous to an inverse time and typically has a Lorentzian shape. The revealed photocurrent \hat{I} is the time convolution of the Poynting vector with the photodetector sensitivity, namely

$$\hat{I} = \int d\omega \frac{\hbar\omega}{\mathcal{A}} R(\omega) a^\dagger(\omega)a(\omega). \quad (55)$$

In Eq. (55) the dependence on the position x of the detector has been dropped. In practice, the zero-frequency component of the Poynting vector is measured when the sensitivity corresponds to an integration time much greater than the inverse bandwidth. The difference photocurrent \hat{I}_D at the output of a balanced homodyne detector is then given by

$$\hat{I}_D = \int d\omega \frac{\hbar\omega}{\mathcal{A}} R(\omega) [a^\dagger(\omega)b(\omega) + b^\dagger(\omega)a(\omega)], \quad (56)$$

where $a(\omega)$ are the signal and $b(\omega)$ the LO field components. The LO is excited in the wideband coherent state $|\{z_{\text{LO}}\}\rangle$ with lineshape $z_{\text{LO}}(\omega)$ [44]

$$|\{z_{\text{LO}}\}\rangle = e^{\int d\omega [z_{\text{LO}}(\omega)a^\dagger(\omega) - \bar{z}_{\text{LO}}(\omega)a(\omega)]} |\{0\}\rangle. \quad (57)$$

The ket $|\{0\}\rangle$ denotes the vacuum state at all frequencies. Tracing over the LO one obtains

$$\hat{I}_D = \int d\omega \frac{\hbar\omega}{\mathcal{A}} R(\omega) [a^\dagger(\omega)z_{\text{LO}}(\omega) + \bar{z}_{\text{LO}}(\omega)a(\omega)]. \quad (58)$$

One can see that the difference photocurrent is proportional to the quadrature $\frac{1}{2}(A + A^\dagger)$ of the field, A being a *wideband mode* satisfying the commutation $[A, A^\dagger] = 1$, which is defined as follows

$$A = \int d\omega f(\omega) a(\omega), \quad (59)$$

where $f(\omega)$ is

$$f(\omega) = \frac{\omega R(\omega) \bar{z}_{\text{LO}}(\omega)}{\sqrt{\int d\omega \omega^2 R^2(\omega) |z_{\text{LO}}(\omega)|^2}}. \quad (60)$$

The appropriate reduced photocurrent \hat{i} which traces A is given by

$$\hat{i} = \frac{\hat{I}_D}{2 \frac{\hbar}{A} \sqrt{\int d\omega \omega^2 R^2(\omega) |z_{\text{LO}}(\omega)|^2}}. \quad (61)$$

A generic coherent wideband state $|\{\alpha\}\rangle$ for any function $\alpha(\omega) \in \mathcal{L}^2(\mathbf{R})$ is an eigenstate of A as follows

$$A|\{\alpha\}\rangle = \left[\int d\omega f(\omega) \alpha(\omega) \right] |\{\alpha\}\rangle. \quad (62)$$

In addition to the two homodyne detectors, the double-homodyne detector needs a quarter wave plate, which in the present context could be considered *achromatic* for simplicity. In this case the two output photocurrents simply trace two conjugated quadratures of A , namely its real and imaginary parts.

Let us now consider the measurement of the phase for this wideband detector. The measurement is operationally defined in a way analogous to the single-mode case, namely the detection of the polar angle between the two output difference photocurrents of the form (61), which correspond to the two quadratures of A . Equation (40) for the single-mode case is rewritten in the form

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{\pi} \int_{\mathbf{C}} d^2z \int_0^\infty \rho d\rho \delta_2(z - \rho e^{i\phi}) |z\rangle\langle z|, \quad (63)$$

representing the delta function in the complex plane by means of the integral

$$\delta_2(w) = \int_{\mathbf{C}} \frac{d^2z}{\pi^2} e^{z\bar{w} - w\bar{z}}. \quad (64)$$

The POM (63) can be easily extended to the wideband case using the functional integration

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{\pi} \int \mathcal{D}^2[\{z\}] \int_0^\infty \rho d\rho \delta_2\left(\rho e^{i\phi} - \int d\omega f(\omega) z(\omega)\right) |\{\alpha\}\rangle\langle\{z\}|, \quad (65)$$

where $\mathcal{D}^2[\{z\}]$ denotes the coherent-state functional measure (in practical situations one has a discrete bounded set of frequencies and the functional integration is not ill defined: here it is only used as a convenient compact notation). From Eqs. (65) and (62) and the resolution of the identity

$$\hat{1} = \int \mathcal{D}^2[\{z\}] |\{z\}\rangle\langle\{z\}|, \quad (66)$$

one gets

$$d\hat{\mu}_D(\phi) = \frac{d\phi}{\pi} \int_0^\infty \rho d\rho e^{-\rho^2} e^{\rho e^{i\phi} A^\dagger} |\{0\}\rangle\langle\{0\}| e^{\rho e^{-i\phi} A}, \quad (67)$$

which is identical to the single-mode formula (40), but now with the wideband operator A in place of the one-mode operator a . For a signal field which is itself in a coherent state $|\{z_S\}\rangle$ this POM provides the phase probability distribution

$$dP(\phi) = \frac{d\phi}{\pi} \int_0^\infty \rho d\rho e^{-|\rho e^{i\phi} - \Lambda|^2}, \quad (68)$$

where

$$\Lambda = \int d\omega f(\omega) z_S(\omega). \quad (69)$$

For simplicity we consider the situation of zero average phase difference with respect to the LO. This corresponds to a suitable choice of the relative phases between the functions $z_{\text{LO}}(\omega)$ and $z_S(\omega)$ in order to obtain a real positive Λ . In the limit of large Λ corresponding to sharp

phase distributions, an expansion up to second order in ϕ leads to the Gaussian approximation

$$P(\phi) \simeq \sqrt{\frac{1}{\pi}} e^{-(\Lambda^2 + \frac{1}{2})\phi^2}, \quad (70)$$

which leads to the sensitivity

$$\Delta\phi^2 \simeq \frac{1}{2\Lambda^2}. \quad (71)$$

Upon minimizing the sensitivity (71) at fixed input energy, namely at fixed Poynting vector $\langle S \rangle$, one has

$$\Delta\phi^2 \simeq \frac{\hbar}{\mathcal{A} \langle S \rangle} \frac{\omega_0^2 + \Delta\omega^2}{\omega_0}, \quad (72)$$

ω_0 being the central frequency of the band. As one can see from Eq. (72) no substantial improvement in phase sensitivity can be gained by widening out the frequency band of the state: typically, one has $\omega_0 \gg \Delta\omega$, and the usual shot-noise expression $\Delta\phi^2 \approx 1/\bar{n}$ is recovered. The phase sensitivity here has been obtained only for wideband coherent states, but a similar argument still holds true for general nonentangled wideband states, namely states which are the direct product of single-mode states. For entangled states, on the other hand, the problem of minimizing the phase sensitivity remains open, due to the technical difficulties encountered in the functional minimization. In Sec. VI the states which optimize the sensitivity in the single-mode case are derived.

V. QUANTUM MEASUREMENT OF PHASE-DEPENDENT OBSERVABLES

In this section we examine the measurement of the customary field quadrature \hat{a}_ϕ and that of the so-called *phase quadrature* \hat{e}_ϕ . The former is measured by means of the homodyne detector, whereas the latter has no corresponding feasible detection scheme. The present measurement of phase-dependent observables exits from the framework of phase estimation theory: we include them both, due to the relevance of homodyne detection in any interferometric setup, whereas the phase-quadrature measurement is, in some sense, an idealized version of it.

The definitions of the field quadrature \hat{a}_ϕ and of the phase quadrature \hat{e}_ϕ are the following

$$\hat{a}_\phi = \frac{1}{2}(ae^{-i\phi} + a^\dagger e^{i\phi}), \quad (73)$$

$$\hat{e}_\phi = \frac{1}{2}(\hat{e}_- e^{-i\phi} + \hat{e}_+ e^{i\phi}), \quad (74)$$

where \hat{e}_\pm denote the raising and lowering operators $\hat{e}_+|n\rangle = |n+1\rangle$, $\hat{e}_- \equiv (\hat{e}_+)^\dagger$. These observables are labeled by the phase difference ϕ between the signal mode and a synchronous coherent highly excited LO. The ϕ dependence of the above operators is not one to one, and thus a phase value cannot actually be obtained by measuring them. However, variations of the phase shift produce variations of the average output, and in this sense one can define a phase sensitivity. From elementary error-propagation calculus one obtains

$$\Delta\phi = \sqrt{\langle \Delta \hat{f}_\phi^2 \rangle} \left| \frac{\delta \langle \hat{f}_\phi \rangle}{\delta \phi} \right|^{-1}. \quad (75)$$

Such sensitivity depends on both on the observable \hat{f}_ϕ and the input state of radiation. Usually, to minimize $\Delta\phi$ the working point is selected which maximizes the derivative in Eq. (75), allowing only very small variations of the phase difference in order to avoid sizeable degradation of $\Delta\phi$: this requires a suitable feedback mechanism which pursues the working point.

A. Field quadrature: homodyne detection

The balanced homodyne scheme in Fig. 5 measures one quadrature of a field mode, which in turn is related to its phase difference with respect to the synchronous LO. Generally one is interested in the measure of the phase shift χ of the signal state

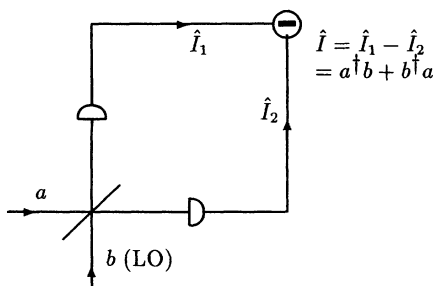


FIG. 5. Outline of the balanced homodyne detector.

$$|\psi\rangle_\chi = \exp(-i\chi\hat{n})|\psi\rangle_0, \quad (76)$$

where, without loss of generality, the input state is assumed to be of the form

$$|\psi\rangle_0 = \sum_{n=0}^{\infty} c_n |n\rangle, \quad c_n \geq 0. \quad (77)$$

The expectation value of the quadrature is given by

$$\begin{aligned} \langle \hat{a}_\phi \rangle_\chi &= \sum_{n=0}^{\infty} \sqrt{n+1} c_n c_{n+1} \cos(\phi - \chi) \\ &= \langle \hat{a}_0 \rangle_0 \cos(\phi - \chi). \end{aligned} \quad (78)$$

The quadrature \hat{a}_ϕ is proportional to the cosine of the phase with a proportionality “constant” $\langle \hat{a}_0 \rangle_0$ which can be evaluated from knowledge of the fixed input state. Notice that, however, when the present scheme is regarded as a measure of the phase of the state $|\psi\rangle_\chi$ itself, the state-dependent “constant” is unknown, and it cannot be preventively measured without destroying the information on phase. In this case the only point which does not need any knowledge of the “constant” is the $\phi - \chi = \pi/2$ point, namely just the maximum-derivative zero-current working point.

A convenient description of the homodyne detector in view of the above considerations is given in Ref. [45], where the zero-point (zero-field) probability distribution is reconsidered as a sort of a phase probability distribution.

B. Phase quadrature

The phase quadrature is defined in Eq. (74) in analogy with the usual field quadrature. A measurement of \hat{e}_ϕ has the expectation value

$$\langle \hat{e}_\phi \rangle_\chi = \sum_{n=0}^{\infty} c_n c_{n+1} \cos(\phi - \chi) = \langle \hat{e}_0 \rangle_0 \cos(\phi - \chi). \quad (79)$$

The same assertions made for the customary field quadrature here hold true for the phase quadrature, as regards the zero-current working point and the proportionality “constant” $\langle \hat{e}_0 \rangle_0$ in Eq. (79) (the latter now approaches unity for semiclassical states: see [41] for more details). For $\phi = 0$ and $\phi = \pi/2$ the phase quadrature (79) coincides with the sine-cosine trigonometric operators \hat{c} and \hat{s} introduced by Susskind-Glogower [16], namely

$$\hat{c} = \frac{1}{2}(\hat{e}^- + \hat{e}^+), \quad \hat{s} = \frac{1}{2i}(\hat{e}^- - \hat{e}^+). \quad (80)$$

Thus the present scheme corresponds to the measurement of one of the self-adjoint operators (80). Here, some remarks are in order regarding two relevant differences between a conventional measurement of a single phase quadrature—say the cosine \hat{c} —and a joint measurement of both sine and cosine quadratures, as in Secs. II and IV. (i) A single-phase-quadrature measurement leads to violation of the trigonometric calculus for expectation values. In fact, for a general density-matrix state $\hat{\rho}$ one has that

$$\text{Tr}[\hat{\rho}(\hat{c}^2 + \hat{s}^2)] = 1 - \frac{1}{2}\langle 0|\hat{\rho}|0\rangle, \quad (81)$$

whereas for a joint measurement one obtains

$$\text{Tr}[\widehat{\rho}(\widehat{\sin^2\phi} + \widehat{\cos^2\phi})] = 1, \quad (82)$$

where $\widehat{\cos^2\phi}$ (and similarly $\widehat{\sin^2\phi}$) is defined according to Eq. (5) as follows

$$\widehat{\cos^2\phi} = \int_{-\pi}^{\pi} d\hat{\mu}(\phi) \cos^2\phi, \quad (83)$$

$d\hat{\mu}(\phi)$ being the POM of the apparatus. Notice that, however, the linear operators coincide in the two cases, and thus one gets the same average values. (ii) The probability distribution of the outcomes from single-phase-

quadrature measurement exhibits unphysical features for nonclassical states, whereas the probability distribution from the joint measurement does not. In the single-quadrature measurement one has

$$P(c) = \text{tr}\{\hat{\rho}|c\rangle\langle c|\}, \quad (84)$$

where the eigenstates of \hat{c} are given by [16, 17]

$$|c\rangle = \sqrt{\frac{2}{\pi}}(1-c^2)^{-1/4} \sum_{n=0}^{\infty} \sin[(n+1)\arccos c]|n\rangle. \quad (85)$$

On the other hand, the Radon-Nikodym derivative of the joint-measurement POM's leads to

$$P(c) = \text{Tr} \left[\hat{\rho} \frac{d\hat{\mu}(\phi)}{d\phi} \frac{d\phi}{dc} \right] = \frac{1}{\pi} (1-c^2)^{-1/2} \sum_{n,m} \langle m|\hat{\rho}|n\rangle \exp[i(n-m)\arccos c], \quad (86)$$

for the ideal case, whereas for double-homodyne detection one obtains

$$P(c) = \text{Tr} \left[\hat{\rho} \frac{d\hat{\mu}_D(\phi)}{d\phi} \frac{d\phi}{dc} \right] = \frac{1}{\pi} (1-c^2)^{-1/2} \sum_{n,m} \langle m|\hat{\rho}|n\rangle \frac{\Gamma(\frac{m+n}{2} + 1)}{\sqrt{n!m!}} \exp[i(n-m)\arccos c]. \quad (87)$$

The differences between the single-quadrature and double-quadrature probabilities become striking for isotropic states, as, for example, the vacuum or a general number state. In this case the above distributions should be compared with the Radon-Nikodym derivative of the constant distribution

$$P(c) = \frac{1}{\pi} \frac{1}{\sqrt{1-c^2}}, \quad (88)$$

which is a concave function and has poles at the $c = \pm 1$ stationary points of the cosine. The probabilities (86) and (87) coincide with (88) for number states, whereas the probability (84) has the opposite curvature for the vacuum state, and oscillates fast around the function (88) for nonvacuum number states. These undesired physical features disappear for highly excited coherent states, where, however, the main quantum features are lost: a comparison between the above probabilities is reported in Fig. 6 for a weakly excited coherent state, along with a simulated histogram of a double-homodyne experiment.

From the above observations we conclude that for nonclassical states the joint phase-quadrature measurements lead to more physical results than the single-quadrature ones.

VI. OPTIMIZING THE STATES IN A PHASE MEASUREMENT

The design of a phase measurement needs optimization of both the detection scheme and of the quantum state which carries the phase information: the physical constraint in optimization is the total power impinged into the state. The ideal measurement provided by quantum estimation theory is the solution of the first problem; however, with no actual scheme achieving it. The optimization of the state depending on the detection scheme

is the main concern of the present section. The state optimization corresponds to minimizing the rms sensitivity

$$\Delta\phi \stackrel{\text{def}}{=} \sqrt{\langle \Delta\phi^2 \rangle} = \sqrt{\langle \hat{\phi}^2 \rangle - \langle \hat{\phi} \rangle^2} \quad (89)$$

for fixed average photon number. In the following we consider for simplicity a zero-average phase state, namely a state with positive real coefficients on the number basis

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad c_n \geq 0. \quad (90)$$

In this case the phase sensitivity has the form

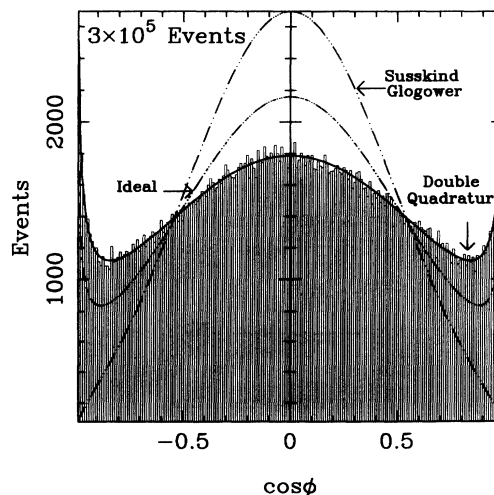


FIG. 6. Probability distributions of the cosine for a weakly excited ($\bar{n} = 1$) coherent state: Susskind-Glogower distribution (84), ideal distribution (86) and double-homodyne distribution (87). The histograms give the results of a simulated experiment.

$$\langle \widehat{\Delta\phi^2} \rangle = \frac{\pi^2}{3} + 2 \sum_{n \neq m} A_{n,m} c_n c_m, \quad (91)$$

and the normalizations and average constraints become

$$\sum_{n=0}^{\infty} c_n^2 = 1, \quad \sum_{n=0}^{\infty} n c_n^2 = \bar{n}. \quad (92)$$

The real symmetric matrix $\mathbf{A} = \{A_{n,m}\}$ depends on the detection scheme. In particular, from Eqs. (15) and (42) one obtains ($A_{n,n} = 0$)

$$A_{n,m} = \begin{cases} \frac{(-1)^{n-m}}{(n-m)^2} & (\text{ideal}) \\ \frac{(-1)^{n-m}}{(n-m)^2} \frac{\Gamma(\frac{n+m}{2}+1)}{\sqrt{n!m!}} & (\text{double homodyne}). \end{cases} \quad (93)$$

$$(94)$$

The method of the Lagrange multipliers reduces the problem to the minimization of the function

$$F(\{c_n\}; \lambda, \beta | \bar{n}) = \frac{\pi^2}{3} + 2 \sum_{n \neq m} A_{n,m} c_n c_m + \lambda \left(\sum_{n=0}^{\infty} c_n^2 - 1 \right) + \beta \left(\sum_{n=0}^{\infty} n c_n^2 - \bar{n} \right) \quad (95)$$

with respect to the coefficients $\{c_n\}$, λ and β being the Lagrange multipliers. The variational problem (95) is equivalent to the diagonalization of the quadratic symmetric form

$$[\mathbf{M}(\beta) + \lambda \mathbf{I}] \cdot \mathbf{c} = 0, \quad \mathbf{c} \equiv (c_0, c_1, \dots), \quad (96)$$

where now the matrix $\mathbf{M} = \{M_{nm}\}$ is given by

$$M_{n,m} = A_{nm} + n\beta\delta_{nm}. \quad (97)$$

Equation (96) can be solved numerically upon a suitable truncation of the Hilbert space \mathcal{H}_s . One can see that the absolute minimum corresponds to the minimum eigenvalue $\lambda = \pi^2/3$, whereas the average number \bar{n} becomes a decreasing function of the running parameter $\beta \in [0, 1]$. In order to avoid the problem of nonvanishing tails of the number distribution near the border of the truncated Hilbert space, we have considered only average values $\bar{n} \ll \dim \mathcal{H}_s/2$.

A. Ideal measurement

For the ideal measurement of the phase, the best phase states which minimize the phase sensitivity lead to the simple power law

$$\Delta\phi \sim \frac{1.36 \pm 0.01}{\bar{n}^{1.00 \pm 0.01}}, \quad (98)$$

in agreement with the results of [7]. The proportionality constant actually increases very slowly as a function of \bar{n} , and one has a variation of a few percent for two decades of $\Delta\phi$. Equation (98) can be compared with the result of Ref. [7], and with the theoretical bound $\Delta\phi \sim 1/(e\bar{n})$

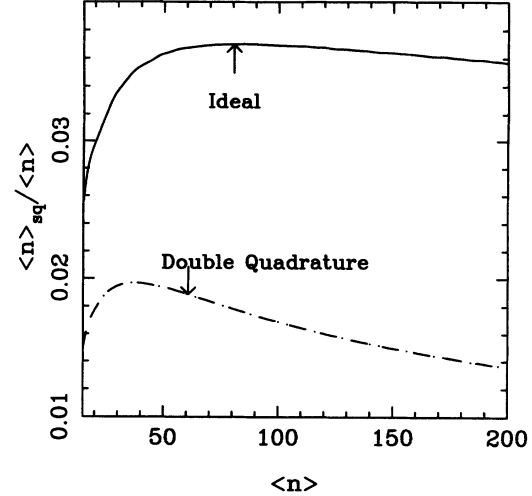


FIG. 7. Optimal fraction of squeezing photons for ideal and double-quadrature phase detection.

[42] obtained by means of information-theory arguments. For large \bar{n} ($\bar{n} > 10$) the optimal states very nearly resemble the customary squeezed states. In fact the same result of Eq. (98) can be obtained using squeezed states which are optimized with respect to the squeezing photon number as a function of \bar{n} . It turns out that the optimal states have only $\sim 3.7\%$ of squeezing photons (see Fig. 7): this result is very different from the customary 50% optimal squeezing number which is obtained for homodyne detection, or in a Mack-Zehnder interferometer [1].

B. Double-homodyne detection

As expected, the actual measurement of the phase does not achieve the same ideal sensitivity as Eq. (98). The double-quadrature phase detection, in fact, is bounded by the following power law:

$$\Delta\phi_D = \frac{1.00 \pm 0.01}{\bar{n}^{0.65 \pm 0.01}}, \quad (99)$$

which is obtained by numerically solving Eq. (96) with the matrix \mathbf{M} given in Eqs. (97) and (93). In Fig. 8 the optimized states for both ideal and double-homodyne detection are compared at equal fixed average photon numbers

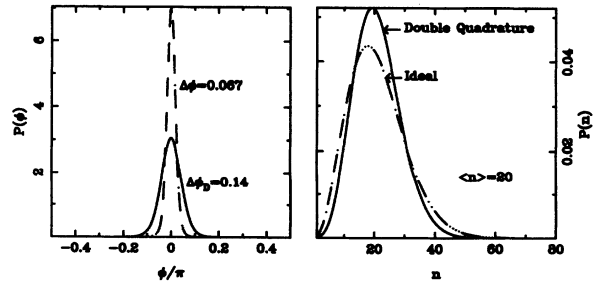


FIG. 8. Number and phase probability distribution of optimal phase states for ideal and double-quadrature phase detection.

ber $\bar{n} = 20$. One can see that the number and phase probability distributions are qualitatively similar in the two cases, but the double-homodyne optimum states have slightly broader phase distribution, whereas the number distribution is sharper. Similarly to the ideal measurement, the best double-quadrature phase states are essentially indistinguishable from optimized squeezed states for large \bar{n} ($\bar{n} > 10$). In this case only less than $\sim 2\%$ of squeezing photons give optimal states, as is shown in Fig. 7.

C. Heisenberg uncertainty product for the phase quadrature

The customary Heisenberg uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} \left| \text{Tr} \left\{ \hat{\rho} [\hat{A}, \hat{B}] \right\} \right| \quad (100)$$

refers to the situation in which the quantum system is prepared in a state with fixed uncertainty say ΔA , whereas the other observable \hat{B} is actually measured. For the case of a joint \hat{A} - \hat{B} measurement, however, a generalized uncertainty relation holds which is identical to Eq. (100) apart from the factor 1/2 on the right side: this enhanced uncertainty corresponds to an added noise of 3 dB [46]. In the case of the phase quadratures one has the commutation relation

$$[\hat{c}, \hat{s}] = -\frac{i}{2} |0\rangle\langle 0|, \quad (101)$$

which corresponds to the joint-measurement uncertainty product

$$\Delta c \Delta s \geq \frac{1}{2} |\langle \psi | 0 \rangle|^2. \quad (102)$$

In Eqs. (100) and (102) the uncertainties are defined in the usual way, namely $\langle \Delta \hat{O}^2 \rangle = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2$. On the other hand, in the POM approach the actually measured uncertainty is defined as $\langle \widehat{\Delta O}^2 \rangle = \langle \widehat{O}^2 \rangle - \langle \widehat{O} \rangle^2$, where $\widehat{O}^2 \neq \hat{O}^2$ is defined as in Eq. (4). The Schwartz inequality leads to the general relation

$$\langle \widehat{O}^2 \rangle - \langle \widehat{O} \rangle^2 \geq \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2, \quad (103)$$

namely the nonorthogonal POM's always lead to uncertainty greater than the customary ones. From the above observations one can deduce the following chain of inequalities for the various minimum-uncertainty products $[\Delta c \Delta s]$:

$$\begin{aligned} [\Delta c \Delta s]_{\text{Heisenberg}} &= 2 [\Delta c \Delta s]_{\text{generalized}} \\ &\leq [\Delta c \Delta s]_{\text{ideal POM}} \\ &\leq [\Delta c \Delta s]_{\text{double-homodyne POM}}. \end{aligned} \quad (104)$$

As a consequence of Eq. (104) one should not expect that the optimum phase states achieve the minimum-uncertainty product (102), even though phase detection itself corresponds to a joint sine-cosine measurement. In Fig. 9, the minimum-uncertainty product (102) is compared with the actual uncertainty product of both ideal and double-quadrature detection. One can see that the

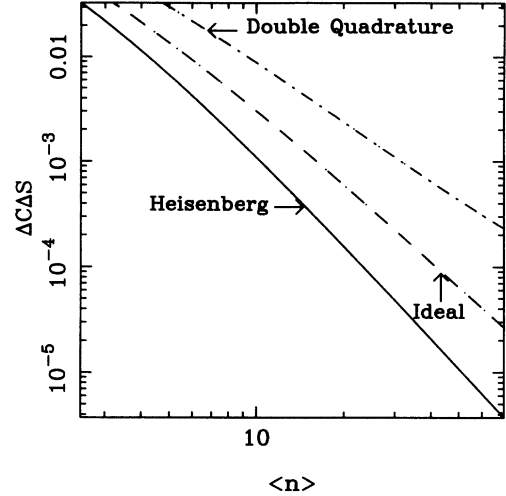


FIG. 9. Uncertainty product sine \times cosine versus the average photon number, the Susskind-Glogower limit of Eq. (102) and the results for the ideal measurement and the double-homodyne measurement are reported.

minimum uncertainties are never achieved, even for ideal detection. On the other hand the difference between uncertainty products for ideal and double-quadrature detection can be obviously ascribed to the noise which is added in the nonideal measurement.

D. Measurement of phase-dependent observables

For homodyne detection the well-known sensitivity $\Delta\phi = \frac{1}{2}\bar{n}^{-1}$ is achieved only near the zero-current working point. Such sensitivity is better than the one achieved by the true phase measurement, either in double-homodyne detection or the ideal case itself. This is due to the fact that the measurement of the field quadrature near the zero-current working point partially underestimates the tails of the phase distribution at $\phi = \pm\pi$. The latter are enhanced by large squeezing, and thus one also finds that the optimal number of squeezing photons is only a few percent of \bar{n} for the true phase measurements, whereas it is 50% for single-homodyne detection. However, it is interesting to notice that the measurement of a single phase quadrature, say $\cos\phi$, also exhibits a small optimal squeezing fraction ($\sim 5\%$), as, in some sense, it is a more faithful observable than the field quadrature. In Table I the above results are reported along with the phase sensitivity for other quantum states and different detection schemes in the limit of large average photon numbers \bar{n} .

VII. CONCLUSIONS

In this paper we have analyzed different detection schemes for the phase. We have considered two main classes of detection, namely the single-quadrature and the two-quadrature schemes, either in the ideal or in the actual cases. We have shown that the former are zero-point measurements, whereas the latter are genuine

TABLE I. Asymptotic sensitivities versus the average photon number (results from numerical calculations are given with error estimation).

	Ideal	Double quadrature	Field quadrature $\hat{a}_{\pi/2}$	Phase quadrature $\hat{e}_{\pi/2}$
	$\Delta\phi$ [Eqs. (41,42,89)]	$\Delta\phi_D$ [Eqs.(41,42,89)]	$\Delta\phi$ [Eq. (75)]	$\Delta\phi$ [Eq. (75)]
Coherent states	$\frac{1}{2}\bar{n}^{-1/2}$	$\frac{1}{\sqrt{2}}\bar{n}^{-1/2}$	$\frac{1}{2}\bar{n}^{-1/2}$	$\frac{1}{2}\bar{n}^{-1/2}$
Phase-coherent states (Ref. [4])	$(\ln 4)^{1/2}\bar{n}^{-1/2}$	$(\frac{1}{2}\ln \bar{n})^{1/2}\bar{n}^{-1/2}$	$(\frac{\ln \bar{n}}{4\pi})^{1/2}\bar{n}^{-1}$	$\frac{1}{2}\bar{n}^{-1/2}$
Optimized squeezed states	$(1.36 \pm 0.01)\bar{n}^{-1.00 \pm 0.01}$	$(1.00 \pm 0.01)\bar{n}^{-0.65 \pm 0.01}$	$\frac{1}{2}\bar{n}^{-1}$	$(1.36 \pm 0.01)\bar{n}^{-1.00 \pm 0.01}$
	$\bar{n}_{\text{sq}}/\bar{n} \sim 4\%$	$\bar{n}_{\text{sq}}/\bar{n} \sim 2\%$	$\bar{n}_{\text{sq}}/\bar{n} = 50\%$	$\bar{n}_{\text{sq}}/\bar{n} \sim 5\%$

phase measurements. Comparing the single-quadrature schemes with the double-quadrature ones, we have concluded that the two-quadrature schemes lead to more reliable results than one-quadrature schemes, because unphysical quantum statistics are avoided and no violation of the trigonometric calculus occurs for expectation values.

We have seen that in an actual measurement the phase shift corresponds to the polar angle between two real output photocurrents. We have analyzed in detail the double-homodyne scheme of Ref. [14], giving the POM of the apparatus: this fully quantum treatment leads to different phase operators corresponding to different measurement schemes, and this clarifies the meaning of the operational approach of Ref. [14].

After a critical revision of the most commonly adopted definitions of sensitivity, we have concluded that the usual rms noise is the right quantity to be considered. We have shown that the sensitivity versus the average photon number \bar{n} is bounded by the ideal limit $\Delta\phi \sim \bar{n}^{-1}$, whereas for double-homodyne detection the bound is $\Delta\phi \sim \bar{n}^{-2/3}$, in between the shot-noise level $\Delta\phi \sim \bar{n}^{-1/2}$ and the ideal bound. The optimal states achieving the

best sensitivity for fixed energy have been numerically obtained, and we have shown that they are very close to coherent states with only 2% of squeezing photons for double-homodyne detection and 3.7% for the ideal measurement. This result, which is in contrast with the 50% of optimal squeezing photons for single-homodyne detection, is due to the sensitivity of the double-quadrature detection to the phase distribution tails, which are enhanced by increasing squeezing. The uncertainty product of two conjugated phase quadratures largely exceeds the Heisenberg limit, even for the ideal joint measurement. Finally, a wideband analysis of the double-homodyne detector has shown that no sensitivity improvement is gained using multimode states, at least for nonentangled states.

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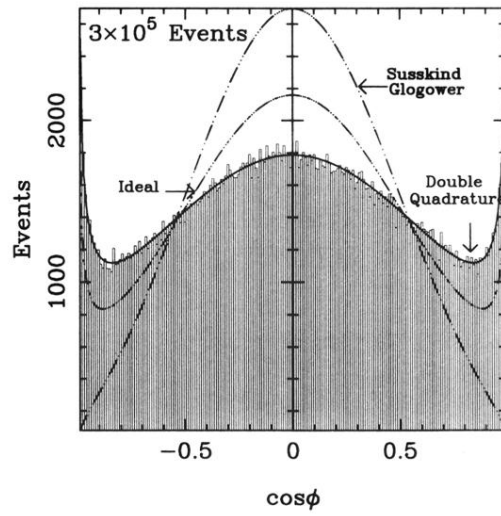


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