RAPID COMMUNICATIONS

PHYSICAL REVIEW A VOLUME 47, NUMBER 4 APRIL 1993

Chaotic and cooperative regimes for the micromaser

R. Bonifacio
Dipartimento di Fisica, via Celoria 16, 20133 Milano, Italy

G. M. D'Ariano and R. Seno
Dipartimento di Fisica "A. Volta," via Bassi 6, 27100 Pavia, Italy

N. Sterpi
Max-Planck-Institut für Quantenoptik, 8046 Garching, Germany
(Received 17 December 1992)

We describe the micromaser in the framework of an exact semiclassical model for regular injection, allowing the case of more than one atom at a time in the cavity to be considered. For high pumping and high atom fluxes we find a new kind of phase transition corresponding to a strongly nonlinear chaotic behavior. In addition to the customary period-doubling route to chaos, unexpected scenarios are exhibited, which are a consequence of the multistable competition between different attractors for time evolution.

PACS number(s): 42.52.+x, 05.45.+b

Besides providing the simplest concrete physical system to test the quantum nature of radiation interacting with matter, the micromaser is also attractive as a toy model for investigations on chaotic behavior, both classically and quantum mechanically [1, 2]. Actually, what makes the micromaser interesting is the fact that the experimental parameters are under good control, and the atom-field interaction time and cavity damping rate can be varied almost at will at extremely low values.

The micromaser has been studied extensively for very low fluxes of atoms, with average time spacing between two consecutive atoms much greater than the flying time in the cavity ($\tau_0 \gg \tau_{\rm int}$), in order to guarantee that there is no more than one atom at a time [3–5]. Such restriction to low atomic fluxes, along with the possibility of neglecting the field decay during the interaction with the atom, allows microscopic quantum-mechanical evaluations. On the other hand, there is no simple theoretical approach for the case of high fluxes, because of the intrinsic difficulty of either treating the atom-atom quantum correlations and/or separating the loss from the gain parts of the density-matrix evolution.

The case of many atoms simultaneously crossing the cavity should be particularly interesting, due to the possibility of cooperative mechanisms. Actually, in order to observe cooperative effects, it is not strictly necessary to have many atoms. In fact, the usual phase transitions predicted by quantum theory become sharper for larger $N_{\rm ex}$—the number of excited atoms crossing the cavity during the photon lifetime. However, for fixed $N_{\rm ex}$ the dynamics should be different when high fluxes are considered—namely, $\tau_0 \simeq \tau_{\rm int}$—as opposed to the extreme case $\tau_0 \gg \tau_{\rm int}$. As is shown in the following, in the high-flux regime new types of "phase transitions" arise for very large values of the pumping parameter θ (see the following for a definition). It should be stressed that in this regime the quantum nature of the system is not so relevant for dynamics as in the case of low fluxes, and a simple semiclassical approach is itself of interest, at least for sufficiently high photon numbers and not too long evolution times [6].

In this paper numerical results are presented, based on an exact semiclassical model for regular injection, including the case of more than one atom in the cavity at a time. In this sense the present model differs from those in Refs. [1, 2], where the limit $\tau_0 \gg \tau_{\rm int}$ describes an almost always empty cavity, with atoms injected at very low rates; hence, coherence and cooperative effects between atoms are totally neglected, whereas they are the main concern of this paper, since we allow at least one atom in the cavity at every time.

Let us briefly summarize the derivation of the model. A monoenergetic beam of two-level atoms crosses a cavity with transit time $\tau_{\rm int}$, all the atoms being injected in the excited state. The Hamiltonian is the usual electric-dipole interaction for atoms resonating with one mode of the cavity in the rotating-wave approximation

$$\hat{H} = \frac{\hbar\omega}{2}\hat{\sigma}_3^j + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\sum_{j=0}^{\infty}\chi_j(t)(\hat{a}^\dagger\hat{\sigma}_-^j + \text{H.c.}) \ , \qquad (1)$$

where $\chi_j(t)$ is the characteristic function on the time interval $[t_j, t_j+\tau_{\rm int}]$ for the jth atom entering the cavity at $t=t_j$. From the Hamiltonian (1) one gets the Heisenberg equations in the interaction picture

$$\begin{aligned}
\frac{d\hat{a}(t)}{dt} &= -ig\sum_{j=0}^{\infty}\chi_j(t)\hat{\sigma}_-^j(t) \ , \\
\frac{d\hat{\sigma}_-^j(t)}{dt} &= ig\chi_j(t)\hat{\sigma}_3^j(t)\hat{a}(t) \ , \\
\frac{d\hat{\sigma}_3^j(t)}{dt} &= 2ig\chi_j(t)[\hat{a}^\dagger(t)\hat{\sigma}_-^j(t) - \text{H.c.}] \ .
\end{aligned} \qquad (2)$$

Cavity losses have to be considered upon introducing a suitable Liouvillian operator in addition to the Hamiltonian (1). In this way the Heisenberg equations take the form of Langevin equations for operators, with a damping term $-(k/2)\hat{a}(t)$ plus a zero average noise operator to be included on the right-hand side of the first of Eqs. (2). The semiclassical limit is obtained by replacing all operators with their expectation values $a(t) = \langle\hat{a}(t)\rangle$, $\sigma_-^j(t) = \langle\hat{\sigma}_-^j(t)\rangle$, $\sigma_3^j(t) = \langle\hat{\sigma}_3^j(t)\rangle$, and consistently dropping the noise term

$$\begin{aligned}\frac{da(t)}{dt} &= -\frac{k}{2}a(t) - ig\sum_{j=0}^{\infty}\chi_j(t)\sigma_-^j(t)\ ,\\ \frac{d\sigma_-^j(t)}{dt} &= ig\chi_j(t)\sigma_3^j(t)a(t)\ ,\\ \frac{d\sigma_3^j(t)}{dt} &= 2ig\chi_j(t)[a^*(t)\,\sigma_-^j(t) - \text{c.c.}]\ .\end{aligned} \tag{3}$$

Since the atoms are injected in the excited state, namely, $\sigma_-^j(t_j) = 0$ and $\sigma_3^j(t_j) = 1$, from Eqs.(3) one has

$$\frac{da(t)}{dt} = -\frac{k}{2}a(t) + \frac{g}{2}\sum_{j=0}^{\infty}\chi_j(t)\sin\left[\int_{t_j}^{t}du\,2ga(u)\right]\ . \tag{4}$$

In a more transparent notation, upon introducing the Bloch rotation angles

$$\phi_j(t) \equiv \int_{t_j}^{t}du\,2ga(u)\ , \tag{5}$$

and the dimensionless time $t' = t/\tau_0$, τ_0 being the time spacing between two consecutive regularly injected atoms, the integro-differential equation (4) is replaced by the following system of differential equations:

$$\frac{dx(t')}{dt'} = -\frac{1}{2N_{\text{ex}}}x(t') + \frac{\theta^2}{2NN_{\text{ex}}}\sum_{j=0}^{\infty}\chi_j(t')\sin\phi_j(t')\ , \tag{6}$$

$$\frac{d\phi_j(t')}{dt'} = \frac{2}{N}x(t')\chi_j(t')\ . \tag{7}$$

In Eqs. (6) and (7) $N = \tau_{\text{int}}/\tau_0$ denotes the number of atoms simultaneously present in the cavity during the flying time τ_{int} (for $N = 1$ one always has one atom in the cavity, namely, a new atom is injected exactly when the previous one leaves the cavity). We also introduced the rescaled field

$$x(t') \equiv g\tau_{\text{int}}a(t')\ , \tag{8}$$

and the parameters

$$N_{\text{ex}} \equiv \frac{1}{k\tau_0}\ , \qquad \theta \equiv g\tau_{\text{int}}\sqrt{N_{\text{ex}}}\ . \tag{9}$$

The system depends on the set of parameters $\{N, N_{\text{ex}}, \theta\}$, where N_{ex} represents the number of atoms crossing the cavity during the photon lifetime k^{-1} and θ is the pumping parameter. In the $N = 1$ case Eqs. (6) and (7) are equivalent to the equation of a damped kicked pendulum of mass $m = k/(g^2\tau_{\text{int}})$

$$m\frac{d^2\phi}{dt'^2} + \frac{m}{2N_{\text{ex}}}\frac{d\phi}{dt'} - \frac{1}{N_{\text{ex}}}\sin\phi = 0\ , \tag{10}$$

where $\phi(t) \equiv \phi_j(t)$ for $t' \in [j, j+1)$ and at each "kick" occurring at $t' = j = 0, 1, \ldots$ the Bloch angle is set equal to zero, whereas the angular velocity $d\phi/dt'$ is continuous.

In the following we focus attention only on the dynamics of the field, and consider the evolution at the "stroboscopic" times $t = n\tau_{\text{int}}$, with n integer, averaging the field on each period τ_{int}. Equations (4) and (5) in the one-atom case ($\tau_0 = \tau_{\text{int}}$) lead to the following energy balance for $t \in [j\tau_0, (j+1)\tau_0 - \epsilon]$:

$$\frac{d}{dt}\left(a^2 + \frac{1}{2}\cos\phi_j(t)\right) = -ka^2\ . \tag{11}$$

Integrating in the interval $[j\tau_0, (j+1)\tau_0 - \epsilon]$ between stroboscopic times $t = j\tau_0$, we get

$$a^2((j+1)\tau_0) - a^2(j\tau_0) + \tfrac{1}{2}\{\cos\phi[(j+1)\tau_0 - \epsilon] - \cos\phi(j\tau_0)\} = -\int_{j\tau_0}^{(j+1)\tau_0}ka^2dt\ . \tag{12}$$

Using the "periodic kick" boundaries $\phi(j\tau_0) = 0$ and the stationary condition $a^2((j+1)\tau_0) = a^2(j\tau_0)$, we obtain

$$\sin^2\bar{x} - m\overline{x^2} = 0\ , \tag{13}$$

where $\bar{f}$ denotes the time average of f over τ_0. In terms of the relative fluctuations $\sigma^2 = \overline{\Delta x^2}/\bar{x}^2$, Eq. (13) is rewritten as follows:

$$\sin^2\bar{x} - m\bar{x}^2\left(1 + \sigma^2\right) = 0\ . \tag{14}$$

Upon neglecting fluctuations σ^2 Eq. (14) becomes the usual steady-state equation, which gives the fixed points of the evolution. This analysis can be extended to the $N > 1$ atom case, leading to the simple generalization [7]

$$N\frac{\sin^2(\bar{x})}{\bar{x}} - m\bar{x} = 0\ . \tag{15}$$

Among the fixed points that are roots of Eq. (15), those with positive derivative of $\sin^2 x$ are unconditionally unstable, whereas the others are conditionally stable, namely their stability depends on the values of the set of parameters $\{N, N_{\text{ex}}, \theta\}$. Typically, one has that for increasing θ, fixed N, N_{ex}, and initial field x_0, the fixed point becomes unstable and is replaced by a period-2 attractor; then it is followed by a period-doubling route to chaos. However, changing the values of the parameters N, N_{ex} and the initial condition x_0, situations different from the customary period doubling can occur; we also find the following scenarios: (i) the fixed point evolves towards a quasiperiodic orbit; (ii) the long-time solution jumps between chaotic orbits; (iii) a fixed point jumps to a different fixed point; (iv) a period-2 jumps to a different period-2. All of these mechanisms are a consequence of the fact that the basins of attraction are very sensitive to the value of θ itself.

In Fig. 1 an example of period doubling is presented

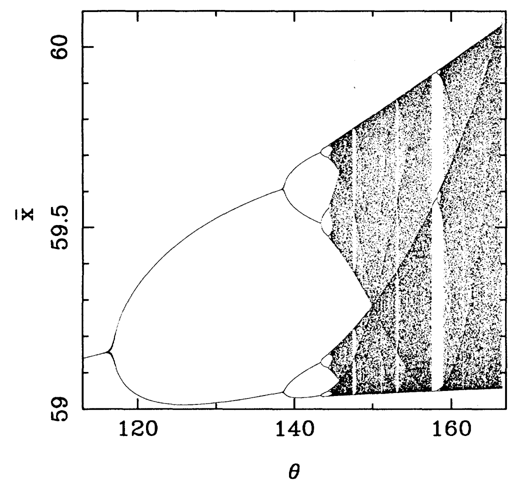

FIG. 1. Onset of chaos for $N = 1$, $N_{\rm ex} = 50$, and initial field $x_0 = 58$. The last 100 stroboscopic points of the time evolution of $\bar{x}$ after a transient $t_0' = 200$ are plotted for each value of the pumping parameter θ.

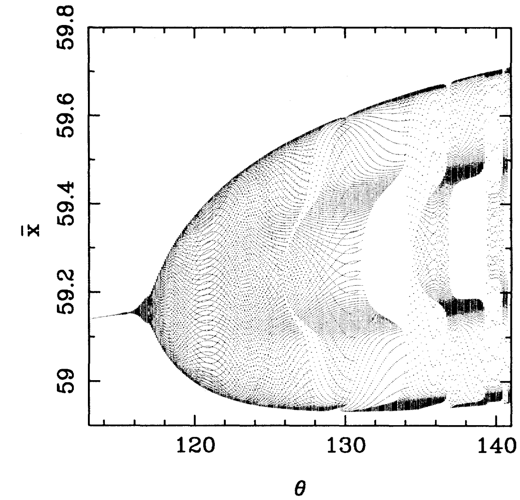

FIG. 3. The same as in Fig. 1 for $N = 2$, $N_{\rm ex} = 50$, and initial field $x_0 = 58$.

for $N = 1$, $N_{\rm ex} = 50$, and $x_0 = 58$. The plot is obtained upon fixing θ and evaluating the evolution of the average field $\bar{x}$ after a transient $t_0' = 200$; then the subsequent stroboscopic 100 values of $\bar{x}$ are plotted on the vertical axis corresponding to the chosen θ. The plot shows a typical Feigenbaum-like [8] sequence, with the first occurrences of bifurcation at $\theta \simeq 116$ and the onset of chaos at $\theta \simeq 144.7$. For greater $\theta \simeq 147.4, 152.9, 157.5$ one has three almost equally spaced windows where the system is nonchaotic again, with attractors of period 3, 6, etc. The present results are very similar to those presented in Ref. [9], even though a quantitative comparison is not possible, because the underlying semiclassical models are quite different (the model of Ref. [9] is more appropriate to the case of very low fluxes).

For very large values of the pumping parameter a totally unexpected behavior appears (see Fig. 2). For $\theta = \theta_c \simeq 166.5$ an abrupt transition of the field is exhibited, through a sequence of jumps between different chaotic attractors centered around the roots of Eq. (15). When θ is increased a greater number of roots of Eq. (15) are available: the system becomes multistable and the roots larger than x_0 attract the field through a sort of "classical tunneling" mechanism. In other words, the basins of attraction of the unstable fixed points—actually chaotic attractors for the current values of the parameters—are so tightly interwoven that the system becomes strongly sensitive to θ; this reflects on a critical dependence on the initial condition, and the system tunnels towards totally unpredictable attractors. In this situation an accurate check of the numerical integration time dt is in order. From Eq. (10) one can see that the time-evolution scale

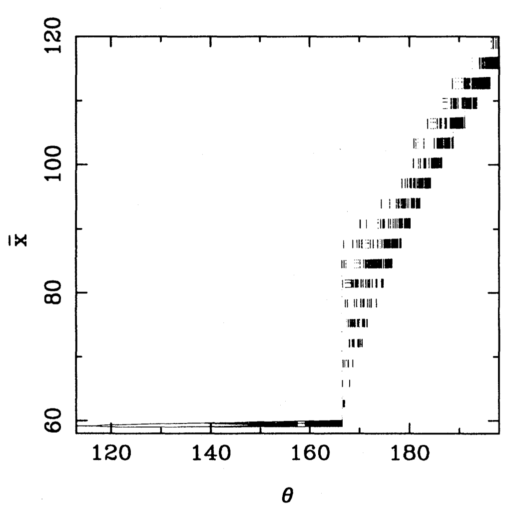

FIG. 2. The same as in Fig. 1 for a wider range of pumping parameter.

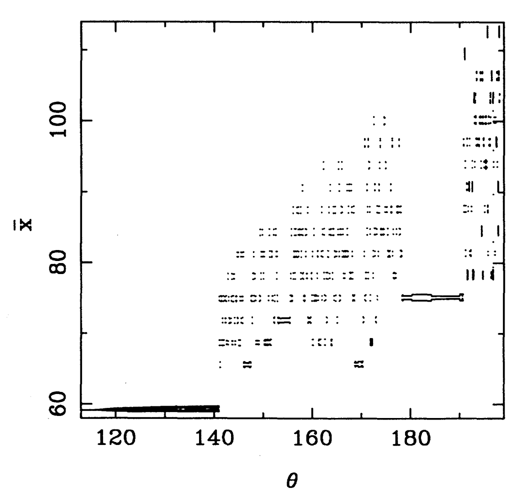

FIG. 4. The same as in Fig. 3 for a wider range of pumping parameter.

is $T \simeq 2\pi\sqrt{mN_{\rm ex}}$. For the numerical results presented here integration steps $dt \ll T$ have been used; in this limit, for varying dt one can check that θ_c is unchanged, whereas both the tiny details of the time evolution and the particular attractor that is reached may be slightly different. Notice that the average tunneling time is of the order of $N_{\rm ex}$, much less than the maximum time scale $t'_{\rm sc} \simeq N_{\rm ex}^2$ needed for agreement between classical and quantum evolutions [6, 10].

As regards the value of θ_c as a function of $N_{\rm ex}$ we have an indication about an analytical dependence in the form of a quadratic power law for x_0 near the center of the set of roots. On the other hand, one should notice that for low fields (where, however, the semiclassical approach is no longer realistic) the transition becomes broadened, and the field starts jumping between fixed points instead of chaotic attractors.

In Fig. 3 the case $N = 2$ for the same previous values of $N_{\rm ex}$ and x_0 are plotted. The period-doubling route to chaos for $N = 1$ is replaced here by an opening up of the fixed point into a quasiperiodic attractor. Then the system undergoes a sharp transition at $\theta_c \simeq 141$ towards a multistable behavior (see Fig. 4). In this case the field jumps between period-2 attractors, and between chaotic attractors for very large θ. A wide window for $178.3 < \theta < 190.7$ contains an isolated very stable period-2 attractor around $\bar{x} = 75$.

In conclusion, we have studied an exact semiclassical model for a regularly injected micromaser. In the high-flux regime, after the customary period-doubling route to chaos, for very large values of the pumping parameter the system undergoes a new type of "phase transition," the field jumping between different chaotic attractors. Also other unexpected scenarios, different from the usual period-doubling route to chaos, have been shown. The system is in a strongly nonlinear regime and exhibits a complicated mixture of periodic orbits, chaos, and multistability, with competition between different attractors (analogous mechanisms of competition between attractors have been found in a study of the bistable behavior of the circular map [11]). The comparison between the $N = 1$ and $N = 2$ cases shows the dramatic differences introduced in the dynamics by the presence of many atoms in the cavity. This is the signature of cooperative-in-time mechanisms underlying the field amplification in a physical situation that lies between the microscopic maser and the ordinary many-atom maser.

[1] P. Filipowicz, J. Javainen, and P. Meystre, J. Opt. Soc. Am. B **3**, 906 (1986).
[2] T. A. B. Kennedy, P. Meystre, and E. M. Wright, in *Fundamentals of Quantum Optics II*, edited by F. Ehlotzky (Springer-Verlag, Heidelberg, 1987), p. 157.
[3] G. Rempe, H. Walter, and N. Klein, Phys. Rev. Lett. **58**, 353 (1987).
[4] G. Rempe, F. Schmidt-Kaler, and H. Walter, Phys. Rev. Lett. **64**, 2783 (1990).
[5] P. Filipowicz, J. Javainen, and P. Meystre, Phys. Rev. A **34**, 3077 (1986).
[6] A. M. Guzman, P. Meystre, and E. M. Wright, Phys. Rev. A **40**, 2471 (1989).
[7] L. Davidovich (private communication).
[8] M. J. Feigenbaum, J. Stat. Phys. **21**, 669 (1979).
[9] P. Meystre and E. M. Wright, in *Chaos, Noise and Fractals*, edited by R. Pike and L. A. Lugiato (Hilger, Bristol, 1987), p. 102.
[10] For a detailed numerical study on the time scale $t_{\rm sc}$ in a kicked top system, see also G. M. D'Ariano, L. R. Evangelista, and M. Saraceno, Phys. Rev. A **45**, 3646 (1992).
[11] R. V. Jensen and E. R. Jessup, J. Stat. Phys. **43**, 369 (1986).