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Hamiltonians for the photon-number-phase amplifiers

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Hamiltonians attaining number-phase amplification are presented. The amplification is driven by a peculiar dependence of the polarization on the phase of the field in a limited frequency range.

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The photon-number amplifier is a device which, ideally, would effect the state transformation

$$|n\rangle \rightarrow |Gn\rangle \ , \tag{1}$$

for an integer $G > 1$, thus preserving the direct-detection signal-to-noise ratio. This kind of amplifier has been recently proposed by Yuen [1–4], who suggested physical approximate schemes and pointed out many applications in quantum optics, using either nonclassical or conventional light. Essentially, the number amplifier would provide the appropriate optical preamplification to make an efficient direct-detection receiver and transceiver [3], enabling the realization of nearly lossless optical taps. In particular, it is suited to design transparent optical networks, especially for short-haul communications which utilize nearly number states in order to achieve the ultimate channel capacity of the field [4].

In this article I present number-amplifying Hamiltonians which could be exploited in planning a real device. In the present treatment the amplifier is considered ideal also with respect to the amplification of the phase $\hat{\Phi}$, the observable which is conjugated with the number. In a way analogous to phase-sensitive amplification—where two conjugated quadrature components of the electric field are inversely amplified—here the phase and number observables both undergo ideal amplification, preserving their uncertainty product. In this fashion the state transformation (1), involving only number eigenstates, becomes too restrictive. Therefore, as a definition of the number amplification, I adopt the Heisenberg evolution

$$\hat{n} \rightarrow G\hat{n} \ , \tag{2}$$

whereas the state of the field is described in general by a density matrix $\hat{\rho}$. Another point which I consider is the possibility of deamplifying $\hat{n}$, namely of amplifying $\hat{n}$ by a noninteger gain $G < 1$. The integer-valued nature of $\hat{n}$ breaks the symmetry between amplification and deamplification, in that it forbids exact deamplification. The transformation (2) can be generalized to the following one:

$$\hat{n} \rightarrow [G\hat{n}] \ , \tag{3}$$

which is defined for real gains G ($[x]$ denotes the integer part of x). The transformation (3) coincides with (2) for integer G. In the following I will show that only the cases $G \equiv$(integer) and $G \equiv$(inverse of integer) lead to unitary evolutions, the second case corresponding to the ideal amplifier used in the output-input reversed direction. [Notice that the composition of maps (3) with different G—which corresponds to put amplifiers in series—does not lead in general to a map of the same form (3) with rational G.]

Regarding the amplification of the phase, a good definition of the phase variable should be adopted, which gives the correct statistics for all functions of the phase variable. This is provided by the optimum *probability operator measure* (POM) [7]

$$dP(\theta) = \sum_{n,n'=0}^{\infty} |n\rangle e^{i(n-n')\theta} \frac{d\theta}{2\pi} \langle n'| \ , \tag{4}$$

which defines the phase operator as

$$\hat{\Phi} = \int_{-\pi}^{\pi} \theta \, dP(\theta) \ , \tag{5}$$

and accordingly gives the functions of the phase variable as

$$\hat{f} = \int_{-\pi}^{\pi} f(\theta) P(d\theta) \ . \tag{6}$$

The mean value of the quantity $\hat{f}$ in a state $\hat{\rho}$ is evaluated, as usual, as $\mathrm{Tr}[\hat{\rho}\hat{f}]$. One can see that the operators proposed previously by Susskind-Glogower [8] give the correct mean value for $f(\phi) = e^{ir\phi}$, for integer r

$$(\hat{E}_{\pm})^r = \int_{-\pi}^{\pi} e^{\mp ir\theta} P(d\theta) \ . \tag{7}$$

$\hat{E}_{\pm}$ being the shift operators $\hat{E}_{\pm}|n\rangle = |n \pm 1\rangle$ ($a = \hat{E}^{-}\hat{n}^{1/2}$ gives the polar decomposition of the annihilation operator). One has also $\frac{1}{2}(\hat{E}_{-} + \hat{E}_{+}) = \hat{f}$ where in $f(\phi) = \cos\phi$, and $\frac{1}{2i}(\hat{E}_{-} - \hat{E}_{+}) = \hat{f}$ where $f(\phi) = \sin\phi$, but $[\frac{1}{2}(\hat{E}_{-} + \hat{E}_{+})]^r \neq \hat{f}$ with $f(\phi) = \cos^r\phi$, and every other operator functional $\hat{f}$ should be evaluated through Eq. (6) in the general case. In the above framework, I say that the transformation $\mathcal{S}_H^{(r)}$ in the Heisenberg picture amplifies the phase variable by the gain r when

$$\mathcal{S}_H^{(r)}(\hat{f}) = \hat{g} \ , \quad g(\phi) = f(r\phi) \ , \tag{8}$$

for periodic functions $f(\theta + 2\pi) = f(\theta)$. Analogously, $\mathcal{S}_H^{(1/r)}$ deamplifies the phase variable by the gain $1/r$

when

$$\mathcal{S}_H^{(1/r)}(\hat{f}) = \hat{g}\ , \quad g(\phi) = f(\phi/r)\ . \tag{9}$$

for $f(\theta + 2\pi/r) = f(\theta)$.

One of the major difficulties encountered in the quantum-mechanical treatment of number amplification is related to the nonunitarity of the transformations (1)–(3). This can be simply understood by considering that $\{|Gn\rangle\}$ span only a proper subspace of the Fock space $\mathcal{F}$ (which is spanned by $\{|n\rangle\}$). As already suggested in Ref. [2], a way to overcome this problem is to consider an auxiliary degree of freedom and construct a unitary operator on an enlarged Hilbert space $\mathcal{F} \otimes \mathcal{H}'$, $\mathcal{H}'$ being infinite dimensional. Here I follow this procedure by first noticing that the transformation (3) is a unit-preserving *completely positive map* (CP map) [5, 6]. CP maps are used to describe the subdynamics of the open quantum systems. A unit-preserving CP map has the general form

$$\mathcal{T}(\hat{O}) = \sum_\alpha \hat{V}_\alpha^\dagger \hat{O} \hat{V}_\alpha\ , \tag{10}$$

where

$$\sum_\alpha \hat{V}_\alpha^\dagger \hat{V}_\alpha = 1\ . \tag{11}$$

The space of the CP maps is closed under (i) convex combination $\sum_i p_i \mathcal{T}_i$; (ii) composition $\mathcal{T}_1\mathcal{T}_2$; (iii) tensor product $\mathcal{T}_1 \otimes \mathcal{T}_2$; (iv) partial trace; namely, if $\mathcal{T}$ is CP on $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $\hat{\rho}_2$ is a density operator on $\mathcal{F}_2$, then

$$\mathcal{T}_1(\hat{O}) = \mathrm{Tr}_2[\hat{\rho}_2 \mathcal{T}(\hat{O} \otimes \hat{1})] \tag{12}$$

is CP on $\mathcal{F}_1$. The last point means that if one has a unitary evolution in a closed system and if subdynamics on a (open) subsystem can be defined—i.e., partial trace on the subsystem degrees of freedom—then these subdynamics are CP maps. In some cases it is also possible to reconstruct unitary evolutions on enlarged systems corresponding to a given CP map. For example, if $\hat{V}_\alpha$ satisfy the orthogonality relations

$$\hat{V}_\alpha \hat{V}_\beta^\dagger = \delta_{\alpha\beta}\ , \tag{13}$$

then the following operator is unitary on $\mathcal{F}_1 \otimes \mathcal{F}_2$, ($\mathcal{F}_1 \equiv \mathcal{F}_2$):

$$\hat{U} = \sum_\alpha \hat{V}_\alpha \otimes \hat{W}_\alpha^\dagger\ , \quad \hat{W}_\alpha = \hat{A}\hat{V}_\alpha\hat{B}\ , \tag{14}$$

$\hat{A}$ and $\hat{B}$ being unitary operators on $\mathcal{F}_2$. The CP map in Eq. (10) corresponds to the partial trace (12) where $\mathcal{T}$ is the unitary evolution given by the operator (14). Here I show that this is exactly the case of the phase-number amplification given by Eqs. (3), (8), and (9).

From Eq. (7) one can see that the transformation attaining the number deamplification (3) ($G^{-1} \equiv r$, an integer) and the phase amplification (8) should transform the shift operators as follows:

$$\mathcal{S}_H^{(r)}(\hat{E}_\pm) = (\hat{E}_\pm)^r\ . \tag{15}$$

An outlook on Eq. (15) leads to the following CP map [9]:

$$\mathcal{S}_H^{(r)}(\hat{O}) = \sum_{\lambda=0}^{r-1} (\hat{S}_\lambda^{(r)})^\dagger \hat{O} \hat{S}_\lambda^{(r)}\ , \qquad \hat{S}_\lambda^{(r)} = \sum_{n=0}^{\infty} |n\rangle\langle nr+\lambda|\ . \tag{16}$$

The operators $\hat{S}_\lambda^{(r)}$ satisfy the relations

$$\sum_{\lambda=0}^{r-1} (\hat{S}_\lambda^{(r)})^\dagger \hat{S}_\lambda^{(r)} = 1 \text{ (completeness)}\ , \tag{17}$$

$$\hat{S}_\lambda^{(r)} (\hat{S}_\mu^{(r)})^\dagger = \delta_{\lambda\mu} \text{ (orthogonality)}\ , \tag{18}$$

$$\hat{S}_\lambda^{(r)} \hat{S}_\mu^{(s)} = \hat{S}_{\lambda s+\mu}^{(rs)} \text{ (composition)}\ . \tag{19}$$

Equation (3), namely $\mathcal{S}_H^{(r)}(\hat{n}) = [\hat{n}/r]$, follows trivially from Eq. (15). The check of the phase amplification (8) is more involved:

$$\begin{aligned}
\mathcal{S}_H^{(1/r)}(\hat{f}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) \sum_{n,m,p,q=0}^{\infty} \sum_{\lambda=0}^{r-1} |c_\lambda|^2 e^{i(n-m)\theta} |p\rangle\langle pr+\lambda|n\rangle\langle m|qr+\lambda\rangle\langle q| \\
&= \sum_{\lambda=0}^{r-1} |c_\lambda|^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta f(\theta) \sum_{p,q=0}^{\infty} e^{i(p-q)r\theta}|p\rangle\langle q| = \frac{1}{2\pi}\int_{-r\pi}^{r\pi} d\theta \frac{1}{r} f(\theta/r) \sum_{p,q=0}^{\infty} e^{i(p-q)\theta}|p\rangle\langle q| \\
&= \hat{g}\ , \quad g(\phi) = f(\phi/r)\ .
\end{aligned} \tag{20}$$

Equations (17) and (18) allow one to obtain a unitary evolution corresponding to $\mathcal{S}_H^{(r)}$ by using the construction (14). I consider the following operator $\hat{U}_{(r)}$ acting on the Fock space $\mathcal{F} \otimes \mathcal{F}$:

$$\hat{U}_{(r)} = \sum_{\lambda=0}^{r-1} \hat{S}_\lambda^{(r)} \otimes (\hat{S}_\lambda^{(r)})^\dagger\ . \tag{21}$$

Here the particular choice $\hat{A} = \hat{B} = 1$ in Eq.(14) leads to a scheme of amplifier where the two input fields are inversely amplified. The subdynamics of the first field correspond to $\mathcal{S}_H^{(r)}$:

$$\langle \hat{U}_{(r)}^\dagger \hat{O}_1 \hat{U}_{(r)} \rangle = \mathrm{Tr}[(\hat{\rho}_1 \otimes \hat{\rho}_2)\hat{U}_{(r)}^\dagger (\hat{O}_1 \otimes \hat{1})\hat{U}_{(r)}] = \mathrm{Tr}_1[\hat{\rho}_1 \mathcal{S}_H^{(r)}(\hat{O}_1)]\ . \tag{22}$$

The action of the operator $\hat{U}_{(r)}$ in Eq. (21) on a number eigenstate is $\hat{U}_{(r)}|n,m\rangle = |[n/r], mr + \langle n/r\rangle\rangle$, where $\langle x\rangle = x - [x]$ denotes the fractional part of x. In a more symmetrical form one has

$$\hat{U}_{(G^{-1})}|n,m\rangle = |[Gn] + G\langle G^{-1}m\rangle, [G^{-1}m] + G^{-1}\langle Gn\rangle\rangle \ , \tag{23}$$

which leads to integer numbers of photons only if either G or G^{-1} is integer. The partial trace on the first field

$$\langle\hat{O}_2\rangle = \mathrm{Tr}[(\hat{\rho}_1\otimes\hat{\rho}_2)\hat{U}^{\dagger}_{(r)}(\hat{1}\otimes\hat{O}_2)\hat{U}_{(r)}] = \mathrm{Tr}_2\left(\hat{\rho}_2\sum_{\lambda=0}^{r-1}(\hat{V}^{(r)}_{\lambda})^{\dagger}\hat{O}_2\hat{V}^{(r)}_{\lambda}\right) \tag{24}$$

produces another CP map (see also Ref. [10])

$$\mathcal{S}^{(1/r)}_{H}(\hat{O}) = \sum_{\lambda=0}^{r-1}(\hat{V}^{(r)}_{\lambda})^{\dagger}\hat{O}\hat{V}^{(r)}_{\lambda}\ , \quad \hat{V}^{(r)}_{\lambda} = c_{\lambda}(\hat{S}^{(r)}_{\lambda})^{\dagger}\ , \quad c_{\lambda} = \{\mathrm{Tr}_1[\hat{\rho}_1(\hat{S}^{(r)}_{\lambda})^{\dagger}\hat{S}^{(r)}_{\lambda}]\}^{1/2}\ , \tag{25}$$

which, due to the form of operators $\hat{V}^{(r)}_{\lambda}$ depends on the state $\hat{\rho}_1$ of the other field. The case of $\hat{\rho}_1$ equal to the vacuum state gives

$$\mathcal{S}^{(1/r)}_{H}(\hat{O}) = \hat{V}_0^{\dagger}\hat{O}\hat{V}_0\ , \quad \hat{V}_0 \equiv (\hat{S}^{(r)}_0)^{\dagger} = \sum_{n=0}^{\infty}|rn\rangle\langle n| \tag{26}$$

and corresponds to exact number amplification $\mathcal{S}^{(1/r)}_{H}(\hat{n}) = r\hat{n}$. The action of $\mathcal{S}^{(1/r)}_{H}$ on the phase variable attains the deamplification (9)

$$\begin{aligned}\mathcal{S}^{(1/r)}_{H}(\hat{f}) &= \int_{-\pi}^{\pi}\frac{d\theta}{2\pi}f(\theta)\sum_{n,m,p,q=0}^{\infty}\sum_{\lambda=0}^{r-1}|c_{\lambda}|^2e^{i(n-m)\theta}|p\rangle\langle pr+\lambda|n\rangle\langle m|qr+\lambda\rangle\langle q|\\ &= \sum_{\lambda=0}^{r-1}|c_{\lambda}|^2\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}f(\theta)\sum_{p,q=0}^{\infty}e^{i(p-q)r\theta}|p\rangle\langle q| = \frac{1}{2\pi}\int_{-r\pi}^{r\pi}d\theta\frac{1}{r}f(\theta/r)\sum_{p,q=0}^{\infty}e^{i(p-q)\theta}|p\rangle\langle q|\\ &= \hat{g}\ , \quad g(\phi) = f(\phi/r)\ .\end{aligned} \tag{27}$$

The operator $\hat{U}_{(r)}$ thus provides a unitary evolution on $\mathcal{F}\otimes\mathcal{F}$ where the subdynamics on the two Fock spaces correspond to inverse phase-number amplifications. Notice that $\hat{U}^{\dagger}_{(G)} = \hat{U}^{-1}_{(G)} = \hat{U}_{(G^{-1})}$, and the operator inversion corresponds to interchanging the roles of the two fields. The broken symmetry between number amplification and deamplification here reflects on the fact that the two fields undergo different CP maps, one of them being dependent on the state of the fields, whereas the other being independent [see Eqs. (16) and (25)].

In order to obtain the Hamiltonian of the amplifier I rewrite the operator (21) in exponential form. Upon denoting the particle operators of the two fields by $a^{\dagger}$ and $b^{\dagger}$, namely

$$|n,m\rangle = \frac{(a^{\dagger})^n}{\sqrt{n!}}\frac{(b^{\dagger})^m}{\sqrt{m!}}|0,0\rangle\ , \tag{28}$$

the operator $\hat{U}_{(r)}$ can be written as follows:

$$\hat{U}_{(r)} = \exp\left(\frac{\pi}{2}(a^{\dagger}b - b^{\dagger}a)\right)\exp\left(-\frac{\pi}{2}(a^{\dagger}_{(r)}b - b^{\dagger}a_{(r)})\right)\ , \tag{29}$$

where $a^{\dagger}_{(r)} = \mathcal{S}^{(r)}_{H}(a^{\dagger})$ is a boson operator creating r photons at a time [11]

$$a^{\dagger}_{(r)}|n\rangle = \sqrt{[n/r]+1}|n+r\rangle\ . \tag{30}$$

The representation (23) of the operator $\hat{U}_{(r)}$ in Eq. (29) can directly be checked using Eqs. (28) and (30). The product of exponentials in Eq. (29) corresponds to the series of two four-port devices. The first exponential describes a parametric frequency converter from ω_a to ω_b, with classical pump at frequency $\Omega' = \omega_a - \omega_b$, $\omega_{a,b}$ being the frequencies of the two quantum fields. The second device corresponds to a phase-number amplifier which simultaneously converts frequencies, having a classical pump at frequency $\Omega = r\omega_a - \omega_b$. The corresponding interaction Hamiltonian has the form

$$\hat{H}_I = -ik(a^{\dagger}_{(r)}be^{-i\Omega t} - b^{\dagger}a_{(r)}e^{i\Omega t}) \tag{31}$$

and the interaction length L is given by $kL = \pi/2$, k being the gain coefficient per unit length.

The Hamiltonian (31) is quite complicated, due to the presence of the multiboson operators $a_{(r)}$ and $a^{\dagger}_{(r)}$. However, for a high average number of photons $\langle a^{\dagger}a\rangle \gg r$ the multiboson operators behave asymptotically as

$$\begin{aligned}a^{\dagger}_{(r)} &= \left(\frac{[a^{\dagger}a/r](a^{\dagger}a-r)!}{a^{\dagger}a!}\right)^{1/2}(a^{\dagger})^r\\ &= [a^{\dagger}a/r]^{1/2}(a^{\dagger}a-r+1)^{-1/2}r^{\frac{1}{2}}\hat{\kappa}_r a^{\dagger} \sim \hat{\kappa}_r a^{\dagger},\end{aligned} \tag{32}$$

where

$$\kappa_r(\theta) = r^{-\frac{1}{2}}e^{-i(r-1)\theta}\ . \tag{33}$$

Taking into account the pumping field also, the phase-number amplifier would require a medium with a $\chi^{(2)}$ susceptibility and an interaction Hamiltonian of the form

$$\hat{H}_I \sim \lambda \hat{\kappa}_r a^\dagger bc + \text{H.c.}\,, \tag{34}$$

c denoting the annihilator of the pumping field. From Eq. (34) it follows that in order to attain phase-number amplification one should use a $\chi^{(2)}$ medium having polarization which depends on the phase of the field according to (33) in a limited frequency range containing ω_a. The amplifier gain r is involved only in the phase factor (33), whereas the interaction length has to be tuned at the complete conversion value $L = \pi/2\lambda I_c^{1/2}$, I_c being the average power flux of the (classical undepleted) pump and $\lambda \propto \chi^{(2)}$. For $r = 1$ the usual parametric frequency converter is obtained, and the two exponentials in (29) cancel each other, leading to the identity operator. For $r > 1$ the phase-dependent coupling in Eq. (34) may also be regarded in terms of an intensity dependent coupling for a $\chi^{(r+1)}$ medium (as one can simply check using the polar decomposition of the particle operators). In practice, for a constant coupling one can tune the interaction length as a function of the intensity, thus obtaining approximate number amplification in the average values. For example, for $r = 2$ one has $\hat{\kappa}_2 = 2^{-\frac{1}{2}}\hat{E}^+ = (2a^\dagger a)^{-\frac{1}{2}}a^\dagger$. In this way, if the usual degenerate-four-wave-mixing Hamiltonian (34) is considered

$$\hat{H}_I = \kappa (a^\dagger)^2 bc + \text{H.c.}\,, \tag{35}$$

($\kappa \propto \chi^{(3)}$) an approximate gain-2 number amplification can be attained upon choosing an interaction length $L = \pi/(2\kappa\sqrt{I_a I_c})$. Similar arguments hold for analogous $\chi^{(r+1)}$ amplifying media for $r > 2$, as in the resonance fluorescence scheme proposed in Ref. [1]. On the other hand, the analytic form of the Hamiltonian (34) also may suggest that improvements in the ideal behavior of the amplifier could be attained through modulation of the nonlinear susceptibility at wavelengths submultiple of that carrying the amplified mode [see Eq. (33)]. Very intense, localized, and highly nonlinear susceptibilities could be obtained using quantum wells: this may prefigure a quasiideal amplifier in the form of a heterostructure-designed device. Work is in progress along these lines.

I conclude with some remarks regarding the ideal photon-*number duplicator*, which in some respect is very similar to the gain-2 photon-number amplifier. Instead of amplifying the number of photons, the ideal duplicator produces two copies of the same input state for eigenstates of the number operator. Such a device would be extremely useful in local area network applications, because it provides a convenient realization of the quantum nondemolition measurement of the photon number, and in addition provides lossless optical taps superior to the amplifier tap [4]. For the number duplicator the unitary transformation can be obtained starting from the CP map $\mathcal{S}_H(\hat{E}_\pm) = \hat{E}_\pm \otimes \hat{E}_\pm$, which is strictly analogous to (15). The technique of enlarging the Fock space by using an extra auxiliary field leads to the Hamiltonian [10]

$$\hat{H}_I = -i\kappa a^\dagger (\max\{b^\dagger b, c^\dagger c\} + 1)^{-1/2} bc e^{-i\Omega t} + \text{H.c.}, \tag{36}$$

where the classical undepleted pump has frequency $\Omega = \omega_a - \omega_b - \omega_c$. For an interaction length $L = \pi/(2\kappa)$ the Hamiltonian (36) attains the duplicating transformation $\hat{U}|n,0,0\rangle = |0,n,n\rangle$ (the general representation of $\hat{U}$ is more involved). When operating in two vacua, one can substitute the function $\max\{b^\dagger b, c^\dagger c\}$ in the Hamiltonian (36) with either $b^\dagger b$ or $c^\dagger c$, without changing the output. In this fashion the Hamiltonian (36) becomes quite similar to the Hamiltonian (34), the main difference being that the field in the phase-dependent frequency range now splits into the two nondegenerate modes bearing the replica states.

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