Amplitude squeezing through photon fractioning

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A prototype mechanism for producing amplitude-squeezed states, based on the so-called "photon fractioning" procedure, is investigated. The coherent state is connected with a quasinumber eigenstate through an intermediate density matrix involving multiphoton processes. The enhancement in the phase fluctuations and the simultaneous reduction of the noise in the photon number result from two competing limits; the "bright limit," which is semiclassical and requires highly excited states, and the photon fractioning limit, which is pure quantum. Asymptotical evaluations of the number probability distribution and of the Q function show that the mixed states are very close to the exact number-phase minimum-uncertainty states.

I. INTRODUCTION

Quantum squeezed and number-phase minimum uncertainty states of light are presently of great interest in quantum optics. Their interest is related to the role of these states in fundamental physics as well as in the possibility of overcoming the limit due to quantum fluctuations on the accuracy of measurements and/or transmissions of information.

Squeezed states[1-3] are a special kind of coherent states, in which the ratio of variances of the electric and magnetic fields can be altered, reducing one variance to the detriment of the other. They can lead to substantial improvements of the sensitivity in high-precision interferometric measurements.

In number-phase minimum uncertainty states[4,5] (NUS)—or amplitude-squeezed states[6]—the quantum noise is shared between the photon number operator $\hat{n}$ and the sine operator $\hat{S}$: this latter, for high mean photon numbers (bright limit), can be interpreted as the quantum analogs of the classical oscillator phase. (As an example of NUS one can consider the number eigenstates, which are nothing but the degenerate case of NUS having zero number fluctuations and completely random phase.) NUS are interesting for communication systems in which information is coded through photon number and detection is obtained by photon counting.[7] In fact, they reduce the number fluctuations without affecting the mean photon number. Therefore NUS states lead to a quantum signal-to-noise ratio which can be made in principle arbitrarily high, whereas, in the squeezed states, the maximum ratio is restricted by the average photon number.[8]

Although the mathematical construction of NUS began long ago,[5] only in recent years was attention on the production of such states stimulated by the aforementioned motivations. Since a mathematically defined NUS cannot be obtained from a coherent state through a unitary evolution, no practical physical scheme has been proposed to prepare these states. However, methods have been suggested to generate NUS-like states which exhibit enhanced phase fluctuations and low number noise. Such methods are based on a nonlinear Mach-Zehnder interferometer having Kerr media placed in one (or both) of the arms.[6,7,9-13] The nonlinear interaction in the Kerr medium produces four-photon processes, which generate self-phase modulation of light via a feedback interferometric mechanism.

The aim of this paper is to give insights into the interplay of multiphoton processes and enhanced fluctuations leading to NUS. I show that the introduction of k-photon observables enhances phase fluctuations and reduces the number noise, by multiplying the phases by a factor k and dividing the numbers correspondingly. The k-photon observables do not correspond to the physical measured ones and one has to construct quantum states having probability distributions for the usual (one-photon) observables reproducing the multiphoton distributions of a given quantum state. This last step has been achieved by D'Ariano and Sterpi[14] and leads to states built up in terms of density matrices (the so-called fractional-photon states). They are characterized in terms of the "fractional-photon index" $t^{-1}=h/k$ (Ref. 15) (the matrix elements of h-photon operators between k-photon states depend only on $t=k/h$) which turns out to be proportional to the ratio between the variances of the number and the phase. This result is in agreement with the interpretation of t as a measure of the sub-Poisson character of the number distribution.[14] However, the physical interpretation of t as a measure of $\Delta n/\Delta\Phi$ is more appropriate since it comes directly from the definition of t in the operator formalism.

The fractional-photon states are briefly presented in Sec. II, where a short recall of the essential ingredients is also given. The mechanism of amplitude squeezing is illustrated in Sec. III, after some remarks about the definition of the phase operator and the related commutations. Particular attention is devoted to the $1/r$-coherent states, i.e., those statistical states which exhibit the r-observables probability distributions of the coherent states. A direct calculation of the number and the phase uncertainties Δn and $\Delta\Phi$ leads to the announced results $\Delta n\,\Delta\Phi\simeq\frac{1}{2}$ and $\Delta n/\Delta\Phi\propto t$.

The probability distributions are analyzed in Sec. IV.

Numerical results show that the quasiprobability distribution (Q-function) is typically expanded in the tangential direction, as can be expected of an amplitude-squeezed state.[11] For large mean photon numbers and small $t=1/r$ the tangential shape is asymptotically Gaussian, with variance increasing as t^{-1}. On the other hand, the number distribution becomes more and more sharply peaked on decreasing t. These results indicate that the mixed $1/r$-coherent states behave very similarly to the mathematical states of Jackiw.[5] The present states, however, involve multiphoton processes and provide a new suggestive physical description of the amplitude squeezing. This phenomenon results from two competing limits, one of which—the bright limit—is semiclassical, whereas the other—the photon fractioning—is a quantum one. In the concluding remarks this last point is discussed in detail.

II. FRACTIONAL-PHOTON STATES

Fractional-photon states[14,15] are introduced through their probability distributions, resorting to the general statement that the matrix elements of h-photon observables acting on k-photon states, depend on h and k only through their ratio $t=k/h$: the corresponding probabilities are referred to as t-fractional-photon probabilities, t being the fractional-photon index. Such arguments lead to defining the fractional-photon states as those quantum states which exhibit exactly the t-fractional-photon probability distributions for all the usual one-photon observables. A straightforward calculations shows that such states can be constructed for every t: for integer t they correspond to the multiphoton states of Ref. 16; for strictly fractional t they are mixed states given in terms of density matrices.[14]

A. h-photon particle operators

h-photon observables $\hat{\mathcal{O}}_{(h)}$ are constructed by using the generalized Bose operators of Brandt and Greenberg[17] $b_{(h)},b^{\dagger}_{(h)}$, satisfying the commutation relations

$$[b_{(h)},b^{\dagger}_{(h)}]=1\ ,\qquad [a^{\dagger}a,b_{(h)}]=-hb_{(h)}\ , \tag{2.1}$$

where $a,a^{\dagger}$ denote the usual annihilation and creation operators. Equations (2.1) lead one to interpret $b_{(h)}$ and $b^{\dagger}_{(h)}$ as annihilation and creation operators of h photons simultaneously. However, it should be noted that $b_{(h)}\neq a^h$ for $h\geq 2$, even though $b_{(1)}=a$.

On the Fock space $b_{(h)}$ and $b^{\dagger}_{(h)}$ operate as follows:

$$b_{(h)}|sh+\lambda\rangle=\sqrt{s}\,|(s-1)h+\lambda\rangle\ ,\qquad b^{\dagger}_{(h)}|sh+\lambda\rangle=\sqrt{s+1}|(s+1)h+\lambda\rangle\ , \tag{2.2}$$

where $0\leq\lambda\leq h-1$. From Eq. (2.1) and (2.2) one can derive the normal-ordered representation

$$b_{(h)}=\sum_{j=0}^{\infty}\alpha_j^{(h)}(a^{\dagger})^j a^{j+h}\ , \tag{2.3}$$

where

$$\alpha_j^{(h)}=\sum_{l=0}^{j}\frac{(-1)^{j-1}}{(j-l)!}\left[\frac{1+[l/h]}{l!(l+h)!}\right]^{1/2}e^{i\phi_l}\ , \tag{2.4}$$

ϕ_m, $m=0,\ldots,j$ being arbitrary real phases ($[x]$ denotes the maximum integer $\geq x$). Equations (2.2) show that the Fock space $\mathcal{F}$ splits into h orthogonal subspaces $\mathcal{F}_\lambda$ which are invariant under the action of the h-photon operators:

$$\mathcal{F}=\bigoplus_{\lambda=0}^{h-1}\mathcal{F}_\lambda,\quad \mathcal{F}_\lambda=\bigoplus_{s=0}^{\infty}\mathrm{span}\{|sh+\lambda\rangle\}\ ,\qquad b_{(h)}\mathcal{F}_\lambda\subset\mathcal{F}_\lambda,\quad b^{\dagger}_{(h)}\mathcal{F}_\lambda\subset\mathcal{F}_\lambda\ . \tag{2.5}$$

Therefore the generic Fock state $|sh+\lambda\rangle$ can be labeled by two quantum numbers s and λ, which are the eigenvalues of the complete set of commuting operators $b^{\dagger}_{(h)}b_{(h)}$ and $\hat{D}_{(h)}=a^{\dagger}a-hb^{\dagger}_{(h)}b_{(h)}$:

$$b^{\dagger}_{(h)}b_{(h)}|sh+\lambda\rangle=s|sh+\lambda\rangle\ ,\qquad \hat{D}_{(h)}|sh+\lambda\rangle=\lambda|sh+\lambda\rangle\ . \tag{2.6}$$

B. h-photon observables

An h-photon observable $\hat{\mathcal{O}}_{(h)}$ is a Hermitian analytic function of h-photon creation and annihilation operators $b_{(h)},b^{\dagger}_{(h)}$,

$$\hat{\mathcal{O}}_{(h)}=\mathcal{F}(b_{(h)},b^{\dagger}_{(h)})\ , \tag{2.7}$$

where $\mathcal{F}$ is an analytic function and $\hat{\mathcal{O}}_{(h)}=\hat{\mathcal{O}}^{\dagger}_{(h)}$. Typical examples of h-photon observables are the "canonical variables" $\hat{Q}_{(h)}$ and $\hat{P}_{(h)}$,

$$\hat{Q}_{(h)}=\frac{1}{\sqrt{2}}(b^{\dagger}_{(h)}+b_{(h)})\ ,\qquad \hat{P}_{(h)}=\frac{i}{\sqrt{2}}(b^{\dagger}_{(h)}-b_{(h)})\ , \tag{2.8}$$

and the "number operator" $\hat{N}_{(h)}$,

$$\hat{N}_{(h)}=b^{\dagger}_{(h)}b_{(h)}\ . \tag{2.9}$$

As a consequence of the general definition (2.7) and the representation (2.2), the h-photon operator $\hat{\mathcal{O}}_{(h)}$ is represented on a fixed Fock sector $\mathcal{F}_\lambda$ in terms of the corresponding one-photon operator $\hat{\mathcal{O}}_{(1)}$ matrix elements

$$\langle nh+\lambda|\hat{\mathcal{O}}_{(h)}|mh+\lambda\rangle=\langle n|\mathcal{O}_{(1)}|m\rangle\ . \tag{2.10}$$

Furthermore, every h-photon observable commutes with the operator $\hat{D}_{(h)}=a^{\dagger}a-hb^{\dagger}_{(h)}b_{(h)}$.

C. Fractional-photon probabilities

The definition of the $\hat{\mathcal{O}}_{(h)}$-probability distributions for fractional-photon states is based on the construction of a complete set of eigenvectors for the two mutually commuting operators $\hat{\mathcal{O}}_{(h)}$ and $\hat{D}_{(h)}$:

$$\hat{\mathcal{O}}_{(h)}|\mathcal{O},\lambda\rangle_{(h)}=\mathcal{O}|\mathcal{O},\lambda\rangle_{(h)}\ ,\qquad \hat{D}_{(h)}|\mathcal{O},\lambda\rangle_{(h)}=\lambda|\mathcal{O}\lambda\rangle_{(h)}\ . \tag{2.11}$$

Equation (2.10) implies that the eigenstates can be expanded as follows:

$$|\mathcal{O},\lambda\rangle_{(h)}=\sum_{l=0}^{\infty}|lh+\lambda\rangle\langle l|\mathcal{O}\rangle\ , \tag{2.12}$$

where $|\mathcal{O}\rangle$ denotes the eigenstate of the one-photon operator

$$\hat{\mathcal{O}}_{(1)}|\mathcal{O}\rangle=\mathcal{O}|\mathcal{O}\rangle\ . \tag{2.13}$$

In fact, using Eq. (2.10), one can verify that the state (2.12) is an eigenstate for $\hat{\mathcal{O}}_{(h)}$,

$$\begin{aligned}\hat{\mathcal{O}}_{(h)}|\mathcal{O},\lambda\rangle_{(h)}&=\sum_{l,n=0}^{\infty}\sum_{\mu=0}^{h-1}|nh+\mu\rangle\langle nh+\mu|\hat{\mathcal{O}}_{(h)}|lh+\lambda\rangle\langle l|\mathcal{O}\rangle\\&=\sum_{l,n=0}^{\infty}|nh+\lambda\rangle\langle n|\hat{\mathcal{O}}_{(1)}|l\rangle\langle l|\mathcal{O}\rangle=\mathcal{O}\sum_{l=0}^{\infty}|lh+\lambda\rangle\langle l|\mathcal{O}\rangle\equiv\mathcal{O}|\mathcal{O},\lambda\rangle_{(h)}\ .\end{aligned} \tag{2.14}$$

The next step in the construction of the fractional probability distribution is to consider a k-photon state of the form

$$|\omega\rangle_{(k)}=\sum_{m=0}^{\infty}\omega_m|km\rangle,\quad \|\omega\|^2\equiv\sum_{m=0}^{\infty}|\omega_m|^2=1\ , \tag{2.15}$$

and then to obtain the probability distribution of the $\hat{\mathcal{O}}_{(h)}$ observable for the k-photon state, $|\omega\rangle_{(k)}$ summing over the hidden degree of freedom λ ($k\neq h,\ t=k/h=s/r,\ s,r$ relativity prime)

$$\mathcal{P}_{\omega}^{(t)}(\mathcal{O})=\sum_{\lambda=0}^{r-1}|{}_{(r)}\langle\mathcal{O},\lambda|\omega\rangle_{(s)}|^2\ . \tag{2.16}$$

D. Density-matrix states

The construction of a density-matrix state $\hat{\rho}_{\omega}^{(t)}$ which exhibits the probability (2.16) for the usual observables $\hat{\mathcal{O}}_{(1)}$

$$\mathcal{P}_{\omega}^{(t)}(\mathcal{O})=\mathrm{Tr}(|\mathcal{O}\rangle\langle\mathcal{O}|\hat{\rho}_{\omega}^{(t)}) \tag{2.17}$$

can be achieved by writing the probability (2.16) in the form

$$\begin{aligned}\mathcal{P}_{\omega}^{(t)}(\mathcal{O})&=\sum_{\lambda=0}^{r-1}\sum_{l,m=0}^{\infty}{}_{(s)}\langle\omega|lr+\lambda\rangle\langle l|\mathcal{O}\rangle\langle\mathcal{O}|m\rangle\langle mr+\lambda|\omega\rangle_{(s)}\\&=\sum_{l,m=0}^{\infty}\langle l|\mathcal{O}\rangle\langle\mathcal{O}|\left[\sum_{\lambda=0}^{r-1}|m\rangle\langle mr+\lambda|\omega\rangle_{(s)}\,{}_{(s)}\langle\omega|lr+\lambda\rangle\langle l|\right]|l\rangle\ .\end{aligned} \tag{2.18}$$

Comparison of Eq. (2.17) with (2.18) leads to the density matrix $\hat{\rho}_{\omega}^{(t)}$

$$\begin{aligned}\hat{\rho}_{\omega}^{(t)}&=\sum_{\lambda=0}^{r-1}|\Omega_{\lambda}^{(t)}\rangle\langle\Omega_{\lambda}^{(t)}|\ ,\\|\Omega_{\lambda}^{(t)}\rangle&=\sum_{m=0}^{\infty}\Omega_{\lambda,m}^{(t)}|m\rangle\ ,\\\Omega_{\lambda,m}^{(t)}&=e^{i\phi_\lambda}\langle mr+\lambda|\omega\rangle_{(s)}\end{aligned} \tag{2.19}$$

(ϕ_λ arbitrary phases). One can see that $\hat{\rho}_{\omega}^{(t)}$ is correctly normalized, since

$$\mathrm{Tr}\hat{\rho}_{\omega}^{(t)}=\sum_{m=0}^{\infty}\sum_{\lambda=0}^{r-1}|\Omega_{\lambda,m}^{(t)}|^2=\|\omega\|^2=1\ . \tag{2.20}$$

The matrix in Eq. (2.19) does not depend on the particular observable $\hat{\mathcal{O}}$, but only on the state $|\omega\rangle_{(k)}$ and on $t=s/r$: consequently, we can refer to it as the fractional-photon state.

E. $1/r$-coherent states

The $1/r$-coherent states are those statistical states which exhibit the probability distributions of the coherent states for the r observables. They are obtained choosing $s=1$ and $|\omega\rangle_{(1)}$ being defined as the coherent state in Eq. (2.19). In this case the density matrix is given by

$$\hat{\rho}_{\omega}^{(1/r)}=e^{-|\omega|^2}\sum_{\lambda=0}^{r-1}\sum_{l,m=0}^{\infty}|m\rangle\frac{\omega^{mr+\lambda}\omega^{*lr+\lambda}}{\sqrt{(mr+\lambda)!(lr+\lambda)!}}\langle l|\ . \tag{2.21}$$

In Ref. 14 it has been shown that the states (2.21) can be squeezed and strongly sub-Poissonian in the photon number distribution. In the next section, I show that they appear as NUS states in the amplitude-squeezing mechanism driven by multiphoton processes.

III. PHASE-NUMBER UNCERTAINTIES

The situation of the conjugate pair of variables $\hat{n}$—number operator—and $\hat{\Phi}$—phase operator—is somewhat more complicated than for other conjugate pairs (such as position $\hat{q}$ and momentum $\hat{p}$). In fact, the phase operator $\hat{\Phi}$ is not Hermitian. In the following I briefly recall this subject,[4] in order to construct the multiphoton generalized operators and to illustrate the mechanism of amplitude-squeezing by fractional coherent states.

A. Phase and number operators

One defines the phase operator $\hat{\Phi}$ through the relation

$$\hat{E}_{\pm}=e^{\mp i\hat{\Phi}} , \tag{3.1}$$

where $\hat{E}_{\pm}$ are the shift operators

$$\begin{aligned} \hat{E}_{-}&=(a^{\dagger}a+1)^{-1/2}a , \\ \hat{E}_{+}&=a^{\dagger}(a^{\dagger}a+1)^{-1/2} . \end{aligned} \tag{3.2}$$

However, $\hat{\Phi}$ is not Hermitian, because $\hat{E}_{\pm}$ are not unitary, as shown by

$$\hat{E}_{+}\hat{E}_{-}=1-|0\rangle\langle 0| , \tag{3.3}$$

even though

$$\hat{E}_{-}\hat{E}_{+}=1 . \tag{3.4}$$

Instead of $\hat{\Phi}$ one can use the sine and cosine Hermitian operators, respectively,

$$\begin{aligned} \hat{S}&=\frac{1}{2i}(\hat{E}_{-}-\hat{E}_{+}) , \\ \hat{C}&=\tfrac{1}{2}(\hat{E}_{-}+\hat{E}_{+}) , \end{aligned} \tag{3.5}$$

having the following commutation relations with $\hat{n}$:

$$\begin{aligned} [\hat{n},\hat{S}]&=i\hat{C} , \\ [\hat{n},\hat{C}]&=-i\hat{S} . \end{aligned} \tag{3.6}$$

Equations (3.6) correspond to the Heisenberg uncertainty relations

$$\begin{aligned} \langle\Delta\hat{n}^2\rangle\langle\Delta\hat{S}^2\rangle &\geq \tfrac{1}{4}\langle\hat{C}\rangle^2 , \\ \langle\Delta\hat{n}^2\rangle\langle\Delta\hat{C}^2\rangle &\geq \tfrac{1}{4}\langle\hat{S}\rangle^2 , \end{aligned} \tag{3.7}$$

where $\Delta\hat{O}\equiv\hat{O}-\langle\hat{O}\rangle$. For highly excited states one can consider $\hat{E}_{\pm}$ as approximately unitary operators, since such states are almost orthogonal to the vacuum state $|0\rangle$ [see Eq. (3.3)]. One can thus treat $\hat{\Phi}$ as an almost Hermitian operator and expand $\hat{S}$ and $\hat{C}$ in powers of $\hat{\Phi}$,

$$\begin{aligned} \hat{S}&=\hat{\Phi}-\frac{\hat{\Phi}^3}{3!}+\cdots , \\ \hat{C}&=1-\frac{\hat{\Phi}^2}{2!}+\cdots . \end{aligned} \tag{3.8}$$

Focusing attention on states with small phase uncertainty $\langle\Delta\hat{\Phi}^2\rangle\ll 1$, and fixing the phase reference point at $\langle\hat{\Phi}\rangle=0$, the sine and cosine operators will be approximated in the form

$$\begin{aligned} \hat{S}&=\Delta\hat{S}\sim\Delta\hat{\Phi} , \\ \hat{C}&\sim 1 . \end{aligned} \tag{3.9}$$

Accordingly, the commutation relations in Eqs. (3.6) can be rewritten

$$[\hat{n},\hat{\Phi}]\sim i , \tag{3.10}$$

and the uncertainly relations (3.7) reduce to

$$\langle\Delta\hat{n}^2\rangle\langle\Delta\hat{S}^2\rangle\sim\langle\Delta\hat{n}^2\rangle\langle\Delta\hat{\Phi}^2\rangle\gtrsim\tfrac{1}{4} . \tag{3.11}$$

Therefore, in the limit of large mean numbers (bright limit) and small mean phases, the phase and number operator can be approximately considered as a conjugate pair of variables.

B. Multiphoton phase and number operators

The multiphoton operators in Eq. (2.3) can be used to construct generalized r-photon shift operators as follows:

$$\begin{aligned} (\hat{E}_{-})_{(r)}&=(b_{(r)}^{\dagger}b_{(r)}+1)^{-1/2}b_{(r)} , \\ (\hat{E}_{+})_{(r)}&=b_{(r)}^{\dagger}(b_{(r)}^{\dagger}b_{(r)}+1)^{-1/2} . \end{aligned} \tag{3.12}$$

The shift operators in (3.12) define a new r-photon phase operator

$$(\hat{E}_{\pm})_{(r)}=e^{\mp i\hat{\Phi}_{(r)}} \tag{3.13}$$

which, as the usual phase operator, is not Hermitian. I now show that it is exactly r times the one-photon phase operator, namely,

$$\hat{\Phi}_{(r)}=r\hat{\Phi} . \tag{3.14}$$

The r-photon shift operators correspond to the rth powers of the usual shift operators

$$(\hat{E}_{\pm})_{(r)}=(\hat{E}_{\pm})^r \tag{3.15}$$

as it appears from the equation

$$\begin{aligned} (\hat{E}_{-})^r&=(a^{\dagger}a+1)^{-1/2}a(a^{\dagger}a+1)^{-1/2}a\cdots(a^{\dagger}a+1)^{-1/2}a \\ &=(a^{\dagger}a+1)^{-1/2}(a^{\dagger}a+2)^{-1/2}\cdots(a^{\dagger}a+r)^{-1/2}a^r \\ &\equiv\left[\frac{\hat{n}!}{(\hat{n}+r)!}\right]^{1/2}a^r=(1+[\hat{n}/r])^{-1/2}b_{(r)}=(1+b_{(r)}^{\dagger}b_{(r)})^{-1/2}b_{(r)} \end{aligned} \tag{3.16}$$

by using the relations

$$b_{(r)}=\left[\frac{(1+[\hat{n}/r])\hat{n}!}{(\hat{n}+r)!}\right]^{1/2}a^r , \tag{3.17}$$

$$b^\dagger_{(r)}b_{(r)}\equiv[\hat{n}/r] . \tag{3.18}$$

The r-photon counterpart of the number operator

$$\hat{N}_{(r)}=b^\dagger_{(r)}b_{(r)} \tag{3.19}$$

satisfies an approximate relation which is the inverse of (3.14),

$$\hat{N}_{(r)}\simeq r^{-1}\hat{n} , \tag{3.20}$$

and can be derived from Eq. (3.18) for highly excited states. Equations (3.14) and (3.20) show that the transition to r-photon operators has the effect of multiplying the phases by the factor r while dividing the numbers by r. These results suggest that the fractional-photon states are good candidates as amplitude-squeezed states, as shown in the following.

C. Fractional-photon uncertainties

The multiphoton variables do not correspond to the physical observed quantities. On the other hand, one can utilize the fractional density matrices to find probability distributions for the observed variables which are exactly the same as the multiphoton ones. Thus, for example, one can start with a coherent state $|\omega\rangle$ satisfying the uncertainty relation $\langle\omega|\Delta\hat{n}^2|\omega\rangle\langle\omega|\Delta\hat{\Phi}^2|\omega\rangle\simeq\frac{1}{4}$, and, by moving to $1/r$-coherent states, reduce the number variance to the detriment of the phase one, while keeping the product constant. Namely, one has

$$\langle\Delta\hat{n}^2\rangle_t=\langle\omega|\Delta\hat{N}^2_{(r)}|\omega\rangle\simeq t^2\langle\omega|\Delta\hat{n}^2|\omega\rangle , \tag{3.21}$$

where the following notation is used:

$$\langle\hat{O}\rangle_t=\mathrm{Tr}(\hat{\rho}^{(t)}_\omega\hat{O}) . \tag{3.22}$$

In an analogous way, one has for the phase operator

$$\langle\Delta\hat{\Phi}^2\rangle_t=\langle\omega|\Delta\hat{\Phi}^2_{(r)}|\omega\rangle=t^{-2}\langle\omega|\Delta\hat{\Phi}^2|\omega\rangle \tag{3.23}$$

and for the uncertainty relations

$$\langle\Delta\hat{n}^2\rangle_t\langle\Delta\hat{\Phi}^2\rangle_t\simeq\langle\omega|\Delta\hat{n}^2|\omega\rangle\langle\omega|\Delta\hat{\Phi}^2|\omega\rangle\simeq\tfrac{1}{4} . \tag{3.24}$$

To ensure the validity of Eq. (3.10) and Eq. (3.20), one has to consider highly excited coherent states. Furthermore, since the photon fractioning reduces the average photon number itself as follows:

$$\langle\hat{n}\rangle_t\simeq t\langle\omega|\hat{n}|\omega\rangle=t|\omega|^2 , \tag{3.25}$$

one should increase the average photon number by exciting the state $|\omega\rangle$ while reducing t; otherwise the vacuum state would be obtained. As is shown in Eq. (3.25), one can attain a constant photon number by keeping constant the product $t|\omega|^2$ in the fractioning process.

D. Asymptotic behavior

The uncertainties in Eqs. (3.21) and (3.23) can be explicitly evaluated for a coherent state $|\omega\rangle$ in the bright limit $|\omega|\gg 1$. The $\hat{n}$ fluctuations are given by

$$\langle\Delta\hat{n}^2\rangle_t\simeq t^2\langle\omega|\Delta\hat{n}^2|\omega\rangle=tn_0 , \tag{3.26}$$

where $n_0=|\omega|^2t\simeq\langle\hat{n}\rangle_t$ denotes the average photon number, constant in the fractioning process. The $\hat{n}$ variance in Eq. (3.26) is identical to the leading term of the asymptotic expansion obtained in Ref. 14 and is in agreement with the $\hat{n}$ moments which will be evaluated in the next section.

As regards the phase uncertainty, one first has to fix the reference phase at $\langle\hat{\Phi}\rangle_t=0$ by restricting ω to positive real values. Then one has to compute the following expression:

$$\langle\Delta\hat{\Phi}^2\rangle_t=t^{-2}\langle\omega|\Delta\hat{\Phi}^2|\omega\rangle\simeq t^{-2}\langle\omega|\Delta\hat{S}^2|\omega\rangle$$
$$=\frac{t^{-2}}{4}\left[2-e^{-|\omega|^2}-2e^{-|\omega|^2}\sum_{n=0}^{\infty}\frac{|\omega|^{2(n+1)}}{n!\sqrt{(n+1)(n+2)}}\right] . \tag{3.27}$$

The last sum is a particular case of the functions (α_i real)

$$H^{[s]}_{\alpha_0,\alpha_1,\ldots,\alpha_{s-1}}(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\prod_{i=0}^{s-1}(n+1+i)^{\alpha_i} \tag{3.28}$$

already studied[14] in the limit $x\gg 1$. The asymptotic expansions are

$$H^{[s]}_{\alpha_0,\alpha_1,\ldots,\alpha_{s-1}}(x)\sim x^{\|\alpha\|}e^x\left\{1+\left[\begin{pmatrix}\|\alpha\|+1\\2\end{pmatrix}+\sum_{k=0}^{s-1}k\alpha_k\right]\frac{1}{x}+O(x^{-2})\right\} , \tag{3.29}$$

where $\|\alpha\|=\sum_{k=0}^{s-1}\alpha_k$. One has

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}[(n+1)(n+2)]^{-1/2}\sim x^{-1}e^x[1-\tfrac{1}{2}x^{-1}+O(x^{-2})] , \tag{3.30}$$

leading to the asymptotic variance for $\hat{\Phi}$

$$\langle\Delta\hat{\Phi}^2\rangle_t\simeq\tfrac{1}{4}t^{-1}n_0^{-1}+O(\rho^{-4}) . \tag{3.31}$$

The same result can also be obtained using the density matrix (2.21). The average sine operator is given by

$$\langle \hat{S} \rangle = \mathrm{Tr}(\hat{\rho}_\omega^{(t)} \hat{S}) = \sum_{\lambda=0}^{r-1} \langle \Omega_\lambda | \hat{S} | \Omega_\lambda \rangle$$

$$= \frac{e^{-|\omega|^2}}{2i} \sum_{\lambda=0}^{r-1} \sum_{n,m=0}^{\infty} \frac{\omega^{*rm+\lambda}\omega^{rn+\lambda}}{\sqrt{(rm+\lambda)!(rn+\lambda)!}} (\delta_{m,n-1} - \delta_{m,n+1})$$

$$= \sin(r\phi) e^{-\rho^2} \rho^r \sum_{l=0}^{\infty} \frac{\rho^{2l}}{l!} [(l+1) \cdots (l+r)]^{-1/2} , \tag{3.32}$$

where $\omega = \rho e^{i\phi}$; therefore, $\phi = 2k\pi$ in order to fix the phase reference point at $\langle \hat{S} \rangle = 0$. The second moment is derived in a way analogous to Eq. (3.32),

$$\langle \Delta \hat{S}^2 \rangle = \langle \hat{S}^2 \rangle = \mathrm{Tr}(\rho_\omega^{(t)} \hat{S}^2)$$

$$= \tfrac{1}{2} - \tfrac{1}{4} e^{-\rho^2} \sum_{\lambda=0}^{r-1} \frac{\rho^{2\lambda}}{\lambda!} - \tfrac{1}{2} e^{-\rho^2} \rho^{2r} \sum_{l=0}^{\infty} \frac{\rho^{2l}}{l!} [(l+1) \cdots (l+2r)]^{-1/2} , \tag{3.33}$$

which, in the bright limit ($\rho \gg 1$) reads

$$\langle \Delta \hat{S}^2 \rangle \sim \tfrac{1}{2} [1 - e^{-\rho^2} \rho^{2r} H^{[2r]}_{-1/2,-1/2,\ldots,-1/2}(\rho^2)] . \tag{3.34}$$

Equation (3.29) yields the asymptotic expansion

$$\sum_{n=0}^{\infty} \frac{\rho^{2l}}{l!} [(l+1) \cdots (l+2r)]^{-1/2}$$

$$\sim \rho^{-2r} e^{\rho^2} \left[1 - \frac{r^2}{2} \rho^{-2} + O(\rho^{-4}) \right] \tag{3.35}$$

which can be substituted into Eq. (3.33), leading to

$$\langle \Delta \hat{S}^2 \rangle = \tfrac{1}{4} t^{-2} \rho^{-2} + O(\rho^{-4}) = \tfrac{1}{4} t^{-1} n_0^{-1} + O(\rho^{-4}) . \tag{3.36}$$

For small phase uncertainty $\langle \Delta \hat{\Phi}^2 \rangle_t \ll 1$ Eq. (3.36) coincides with Eq. (3.31). One can see that the product of the uncertainties (3.21) and (3.31) is equal to $\frac{1}{4}$ up to the second order in ρ^{-1}: this emphasizes that, in the bright limit, the $1/r$-coherent states are number-phase minimum uncertainty states.

IV. PROBABILITY DISTRIBUTIONS

In this section, I derive some asymptotic evaluations of the density matrix, the $\hat{n}$-probability distribution, and the quasiprobability distribution (Q function)

$$Q_\omega^{(t)}(z^*, z) \equiv \langle z | \hat{\rho}_\omega^{(t)} | z \rangle \equiv \mathrm{Tr}(|z\rangle\langle z| \hat{\rho}) \tag{4.1}$$

for the $1/r$-coherent states. Some numerical results for the Q function will also be given and discussed in comparison with the analytical evaluations.

A. Density matrix

In the bright limit—i.e., for large coherent mean field $|\omega| \gg 1$—the density matrix ρ_{ml} (2.21) becomes sharply peaked on the main diagonal, around the most probable indices $m = l = \langle \hat{n} \rangle_t = n_0 \equiv |\omega|^2 t$. This allows the matrix to be approximated by taking advantage of the Gaussian limit of the Poisson distribution:

$$e^{-\alpha} \frac{\alpha^n}{n!} \sim \mathcal{G}_{\alpha,\alpha}(n) \quad (\text{ for } \alpha \gg 1, n \sim \alpha) , \tag{4.2}$$

where the following shorthand notation is used:

$$\mathcal{G}_{\bar{x},\Delta^2}(x) = (2\pi\Delta^2)^{-1/2} \exp\left[-\frac{1}{2} \frac{(x-\bar{x})^2}{\Delta^2} \right] . \tag{4.3}$$

The asymptotic behavior is

$$\rho_{n,m} = \rho^*_{m,n} = \sum_{\lambda=0}^{r-1} e^{-ir(m-n)\phi} \frac{e^{-(1/2)|\omega|^2} |\omega|^{nr+\lambda}}{\sqrt{(nr+\lambda)!}} \frac{e^{-(1/2)|\omega|^2} |\omega|^{mr+\lambda}}{\sqrt{(mr+\lambda)!}}$$

$$\sim e^{-i(m-n)\phi/t} t \sum_{\lambda=0}^{r-1} [\mathcal{G}_{n_0-\lambda t, n_0 t}(n) \mathcal{G}_{n_0-\lambda t, n_0 t}(m)]^{1/2} \tag{4.4}$$

($\omega=\rho e^{i\phi}$) and, after the substitution $m=n+d$, it reduces to

$$\rho_{n,n+d}=\rho^*_{n+d,n} \sim (8\pi n_0 t)^{1/2} e^{-id\phi/t} \mathcal{G}_{0,4n_0 t}(d) t \times \sum_{\lambda=0}^{r-1} \mathcal{G}_{n_0-\lambda t-d/2,n_0 t}(n) \; . \tag{4.5}$$

One should note that the approximate expression in Eq. (4.5) preserves the normalization of the density matrix

$$\mathrm{Tr}\hat{\rho}=\sum_{n=0}^{\infty}\rho_{n,n} \sim \sum_{n=0}^{\infty} t \sum_{\lambda=0}^{r-1} \mathcal{G}_{n_0-\lambda t,n_0 t}(n) \sim t\sum_{\lambda=0}^{r-1} \langle\langle 1 \rangle\rangle_{n_0-\lambda t,n_0 t} = tr = 1 \; . \tag{4.6}$$

The double angular brackets in (4.6) denote the Gaussian average

$$\langle\langle f(x) \rangle\rangle_{\bar{x},\Delta^2} = \int_{-\infty}^{+\infty} dx\, \mathcal{G}_{\bar{x},\Delta^2}(x) f(x) \; . \tag{4.7}$$

B. Number distribution

Using the asymptotic evaluation in Eq. (4.5), one can derive the asymptotic form of the probabilities. In particular, the number probability distribution corresponds to the diagonal of the density matrix

$$\mathcal{N}^{(t)}_{\omega}(n) \sim t \sum_{\lambda=0}^{r-1} \mathcal{G}_{n_0-\lambda t,n_0 t}(n) \; . \tag{4.8}$$

By means of Eq. (4.8) the qth moment can be evaluated as follows:

$$\langle \hat{n}^q \rangle_t \sim t \sum_{\lambda=0}^{r-1} \langle\langle (n-\lambda t)^q \rangle\rangle_{n_0,n_0 t} \; . \tag{4.9}$$

One obtains

$$\langle \hat{n}^q \rangle_t \sim t \sum_{\lambda=0}^{r-1} \langle\langle (n-n_0+n_0-\lambda t)^q \rangle\rangle_{n_0,n_0 t} = t \sum_{\lambda=0}^{r-1} \sum_{l=0}^{[q/2]} \frac{2^{-l} q!}{l!(q-2l)!} (n_0 t)^l (n_0-\lambda t)^{q-2l} \sim n_0^q \left[1 - n_0^{-1} \frac{q}{2}(qt-1) + \mathcal{O}(n_0^{-2}) \right] . \tag{4.10}$$

The last result is identical to the asymptotic expansion obtained in Ref. 14 using a completely different approach. Furthermore, one can see that the leading terms in the expansion (4.10) for $\langle \hat{n} \rangle_t$ and $\langle \Delta\hat{n} \rangle_t$ are identical to the expressions (3.25) and (3.26).

C. Quasiprobability distribution

The Q function defined in (4.1)

$$\mathcal{Q}^{(t)}_{\omega}(z^*,z) \equiv \mathrm{Tr}(|z\rangle\langle z|\hat{\rho}) = e^{-|\omega|^2-|z|^2} \sum_{\lambda=0}^{r-1} \left| \sum_{n=0}^{\infty} \frac{z^{*n}\omega^{rn+\lambda}}{\sqrt{n!(rn+\lambda)!}} \right|^2 , \tag{4.11}$$

can be evaluated using the Gaussian approximation (4.2) for the projector matrix $(|z\rangle\langle z|)_{n,m}$

$$(|z\rangle\langle z|)_{n+d,n} \sim (8\pi|z|^2)^{1/2} e^{id\psi} \mathcal{G}_{0,4|z|^2}(d) \times \mathcal{G}_{|z|^2-d/2,|z|^2}(n) \; . \tag{4.12}$$

As a matter of fact, only the terms with large entries $n\sim m\sim n_0 \gg 1$ will contribute in the trace (4.11). Therefore this latter can be rewritten in terms of Gaussian averages

$$\mathcal{P}^{(t)}_{\omega}(z^*,z) \sim (2\pi\Gamma^2)^{1/2} \langle\langle \exp[id(\psi-\phi/t)] \rangle\rangle_{0,\Gamma^2} \times t \sum_{\lambda=0}^{r-1} \langle\langle \mathcal{G}_{|z|^2+\lambda t,|z|^2}(n) \rangle\rangle_{n_0,n_0 t} \; , \tag{4.13}$$

where

$$\Gamma^2 = 4\frac{n_0 t|z|^2}{n_0 t+|z|^2} \; . \tag{4.14}$$

Using the identities

$$\langle\langle \mathcal{G}_{x',\Delta'^2}(x) \rangle\rangle_{x'',\Delta''^2} = \mathcal{G}_{0,\Delta'^2+\Delta''^2}(x'-x'') \; , \tag{4.15}$$

$$\langle\langle \exp(ix\psi) \rangle\rangle_{0,\Delta^2} = (2\pi\Delta^{-2})^{1/2} \mathcal{G}_{0,\Delta^{-2}}(\psi) \; , \tag{4.16}$$

the quasiprobability is decomposed in a z-phase and a z-modulus factor

$$\mathcal{Q}^{(t)}_{\omega}(z^*,z) \sim 2\pi \mathcal{G}_{\phi/t,\Delta^2_\psi}(\psi) t \sum_{\lambda=0}^{r-1} \mathcal{G}_{\lambda t,n_0+|z|^2}(n_0-|z|^2) \; . \tag{4.17}$$

The Q function (4.17) is Gaussian in the phase of z, with variance

$$\Delta^2_\psi = \Gamma^{-2} = \frac{n_0 t+|z|^2}{4|z|^2 n_0 t} \tag{4.18}$$

and average value ϕ/t. On the other hand, regarding the z-modulus behavior, the probability (4.17) is no more Gaussian and exhibits its maximum value near to $|z|^2 \sim n_0$. Substituting this value in Eq. (4.18) and taking the limit for $t\to 0$, one obtains

$$\Delta^2_\psi \sim \tfrac{1}{4}(n_0 t)^{-2} = \tfrac{1}{4} t^{-2} |\omega|^{-2} \; , \tag{4.19}$$

which corresponds to the second moment of the phase operator (3.31).

Some numerical calculations of the summations in (4.11) have been performed for fixed average photon number $\langle \hat{n} \rangle_t \sim |\omega|^2 t$. The series sum has been cut at sufficiently high index $n=30$, while convergence and completeness of the scanning on the z plane have been checked up to the fourth digit using the identity

$$1 = \mathrm{Tr}(\rho^{(t)}_{\omega}) = \frac{1}{\pi} \int d^2z\, \mathcal{Q}^{(t)}_{\omega}(z^*,z) \; . \tag{4.20}$$

A three-dimensional plot for the lowest t value ($r=1/t=50$) is shown in Fig. 1. The results for several t values are shown in Fig. 2, where the contours of 0.8, 0.6, 0.4, and 0.2 times the maximum value are represented [the limiting case $r=1$ of Fig. 2(a) corresponds to the usual coherent state]. One can see that the asymptotic

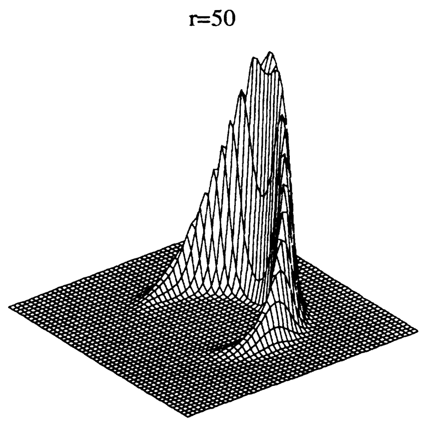

FIG. 1. Three-dimensional plot of the Q function $Q_\omega^{(t)}(z^*,z)$ of Eq. (4.11) for fixed average number of photons $n_0=25$ and for $r=1/t=50$.

factorization (4.17) fits the numerical results quite well in the $t\rightarrow 0$ limit. For decreasing t (increasing r) the quasiprobability is expanded in the phase (tangential) direction while being slightly "squeezed" in the number (radial) direction. The expanded tangential distribution for the Q function is intimately connected with the enhancement in the phase operator variance. On the other hand, the lack of correspondence between the radial squeezing and the $\hat{n}$ distribution squeezing is only an artifact related to the Q function (see also the numerical plots in Ref. 11) and the sub-Poisson character of the state should be inferred only from the $\hat{n}$ distribution shape.

Comparing the present results with those in Ref. 11, one can notice that the $1/r$-coherent states exhibit a Q function which is symmetrically distributed on the tangential direction around its maximum value, while the states of Kitagawa and Yamamoto[11] produce a quasiprobability which is asymmetrical and slightly tilted with respect to the tangential direction. In this sense, the $1/r$-coherent states are more similar to the mathematically constructed NUS of Jackiw.[5] This observation is also supported by the fact that the sub-Poisson parameter $\langle \Delta\hat{n}^2\rangle/\langle\hat{n}\rangle$ is equal to t for the $1/r$-coherent states—i.e., it can be reduced to zero without affecting the average number of photons—whereas, for the states in Ref. 11, it is proportional to $\langle\hat{n}\rangle^{-2/3}$.

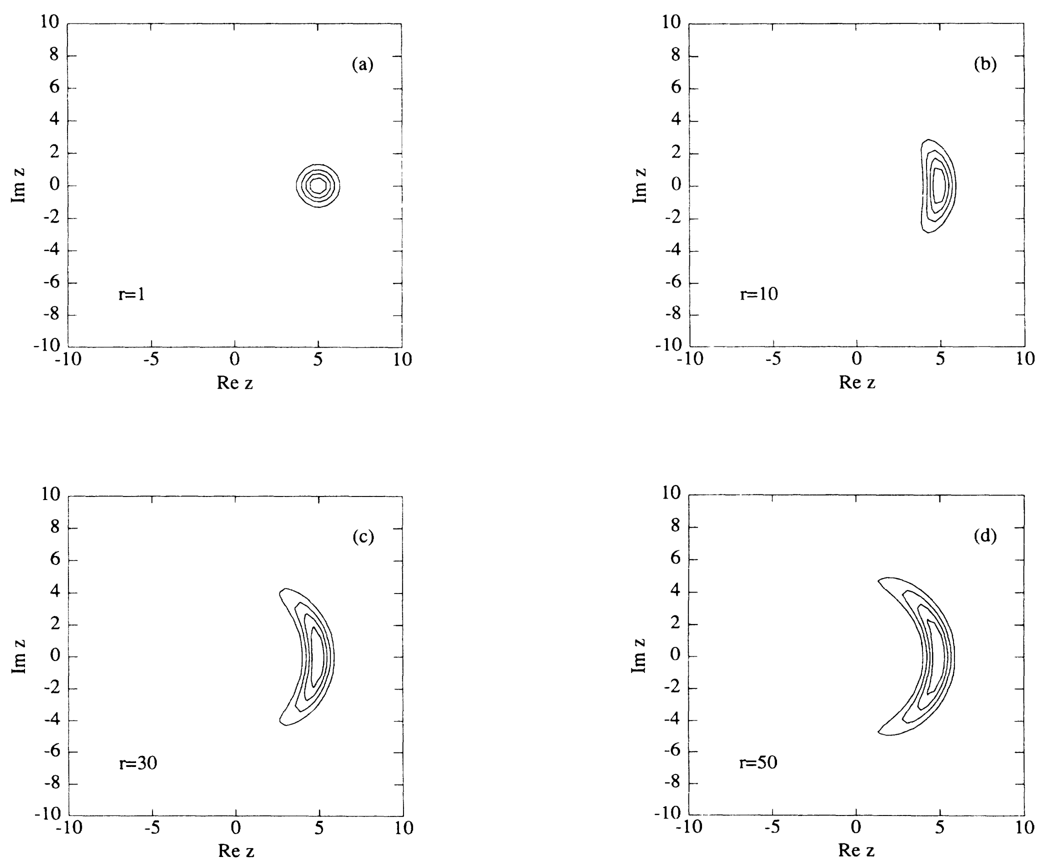

FIG. 2. The Q function $Q_\omega^{(t)}(z^*,z)$ of Eq. (4.11) for fixed average number of photons $n_0=25$ and for $r=1/t=1,10,30,50$. Contours correspond to 0.8, 0.6, 0.4, and 0.2 times the maximum value. (d) corresponds to Fig. 1. (a) ($r=1$) represents the limiting case of the pure coherent state.

V. CONCLUSIONS

The prototype mechanism of producing amplitude squeezing presented here is highly nonlinear and involves many well-balanced multiphoton processes. The amplitude-squeezing phenomenon turns out to result from two competing limits: the bright limit and the photon fractioning ($t \to 0$) limit. The former has a semiclassical character, and it is needed in order to well define the phase and to minimize the number-phase uncertainty product. The latter has strictly quantum character—it goes towards the vacuum state—and it is responsible for the enhancement of the phase fluctuations and the simultaneous reduction of the noise in the number. Since the mechanism, starting from a coherent state, results in a density-matrix state, it cannot be implemented through a unitary time evolution. This last feature is common to the mathematically constructed NUS of Jackiw[5] and is related to the change in the commutation algebra, from the Heisenberg-Weyl type (connecting a and $a^{\dagger}$), to the larger algebra generated by $\hat{n}$, $\hat{S}$, and $\hat{C}$ (for this argument see also Ref. 11). The bright limit produces density matrices which become sharply peaked around a number eigenstate, thus recovering an almost pure state. Nevertheless, the present analysis seems to indicate that the passage from the coherent state to the number eigenstate could involve a dissipative mechanism. This last point will be further investigated in view of physical applications.

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