Evaluation of multiphoton processes by means of Gaussian averages

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A method to evaluate matrix elements of anharmonic unitary operators when the usual expansion techniques lead to nonconvergent series is suggested. This problem is particularly relevant in nonlinear quantum optics, when multiphoton processes are analyzed, and two examples of applications in this case are given.

Multiphoton processes are presently of great interest in quantum optics. For example, they are involved in the optical Kerr effect, where the third-order nonlinear susceptibility leads to Hamiltonians that are quartic in the creation or annihilation operators. Correspondingly, strongly anharmonic interactions lead to non-Gaussian wave packets and to non-Poissonian fields in the number representation. The problem now is the knowledge of the best tuning for all the parameters in the interaction Hamiltonian, in order to get highly squeezed and/or strongly sub-Poissonian radiation. These states of the field are the best candidates in a quantum nondemolition measurement or in order to improve the quantum limit to the signal-to-noise ratio.[1,2] The study of such many-photon processes usually involves the problem of evaluating matrix elements of anharmonic unitary operators between coherent states, namely, matrix elements of the form

$$\langle \alpha | \hat{S}_k(g) | \beta \rangle = \langle \alpha | \exp(ig\hat{H}_k) | \beta \rangle \ , \tag{1}$$

where $\hat{H}_k$ is a Hermitian polynomial in the Bose operators a_λ and $a_\lambda^\dagger$ ($\lambda = 1, \ldots, N$) with powers up to k, and $|\alpha\rangle$ and $|\beta\rangle$ are coherent states.

For $k > 2$ (i.e., in the strictly anharmonic case) the generally adopted procedure to compute the matrix element in Eq. (1) is the following: (i) Taylor expansion of the exponential; (ii) normal ordering of the boson operators; (iii) evaluation of the generic matrix element of the expansion; and then (iv) summation of the series.

However, the above procedure leads, in general, to series that have zero radius of convergence.[3] (This feature is typical, for example, of the perturbation series for the energy of quartic anharmonic oscillator.[4])

The first issue to emphasize is that the divergent series is only a mathematical artifact. Indeed, the operator $\hat{S}_k$ is unitary and the matrix element *does exist* even though, as a function of the coupling parameter g, it is singular in $g = 0$, and so it cannot be Taylor expanded [mathematically, one says that the vector $|\beta\rangle \in \mathcal{B}$, where $\mathcal{B}$ is a Banach space, is not analytical for the operator $\hat{H}_k$, as the series expansion $\exp(t\hat{H}_k)|\beta\rangle$ has zero radius of convergence[5]]. Despite the fact that the divergence is only an artifact, it still remains a serious problem. In fact, one has only two possibilities: (a) to try to extract quantitative informations from the divergent series expansion or (b) to adopt a quite different procedure, avoiding in particular steps (i) and (iv).

Concerning the former option (i.e., extracting informations from the series) in some cases one can recognize the divergent expansion as the asymptotic series of some known function in the neighborhood of $g = 0$. Furthermore, if this is not the case, there are some methods of analytic continuation available, like, for example, the Borel summations,[6] in which the divergent series is rewritten in terms of integrals of (the analytic continuation of nonuniformly) convergent series; or the Padé approximants, where the Taylor truncated series is matched with a rational function, which tends to reproduce the pole structure limiting the Taylor expansion convergence. (An application of the latter method in the case of a cubic and quartic operator can be found in Ref. 7.)

If one chooses the latter option, i.e., avoiding Taylor expansion, one is forced to avoid step (ii) too. In fact, normal ordering techniques can be applied in this case only when it is possible to compute the inverse Baker-Campbell-Hausdorff formula. As it is know, this can not be achieved for $k < 2$ (the boson algebra that one should handle is actually infinitely dimensional), the only nonharmonic Hamiltonian that can be treated in this way involves multiphoton operators (see Ref. 8 and references therein). Furthermore, D'Ariano, Rasetti, and Vadacchino[9] pointed out that there is a deep relation between the singular behavior of the S matrix and the phenomenon of breaking coherence. Indeed the coherence-preserving Hamiltonians $\hat{H}_k$ are bilinear in a_λ and $a_\lambda^\dagger$, i.e., they are harmonical.

The last remaining possibility is now to avoid the normal ordering step. Of course, this can be achieved when, for example, one is able to diagonalize the operator $\hat{H}_k$.

Here we want to suggest an alternative procedure avoiding expansion in the coupling parameter. The essential idea of such a procedure can be already found in Ref. 10, where, however, the method is analyzed no further, and a different approach is used in the analytical solution of a four-photon model. As a matter of fact, the major advantages of the present procedure can be found in performing numerical evaluations.

The method reduces the problem to integration of rapidly converging Gaussians that can be easily carried out (whereas the Padé approximants or the Borel summability require sophisticated procedure and software). The generality of the method suffers the restriction that the Hamiltonian $\hat{H}_k$ is required to be the square of another Hermitian operator $\hat{A}_{k/2}$,

$$\hat{H}_k = (\hat{A}_{k/2})^2 , \quad (2)$$

and the procedure is very easy to handle in the case of Hamiltonians which are *perfect squares* of polynomial operators in a_λ and $a_\lambda^\dagger$. However, this special class of Hamiltonians is interesting in itself since some relevant high-g behaviors of the S matrix can be generally inferred from the higher-degree part of the Hamiltonians.[3] A further limitation of the method is the fact that, in order to concretely improve convergence, one needs to know the analytic continuation $\Phi(x)$ of the matrix element of $\langle\alpha|\exp(ix\hat{A}_{k/2})|\beta\rangle$ in the complex x plane. (For cases in which the analytic continuation cannot be done by inspection, one may use a conformal transformation.[6]) The method could certainly be extended to more general situations, to the disadvantage of simplicity. Therefore we prefer to present it in its crudest form, also considering that the simple examples which we will show are already suitable for some interesting applications in quantum optics.

The key idea is that the Gaussian averaging procedure maps $g \to g^{-1}$, and one can take advantage of the following identity:

$$\exp[ig(\hat{A}_{k/2})^2] = \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{4\pi g}} \exp\left[i\left[x\hat{A}_{k/2} - \frac{x^2}{4g} + \frac{\pi}{4}\right]\right] . \quad (3)$$

Equation (3) holds trivially true if $\hat{A}_{k/2}$ is a real number; nevertheless, it still holds if $\hat{A}_{k/2}$ is a Hermitian operator because all operators in the equation commute. One can observe that Eq. (3) is nothing but the usual Gaussian identity (like the Hubbard-Stratanovic mapping[11]), but *Wick rotated,* in the complex g plane to imaginary "temperatures" g^{-1}

$$\exp[-g(\hat{A}_{k/2})^2] = \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{4\pi g}} \exp\left[ix\hat{A}_{k/2} - \frac{x^2}{4g}\right] . \quad (4)$$

If one takes matrix elements—in the domain of the operator $\hat{A}_{k/2}$—on both sides of the identity (4),

$$\langle\alpha|\exp[-g(\hat{A}_{k/2})^2]|\beta\rangle = \int_{-\infty}^{+\infty} dx \frac{\exp(-x^2/4g)}{\sqrt{4\pi g}} \langle\alpha|\exp(ix\hat{A}_{k/2})|\beta\rangle , \quad (5)$$

one can easily check that the integral in the right-hand side of Eq. (5) is bounded because the integrand matrix element has a modulus smaller than 1. On the other hand, the *non-Euclidean* Wick rotated version of Eq. (5)

$$\langle\alpha|\exp[ig(\hat{A}_{k/2})^2]|\beta\rangle = \int_{-\infty}^{+\infty} dx \frac{\exp\left[-i\left[\frac{x^2}{4g} - \frac{\pi}{4}\right]\right]}{\sqrt{4\pi g}} \times \langle\alpha|\exp(ix\hat{A}_{k/2})|\beta\rangle \quad (6)$$

leads to a convergent oscillating integral, if the same matrix element does not oscillate as fast as to compensate Gaussian exponential. However, the convergence of the integral in Eq. (3) could be, in general, a serious numerical problem. In fact, the integral is an improper one and, at least, the pure Gaussian oscillating phase converges as $t^{-1/2}e^{it}$, which requires a quadratic number of steps to linearly improve convergence. The elimination of Gaussian oscillations could be attained performing a new kind of Wick's rotation which now involves the coupling parameter of the operator $\hat{A}_{k/2}$— instead of $(\hat{A}_{k/2})^2$— namely, a rotation in the complex x plane

$$x \to e^{-i\phi}x, \quad 0 \le \phi \le \pi/2 . \quad (7)$$

One also needs the analytic continuation $\Phi(x)$ of the matrix element on the right-hand side of Eq. (6) from the real axis to the complex plane

$$\Phi(x) = \langle\alpha|\exp(ix\hat{A}_{k/2})|\beta\rangle (\text{analytic continuation}) . \quad (8)$$

When $\Phi(x)$ satisfies the following requirements: (i) $\Phi(x)$ has no branching points or poles in the two acute angles of the x plane between the real axis and the straight line forming an angle $-\phi$ with it and (ii) $|\Phi(x)|$ does not increase faster than the Gaussian weight itself for $|x| \to \infty$, at least in the region of plane of item (i); then one can subsitute the integral on the real axis with the integral (of the analytic continuation) on the straight line forming an angle $-\phi$ with the real axis itself, optimizing the free parameter ϕ in order to improve convergence. In the typical case of nonoscillating $\Phi(x)$, one chooses $\phi = \pi/4$. After further rescaling x, one can put the identity in the form

$$\langle\alpha|\exp[ig(\hat{A}_{k/2})^2]|\beta\rangle = \int_{-\infty}^{+\infty} dx \frac{e^{-x^2}}{\sqrt{\pi}} \Phi(2\sqrt{g}e^{-i\pi/4}x) . \quad (9)$$

Two simple examples of applications that are relevant in the field of nonlinear quantum optics are now discussed. In these examples we analyze quartic Hamiltonians $\hat{H}_4$ which are perfect squares of polynomial Hamiltonians $\hat{A}_2$, nevertheless describing interactions between many oscillator modes. Only vacuum-to-vacuum S matrix elements are presented, even though the general matrix elements between coherent states can be computed without adding any serious complication. Such matrix elements allow the calculation of the so called *quasiprobability* distribution function in several cases of interest.

(a) Many-modes degenerate four-photon parametric amplifier. The interaction Hamiltonian is

$$\hat{H}_4 = \tfrac{1}{4}\left[\sum_{\lambda=1}^{N} (a_\lambda + a_\lambda^\dagger)^2\right]^2 . \quad (10)$$

[For $N=2$ the Hamiltonian (10) can be used to analyze an interesting case of nonlinear Mach-Zehender interferometer.[2]] The Taylor expansion for the S matrix vacuum expectation value leads to the series[9]

$$\langle 0_N|\hat{S}_4|0_N\rangle = \sum_{n=0}^{\infty} g^n \left[\frac{i}{4}\right]^n \frac{(N+4n-2)!!}{n!(N-2)!!} , \quad (11)$$

which has zero radius of convergence. The series in Eq. (11) can be recognized as the asymptotic expansion about $g=0$ of the function

$$\langle 0_N|\hat{S}_4|0_N\rangle=(-2ig)^{-N/4}\exp\left[\frac{i}{8g}\right]\times D_{-n/2}((-2ig)^{-1/2}) \ , \qquad (12)$$

where $D_\nu(z)$ denotes the parabolic cylinder function,[12] which is regular at infinity ($g\to 0$). Using the method I propose, or the *Gaussian average method,* one has to compute the integral

$$\langle 0_N|\hat{S}_4|0_N\rangle=\int_{-\infty}^{+\infty}dx\frac{\exp -i\left[\frac{x^2}{4g}-\frac{\pi}{4}\right]}{\sqrt{4\pi g}}\times\langle 0_N|\exp\left[ix\sum_{\lambda=1}^{N}\hat{q}_\lambda^2\right]|0_N\rangle \ , \qquad (13)$$

where $\hat{q}_\lambda=1/\sqrt{2}(a_\lambda+a_\lambda^\dagger)$. The matrix element on the right-hand side of Eq. (13) can be factorized into single-mode terms

$$\left\langle 0_N\left|\exp\left[ix\sum_{\lambda=1}^{N}\hat{q}_\lambda^2\right]\right|0_N\right\rangle=\langle 0|\exp(ix\hat{q}^2)|0\rangle^N \ , \qquad (14)$$

and the single-mode vacuum expectation value can be evaluated by means of the spectral decomposition of the operator $\hat{q}$

$$\langle 0|e^{ix\hat{q}^2}|0\rangle=\int_{-\infty}^{+\infty}dq\frac{1}{\sqrt{\pi}}e^{-q^2+ixq^2} = \begin{cases}\frac{1}{\sqrt{1-ix}}, & \text{if Im } x>-1\\ \text{otherwise does not exist} \ . & \end{cases} \qquad (15)$$

From Eqs. (14) and (15) one obtains the analytic continuation $\Phi(x)$ of the N-modes matrix element

$$\Phi(x)=(1-ix)^{-N/2} \quad (\text{complex } x) \ , \qquad (16)$$

which satisfies the right requirements to perform the x Wick's rotation (it has only a branching pole on the negative imaginary axis and decreases to zero for $|x|\to\infty$). Using a $\phi=\pi/4$ rotation, one obtains the final result suitable for numeric integration

$$\langle 0_N|\hat{S}_4|0_N\rangle=\int_{-\infty}^{+\infty}dx\frac{e^{-x^2}}{\sqrt{\pi}}(1+2\sqrt{g}e^{-i3\pi/4}x)^{-N/2} \ . \qquad (17)$$

This example can be straightforwardly generalized putting different real weights for the oscillator modes; the resulting function $\Phi(x)$ becomes simply a product of inverse square roots as in Eq. (15). New singular points will appear, but all of them lie on the imaginary axis (see also the next example). Consequently, also in this case the x Wick's rotation can be performed. In Fig. 1 we show the numerical evaluation of the matrix element in Eq. (17) versus g for some values of N; the calculation needed only 2 h of a personal computer CPU time (and approximately the same time will be necessary for any other of the present examples).

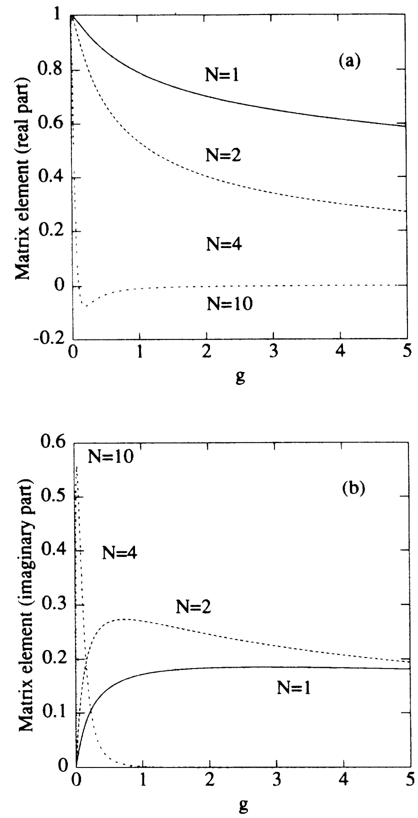

FIG. 1. Numerical evaluation of the vacuum expectation value of the S operator corresponding to the Hamiltonian in Eq. (17) vs the coupling parameter g and for some values of N (=1,2,4,10). The calculation utilizes Eq. (20).

(b) Many-modes fourth-order parametric down conversion. The interaction Hamiltonian is

$$\hat{H}_4=\frac{1}{4}\left[\sum_{\lambda=1}^{N}z_\lambda(a_\lambda^\dagger)^2+z_\lambda^* a_\lambda^2\right]^2 \ , \qquad (18)$$

where z_λ are complex number. [Some cases in the same class of Hamiltonian (18) are studied in Refs. 10 and 13.] The analytic continuation $\Phi(x)$ can be immediately evaluated using the single oscillator matrix element[3]

$$\left\langle 0\left|\exp\left[i\frac{\pi}{2}[z(a^\dagger)^2+z^*a^2]\right]\right|0\right\rangle=\cosh(|z|x) \qquad (19)$$

leading to

$$\Phi(x)=\prod_{\lambda=1}^{N}[\cosh(|z_\lambda|x)]^{-1/2} \ . \qquad (20)$$

Also in this case the analytic continuation satisfies the right requirements which allow the x Wick's rotation. [In fact, the function in Eq. (20) has only branching poles on the imaginary axis and grows slower than the Gaussian weight.] So one can improve convergence using a Wick's rotation with $\phi=\pi/4$, obtaining the final result

$$\langle 0_N|\hat{S}_4|0_N\rangle = \int_{-\infty}^{+\infty} dx \frac{e^{-x^2}}{\sqrt{\pi}} \prod_{\lambda=1}^{N} [\cosh(2\sqrt{g}\,|z_\lambda|e^{-i\pi/4}x)]^{-1/2} . \quad (21)$$

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