New type of two-photon squeezed coherent states

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A new set of non-naive generalizations of the squeezed coherent states recently discussed by Fisher, Nieto, and Sandberg is given, based on generalized Bose operators.

In a recent paper,[1] Fisher, Nieto, and Sandberg discuss the properties of the unitary operators that generate the squeezed coherent states. In particular they demonstrate the impossibility of generalizing such states by means of generalized squeeze operators of the form

$$U_k=\exp(iA_k)=\exp[z_k(a^\dagger)^k+ih_{k-1}-(z_k)^*a^k], \quad k>2 \ , \tag{1}$$

where

$h_q=$ Hermitian polynomial in a and $a^\dagger$ with power up to q ,

due to the divergence of their vacuum expectation value $\langle 0|U_k|0\rangle$ induced by the fact that $|0\rangle$ is not an analytic vector of the generator A_k in the Fock space.

In the present Comment we want to show that the conclusion of Ref. 1 can be modified if the k-photon creator a^k [and annihilator $(a^\dagger)^k$] as well as h_{k-1} in (1) are replaced by infinite series. The puzzling feature is that the generalized states one obtains give one-photon and two-photon squeezed states, whereas for many-photon states ($k>2$) the squeezing disappears. The construction of the new type of unitary squeeze operator is based on the generalized multiphoton Bose operators of Brandt and Greenberg[2] which in a normal-ordered representation are written as follows:

$$b_{(k)}=\sum_{j=0}^{\infty}\alpha_j^{(k)}(a^\dagger)^j a^{j+k} \ , \tag{2}$$

$$\alpha_j^{(k)}=\sum_{l=0}^{j}\frac{(-)^{j-l}}{(j-l)!}\left(\frac{1+[l/k]}{l!(l+k)!}\right)^{1/2}e^{i\theta_l} \tag{3}$$

(where $[x]$ denotes the greatest integer not exceeding x; the phases θ_l are arbitrary real numbers).

It should be noted that $b_{(1)}=a$ but $b_{(k)}\neq a^k$ for $k\geq 2$. Generalized Bose operators satisfy the commutation relations

$$[b_{(k)},b_{(k)}^\dagger]=1 \ , \tag{4}$$

$$[N,b_{(k)}]=-kb_{(k)} \ , \tag{5}$$

where $N=a^\dagger a$ is the usual number operator. In the Fock space they operate as k-particle creators or annihilators as follows:

$$b_{(k)}|sk+\lambda\rangle=\sqrt{s}\,|(s-1)k+\lambda\rangle \ , \tag{6a}$$

$$b_{(k)}^\dagger|sk+\lambda\rangle=(s+1)^{1/2}|(s+1)k+\lambda\rangle \ , \tag{6b}$$

where $0\leq\lambda\leq k$.

Our generic many-photon squeezed state can be written

$$|\alpha,(z,w)_{(k)}\rangle\equiv D(\alpha)S_{(k)}(z,w)|0\rangle \ , \tag{7}$$

where the new squeezed operator is defined as

$$S_{(k)}(z,w)=\exp(zb_{(k)}^\dagger+iwN-z^*b_{(k)}) \ , \quad z\in\mathbb{C}, \ w\in\mathbb{R} \ , \tag{8}$$

whereas $D(\alpha)$ is the usual displacement operator:

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a), \quad \alpha\in\mathbb{C} \ . \tag{9}$$

The algebra of the operators $b_{(k)},b_{(k)}^\dagger$, given by (4) and (5), is isomorphic to the Weyl algebra: it is, therefore, straightforward to show that the time evolution of the state (7) under the action of the usual harmonic-oscillator Hamiltonian

$$H=\hbar\omega_0(N+\tfrac{1}{2}) \tag{10}$$

is given by

$$|\alpha(t),\{z(t),w\}\rangle_{(k)}=e^{-(i/2)\omega_0 t}|e^{i\omega_0 t}\alpha,(e^{ik\omega_0 t}z,w)\rangle_{(k)} \ . \tag{11}$$

In other words the states $|\alpha,(z,w)\rangle_{(k)}$ evolve in time according to the classical equations of motion. Moreover, the following formulas, which are derived in Appendix A, hold:

$$\langle a\rangle=_{(k)}\langle\alpha,(z,w)|a|\alpha,(z,w)\rangle_{(k)}=z \ , \tag{12}$$

$$\langle a^2\rangle=z^2+\sqrt{2}\delta_{k,2}\rho F(\rho^2) \ , \tag{13}$$

$$\langle N\rangle=|z|^2+k\rho^2 \ , \tag{14}$$

where the function F is defined by

$$F(x)=e^{-x}\sum_{n=0}^{\infty}\frac{\sqrt{2n+1}}{n!}x^n \ , \tag{15}$$

and

$$\rho=|z|\frac{\sin\frac{1}{2}kw}{\frac{1}{2}kw} \ . \tag{16}$$

In configuration space these new squeezed states are not Gaussian wave packets. Their uncertainties can be easily derived using Eqs. (12)–(16) and are given by

$$(\Delta q)^2=\frac{1}{2a_0}\Delta_{(k)}\{\phi(t),\rho\} \ , \tag{17}$$

$$(\Delta p)^2=\frac{\hbar^2 a_0^2}{2}\Delta_{(k)}\{\phi(t)+\pi,\rho\} \ , \tag{18}$$

where

$$a_0=\frac{m\omega_0}{\hbar} \ , \tag{19}$$

$$\Delta_{(k)}\{\phi(t),\rho\}=1+2k\rho^2+2\sqrt{2}\,\delta_{k,2}\cos(\phi)\rho F(\rho^2) \ , \tag{20}$$

and

$$\phi=\phi(t)=\arg z+\tfrac{1}{2}kw+k\omega_0 t \ . \tag{21}$$

The product of the uncertainties is therefore given by

$$(\Delta p)(\Delta q)=\frac{\hbar}{2}[(1+2k\rho^2)^2-\delta_{k,2}8\cos^2(\phi)\rho^2F^2(\rho^2)]^{1/2} \ . \tag{22}$$

From the inequality

$$F(x)<(1+2x)^{1/2} \ \forall x>0 \tag{23}$$

proved in Appendix B, it follows that the product of the uncertainties attains the minimum value $\hbar/2$ for $\rho=0$, i.e.,

$$\Delta q\Delta p=\frac{\hbar}{2} \text{ for } w=\frac{2n\pi}{k}, \ n\in\mathbb{Z} \text{ or } z=0 \ . \tag{24}$$

It is interesting to notice how, not only for $z=0$, the squeezed state (7) coincides with the usual coherent state $|\alpha\rangle=D(\alpha)|0\rangle$ (in that the squeezing operator $S_{(k)}$ becomes trivially equivalent to the identity in its action over $|0\rangle$); but also, for arbitrary $z\neq 0$, $|\alpha,(z,2n\pi/k)\rangle$ is a minimum-uncertainty coherent state, even though different from $|\alpha\rangle$. The latter property has very interesting bearings on the question of integrability of some highly nontrivial nonlinear dynamical systems.[4]

On the other hand, for nonzero squeezing ($\rho\neq 0$) the product of uncertainties has a minimum value $c(\hbar/2)$ with c arbitrarily close to one. The time dependence of the uncertainties is driven by the phase $\phi(t)$. From Eqs. (17), (18), and (22) we see that only for $k=2$ are the uncertainties not constant in time and oscillate with a frequency $k\omega_0$, as for the "two-photon" squeezed state of quantum optics. Furthermore, only for $k=2$ can one obtain for Δq or Δp values separately lower than those of nonsqueezed states. In order to illustrate more clearly the behavior of the uncertainties with squeezing, we give the plots of the dimensionless minimum uncertainty for q (or p) $\Delta_{(2)}(\pi,\rho)$ vs ρ (Fig. 1), the minimum value of the product of the uncertainties, $[\Delta_{(2)}(0,\rho)\Delta_{(2)}(\pi,\rho)]$ vs ρ (Fig. 2) and the loci in the complex ζ plane, $\zeta=\rho e^{i\phi}$, of constant values for $\Delta_{(2)}(\phi,\rho)$ (Fig. 3), and for $[\Delta_{(2)}(\phi,\rho)\Delta_{(2)}(\phi+\pi,\rho)]$ (Fig. 4). In Fig. 1 one observes that the coherent states $|\alpha,(z,w)\rangle$ are indeed squeezed in that the uncertainty of one single canonical variable can be reduced to values smaller than that of the usual coherent states. The most effective value ρ_0 of the squeezing parameter is finite ($\rho_0\sim 0.7$) and corresponds to a squeezing factor $\Delta_{(2)}$ approximately equal to $\frac{1}{3}$.

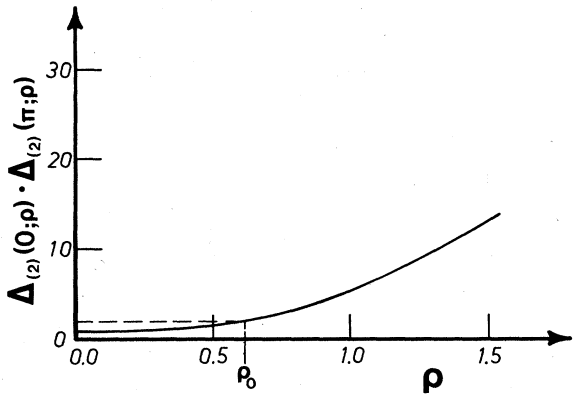

FIG. 2. Plot of the dimensionless minimum product of square uncertainties $\Delta_{(2)}(0,\rho)\Delta_{(2)}(\pi,\rho)$ vs ρ.

On the other hand, for larger values of ρ the squeezing factor becomes asymptotically constant, equal to $\frac{1}{2}$ (see Appendix B). Such a behavior is sensibly different from that of the usual squeezed states[1,3] for which the squeezing factor can be reduced to zero only for infinite value of the squeezing parameter. From Fig. 2 and Eqs. (11) and (22) one can notice that the product of uncertainties oscillates in time with a minimum value which is always greater than $\hbar/2$; but corresponding to $\rho=\rho_0$ the product of uncertainties has a minimum value of about $\hbar/\sqrt{2}$. The frequency of oscillations in the uncertainties is twice that of the motion of the wave packet in the complex plane (k times in general).

The curves of Fig. 4 are compared with the analogous

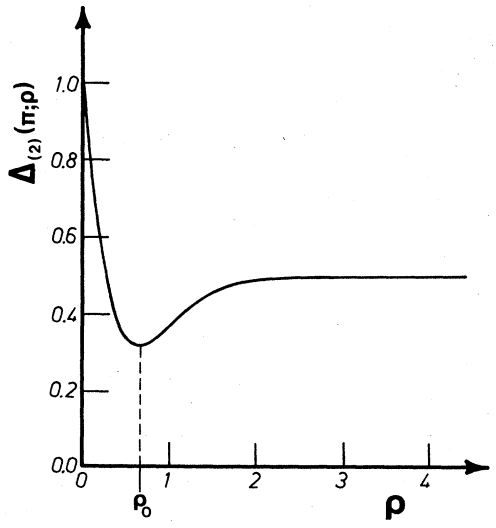

FIG. 1. Plot of the dimensionless minimum uncertainty for q (or p) $\Delta_{(2)}(\pi,\rho)$ vs ρ.

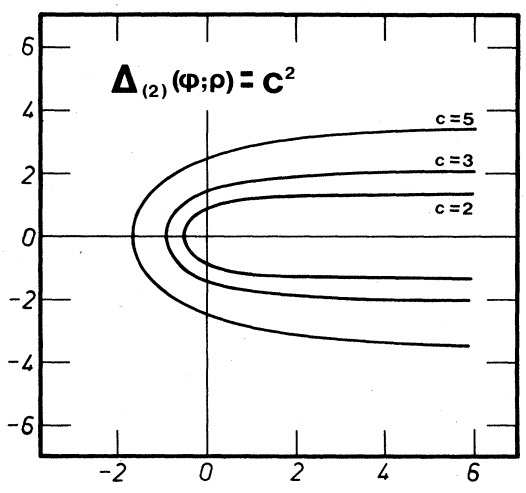

FIG. 3. The locus in the complex plane $\rho e^{i\phi}$ of the solutions of the equations $\Delta_{(2)}(\phi,\rho)=c^2$ for $c=2$ (curve 1), $c=3$ (curve 2), and $c=5$ (curve 3).

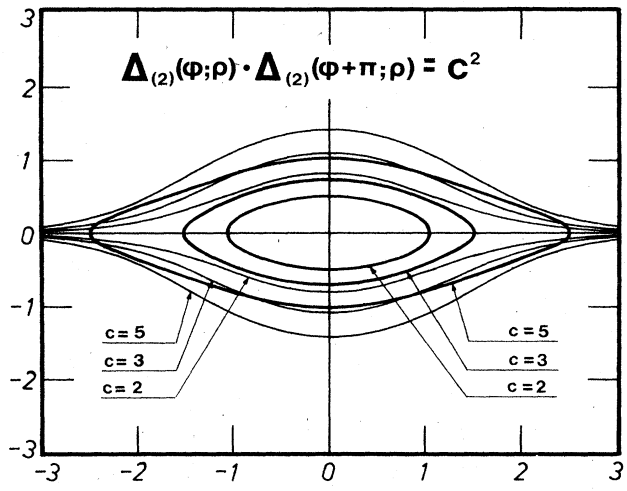

FIG. 4. Full line: the locus in the complex plane $\rho e^{i\phi}$ of the solutions of the equations $\Delta_{(2)}(\phi,\rho)\Delta_{(2)}(\phi+\pi,\rho)=c^2$ for $c=2$ (curve 1), $c=3$ (curve 2), and $c=5$ (curve 3). Broken line: the same for the product of indeterminacies of Fisher, Nieto, and Sandberg (Ref. 1).

curves for the two-boson squeezed states of Fisher, Nieto, and Sandberg. It is interesting to notice how the latter are not compact and show an asymptote for $\phi=n\pi$, $n\in\mathbb{Z}$.

As a last remark, it should be mentioned that preliminary results[4] on the moment-generating function for the states (7) suggest that the latter might be squeezed in the generalized sense of Hong and Mandel[5] even for $k>2$.

This seems to depend mathematically on the finite dimension of the group generated by the squeezing operator $S_{(k)}$ [notice that the algebras generated by the operator $U_{(k)}$ of (1) is finite only for $k\leq 2$] and to depend physically on the renormalization induced by the "dressing" of the photon pairs realized by the infinite series (2).

APPENDIX A

The squeezed vacuum

$$S_{(k)}(z,w)|0\rangle=\exp(iwN+zb^{\dagger}_{(k)}-z^{*}b_{(k)})|0\rangle \tag{A1}$$

can be rewritten in the more convenient form

$$S_{(k)}(z,w)|0\rangle=\exp[-(i\theta_{(k)}+|\phi_{(k)}|^2)|z|^2]e^{\phi_{(k)}zb^{\dagger}_{(k)}}|0\rangle\ , \tag{A2a}$$

with

$$\phi_{(k)}=\frac{\sin(\frac{1}{2}kw)}{\frac{1}{2}kw}e^{ikw/2}\ , \tag{A2b}$$

$$\theta_{(k)}=\frac{kw-\sin kw}{(kw)^2}\ , \tag{A2c}$$

using the identities

$$\exp(iwN+zb^{\dagger}_{(k)}-z^{*}b_{(k)})=\exp\left[-i\frac{|z|^2}{kw}\exp\left(\frac{i}{kw}Ad(zb^{\dagger}_{(k)}+z^{*}b_{(k)})\right)N\right]=\exp\left[-\frac{|z|^2}{(kw)^2}[ikw+(1-e^{ikw})]\right]$$
$$\times\exp\left(i\frac{z}{kw}(1-e^{ikw})b^{\dagger}_{(k)}\right)\exp\left(i\frac{z^{*}}{kw}b_{(k)}\right)e^{iwN}\exp\left(-i\frac{z^{*}}{kw}b_{(k)}\right)\ . \tag{A3}$$

Based on the definition

$$\langle a^n\rangle=\langle 0|S^{\dagger}_{(k)}(z,w)a^nS_{(k)}(z,w)|0\rangle+z^n \tag{A4}$$

and the relations (6a) and (6b) as well as (A2a), one has

$$aS_{(k)}(z,w)|0\rangle=e^{-i\theta_{(k)}|z|^2}\sqrt{k}\,\zeta e^{-\zeta^2}\sum_{n=0}^{\infty}\frac{\zeta^n}{\sqrt{n!}}|kn+(k-1)\rangle\ , \tag{A5}$$

$$a^2S_{(k)}(z,w)|0\rangle=e^{-i\theta_{(k)}|z|^2}\sqrt{k}\,\zeta e^{-\zeta^2}\sum_{n=0}^{\infty}\left[\frac{k(n+1)-1}{n!}\right]^{1/2}\zeta^n|kn+(k-2)\rangle\ , \tag{A6}$$

where

$$\zeta=\phi_{(k)}z\ , \tag{A7}$$

which follow in a straightforward manner from (A4)–(A6).

APPENDIX B

In this appendix we study the asymptotic behavior of the function introduced in Eq. (15):

$$F(x)=e^{-x}\sum_{n=0}^{\infty}\frac{\sqrt{2n+1}}{n!}x^n\equiv e^{-x}[f(x)+2x\partial_x f(x)]\ . \tag{B1}$$

Equation (B1) defines the more convenient auxiliary function

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!\sqrt{2n+1}}\ . \tag{B2}$$

Using the integral representation

$$\frac{1}{\sqrt{p}}=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-pt}}{\sqrt{t}}\,dt \ , \tag{B3}$$

the function $f(x)$ can be written

$$f(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty dt\frac{e^{-t}}{\sqrt{t}}\exp(xe^{-2t}) \ . \tag{B4}$$

By a straightforward change of variables (B4) transforms to

$$f(x)=\frac{e^x}{\sqrt{2\pi}}\int_0^1 \frac{dze^{-xz}}{[(z-1)\ln(1-z)]^{1/2}}=\frac{e^x}{\sqrt{2\pi}}\int_0^1 \frac{dz}{\sqrt{z}}e^{-xz}\left[1-\sum_{n=1}^{\infty}\frac{z^n}{n(n+1)}\right]^{-1/2}$$

$$=\frac{e^x}{\sqrt{2\pi}}\int_0^1 \frac{dze^{-xz}}{\sqrt{z}}[1+\tfrac{1}{4}z+O(z^2)]\sim\frac{e^x}{\sqrt{2\pi}}\left[1+\frac{1}{8x}+O\left(\frac{1}{x^2}\right)\right]\int_0^1\frac{dze^{-xz}}{\sqrt{z}} \ , \tag{B5}$$

which gives the desired asymptotic expansion provided one notices that

$$\frac{1}{\sqrt{\pi}}\int_0^1\frac{dze^{-xz}}{\sqrt{z}}=\frac{1}{\sqrt{x}}\,\mathrm{erf}(\sqrt{x})\sim\frac{1}{\sqrt{x}} \ . \tag{B6}$$

Inserting (B5) into (B1) one finds

$$F(x)=\sqrt{2x}+\frac{1}{4\sqrt{2x}}+O(x^{-3/2}) \ . \tag{B7}$$

Both the asymptotic expansion (B6) and the series in (B1) show that for all x

$$F(x)<\sqrt{1+2x} \ . \tag{B8}$$

From Eq. (B7) one easily derives that

$$\Delta_{(2)}(\pi,\rho)=\frac{1}{2}+\frac{5}{4\rho^2}+O(\rho^{-4}) \ , \tag{B9}$$

given the asymptotic behavior shown in Fig. 1.

By the same token the asymptotes of the curves in Fig. 3 can be seen to be $\pm[\frac{1}{4}(2c^2-1)]^{1/2}$.

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